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On AutoGraphiX Conjecture Regarding Domination Number and Average Eccentricity

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Abstract. The eccentricity of a vertex is the maximal distance from it to another vertex and the average eccentricity $ecc(G)$ of a graph G is the mean value of eccentricities of all vertices of G. A set $S \subseteq V(G)$ is a dominating set of a graph G if $N_G(v) \cap S \neq \emptyset$ for any vertex $v \in V(G) \setminus S$. The domination number $\gamma(G)$ of G is the minimum cardinality of all dominating sets of G. In this paper, we correct an AutoGraphiX conjecture regarding the domination number and average eccentricity, and present a proof of the revised conjecture. In addition, we establish an upper bound on $\gamma(T) - ecc(T)$ for an n-vertex tree T.

1. Introduction

For terminology and notation not defined here, we refer to [1]. Let $G = (V(G), E(G))$ be a simple and connected graph with vertex set $V(G)$ and edge set $E(G)$. A graph H is called a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph H is a *spanning subgraph* of G if $V(H) = V(G)$. A spanning subgraph is a *spanning tree* of G if it is a tree. For $u, v \in V(G)$, the *distance* $d_G(u, v)$ between two vertices u and v is the length of the shortest path between u and v. The *open neighborhood* of v is the set $N_G(v) = \{u \in V(G) \mid uv \in E(G)\}$, and the *closed neighborhood* of v is $N_G[v] = N_G(v) \cup \{v\}$. The *eccentricity* $\varepsilon_G(v)$ of a vertex v is defined as $\varepsilon_G(v) = max\{d_G(u, v) \mid u \in V(G)\}$. Denote by $\zeta(G)$ the sum of the eccentricity over all vertices in $V(G)$, that is $\zeta(G) = \sum_{v\in V(G)} \varepsilon_G(v)$. The *average eccentricity* $ecc(G)$ of G is $ecc(G) = \frac{1}{n}\sum_{v\in V(G)} \varepsilon_G(v) = \frac{\zeta(G)}{n}$. The *degree* $d_G(v)$ of a vertex v is the number of edges incident with v in G. We call v a *pendent vertex* if $d_G(v)$=1. The minimum and maximum vertex degree of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For $S \subseteq V(G)$ and $D \subseteq E(G)$, $G - S$ is the subgraph of G obtained by deleting the vertices in S and their incident edges, and $G - D$ is obtained by deleting the edges in D.

A subset $M \subseteq E(G)$ is called a *matching* of G if each pair of edges in M are not adjacent. The *matching number* $m(G)$ of G is the maximum cardinality of all matchings of G. A set $S \subseteq V(G)$ is a dominating set of a graph G if $N_G(v) \cap S \neq \emptyset$ for any vertex $v \in V(G) \setminus S$. The *domination number* $\gamma(G)$ of G is the minimum cardinality of dominating sets of G. Ore [2] proved that $\gamma(G) \leq \lfloor \frac{n}{2} \rfloor$ if the n-vertex graph G has no islolated

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vertex. The n-vertex graphs with domination number $\lfloor \frac{n}{2} \rfloor$ were characterized in [3, 4]. Unfortunately, determining the domination number is very difficult and it was verified to be NP-complete [5]. For the comprehensive study of domination one may refer to two excellent books by Haynes et al. [6, 7].

In graph theory, a communications network can be viewed as a graph. The vertices of a network graph represent components of the network and the edges represent communication links. The eccentricity of a component in communications network can be interpreted as the maximum time delay of a message emitting from it. Then the average eccentricity of a communications network is the average of eccentricities of all components. It is attractive to study the properties of the average eccentricity. Dankelmann et al. [8] determined the average eccentricity of some graphs:

$$ecc(K_n) = 1,\ ecc(C_n) = \lfloor \frac{n}{2} \rfloor,\ ecc(P_n) = \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \rfloor,\ ecc(K_{n_1,n_2}) = 2.$$

Besides, the authors established some upper bounds on the average eccentricity of a graph and they examined the change in the average eccentricity when a graph is replaced by a spanning subgraph, in particular the two extreme cases: taking a spanning tree and removing one edge. Dankelmann and Mukwembi [9] presented some sharp upper bounds on the average eccentricity of a connected graph with given order in terms of its independence number, chromatic number, domination number or connected domination number. Tang and Zhou [10] gave some lower and upper bounds for average eccentricity of trees with fixed diameter, fixed number of pendent vertices and fixed matching number, respectively, and characterized the n-vertex trees with the first four smallest and the first $\lfloor \frac{n}{2} \rfloor$th-largest average eccentricities for $n \geq 6$. They [11] also determined the n-vertex unicyclic graphs with the first $(\lfloor \frac{n}{2} \rfloor - 1)$th-largest average eccentricities for $n \geq 6$. For more recent results of average eccentricity one may be referred to [12–14].

AutoGraphiX (AGX) [15, 16] computer system is used for finding extremal graphs in regard to graph invariants by applying the variable neighborhood search metaheuristic and data analysis methods. A number of AGX conjectures on various graph invariants have been investigated [17–26]. In particular, many researchers are interested in studying AutoGraphiX conjectures about the average eccentricity. Ilić [27] resolved four conjectures, obtained by the system AutoGraphiX, about the average eccentricity and other graph parameters (the clique number and the independence number) and refuted one AutoGraphiX conjecture about the average eccentricity and the minimum vertex degree. What is important, the author corrected one AutoGraphiX conjecture about the domination number, which was proved by Du and Ilić [28, 29] later. Du and Ilić [30] resolved another five AutoGraphiX conjectures about the average eccentricity and other graph parameters (independence number, chromatic number and the Randić index), and refuted two AutoGraphiX conjectures about the average eccentricity and the spectral radius. In [31, 32], the authors resolved two AutoGraphiX conjectures about the average eccentricity and the Randić index.

This paper is organized as follows. In section 2 we make a minor modification of the low bound on $\gamma(G) - ecc(G)$ in AutoGraphiX Conjecture A.481. In section 3 we study the AutoGraphiX Conjecture A.481 regarding the maximum values of $\gamma(G) - ecc(G)$. In section 4 we present an upper bound on $\gamma(T) - ecc(T)$ for n-vertex trees which is less than that for n-vertex connected graphs.

2. Conjecture A.481-L

Conjecture 2.1 (Conjecture A.481-L) ([15]). *Let G be a connected graph of order n. Then*

$$\gamma(G) - ecc(G) \geq \begin{cases} \lfloor \frac{n+1}{3} \rfloor - \frac{n(3n+1)}{4(n-1)} & if\ n\ is\ odd\ and\ n \not\equiv 1(mod\ 3) \\ \lfloor \frac{n+1}{3} \rfloor - \frac{3n-2}{4} & if\ n\ is\ even\ and\ n \not\equiv 1(mod\ 3) \\ \frac{3}{4n} - \frac{5n-8}{12} & if\ n\ is\ odd\ and\ n \equiv 1(mod\ 3) \\ \frac{1}{n} - \frac{5n-8}{12} & if\ n\ is\ even\ and\ n \equiv 1(mod\ 3) \end{cases}$$

with equality if and only if $G \cong P_n$ for $n \not\equiv 1(mod\ 3)$ or G is a tree with $diam(G) = n - 2$ and $\gamma = \lfloor \frac{n+1}{3} \rfloor$ for $n \equiv 1(mod\ 3)$.

When $n \not\equiv 1(mod\ 3)$, we have $\gamma(P_n) = \lceil \frac{n}{3} \rceil = \lfloor \frac{n+1}{3} \rfloor$. If n is odd, then

$$ecc(P_n) = \frac{1}{n}\Big\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \Big\rfloor = \frac{1}{n}\Big\lfloor \frac{3}{4}(n-1)^2 + n - 1 + \frac{1}{4} \Big\rfloor = \frac{(n-1)(3n+1)}{4n}.$$

Hence, $\gamma(P_n) - ecc(P_n) = \lfloor \frac{n+1}{3} \rfloor - \frac{(n-1)(3n+1)}{4n}$ when n is odd and $n \not\equiv 1(mod\ 3)$, which implies that the lower bound needs modifying in Conjecture 2.1. In the following Theorem 2.2, we give an improvement of Conjecture 2.1 which present a corrected lower bound on $\gamma(G) - ecc(G)$ and characterize the graphs attaining the lower bound of $\gamma(G) - ecc(G)$ when $n \equiv 1(mod\ 3)$.

Let G_1 be an n-vertex graph obtained from K_3 by attaching a path P_{n-3} to a vertex of K_3, and G_i be the graph of order n obtained from a path $P = v_1 v_2 \cdots v_{n-1}$ by attaching a pendent vertex v_n to v_i, where $i = 2, \ldots, \lfloor \frac{n}{2} \rfloor$.

Theorem 2.2. *Let G be a connected graph of order $n \geq 3$. Then*

$$\gamma(G) - ecc(G) \geq \begin{cases} \lceil \frac{n}{3} \rceil - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \rfloor & if\ n \equiv 0, 2(mod\ 3) \\ \frac{n-1}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - n - \frac{3}{4} \rfloor & if\ n \equiv 1(mod\ 3) \end{cases}$$

with equality if and only if $G \cong P_n$ for $n \equiv 0, 2(mod\ 3)$ or $G \in \{G_1, G_2\}$ for $n \equiv 1(mod\ 3)$.

In order to prove Theorem 2.2, we show some lemmas as requisite preliminaries.

Lemma 2.3. *Assume that G is an n-vertex connected graph but not a tree, where $n \geq 5$.*
(1) If $P_n = v_1 \ldots v_n$ is a spanning tree of G, then $ecc(G) < ecc(P_n)$.
(2) If G_2 is a spanning tree of G, then $ecc(G) \leq ecc(G_2)$, and the equality if and only if $G \cong G_1$, where G_1 and G_2 are the graphs defined above.

Proof. Let T be a spanning tree of G. It is obvious that $\varepsilon_G(v) \leq \varepsilon_T(v)$ for all $v \in V(G)$.
(1) If $P_n = v_1 \ldots v_n$ is a spanning tree of G, then $v_s v_t \in E(G)$, where $1 \leq s, t \leq n$ and $|s - t| \geq 2$. It follows that $\varepsilon_G(v_1) = \varepsilon_G(v_n) \leq n - 2 < n - 1 = \varepsilon_{P_n}(v_1) = \varepsilon_{P_n}(v_n)$. Hence, $ecc(G) < ecc(P_n)$.
(2) If $G \cong G_1$, then $ecc(G) = ecc(G_2)$ by a direct calculation. If $G \not\cong G_1$, then $v_s v_n \in E(G)$ for some $s \in \{3, \ldots, n-1\}$, or $v_s v_t \in E(G)$, where $1 \leq s, t \leq n-1$ and $|s - t| \geq 2$. When $v_s v_n \in E(G)$ for some $s \in \{3, \ldots, n-1\}$, we have $\varepsilon_G(v_n) < \varepsilon_{G_2}(v_n)$. In addition, $\varepsilon_G(v_1) \leq n - 3 < n - 2 = \varepsilon_{G_2}(v_1)$ when $v_s v_t \in E(G)$ for $1 \leq s, t \leq n-1$ and $|s - t| \geq 2$. Therefore, $ecc(G) < ecc(G_2)$, as desired. □

Lemma 2.4([9]). *Let G be a connected graph of order n and domination number $\gamma \leq \frac{n}{3}$. Then*

$$\zeta(G) \leq \lfloor 3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma \rfloor,$$

and this bound is sharp.

Suppose that G is an n-vertex connected graph with domination number $\gamma \leq \frac{n}{3}$, and $\zeta(G) = \lfloor 3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma \rfloor$. From the proof of Lemma 2.4 (Theorem 3.4 in [9]), it is easy to see that $diam(G) = 3\gamma(G) - 1$ if G is a tree, or every spanning tree T of G with the same domination number as G such that $diam(T) = 3\gamma(T) - 1$ if G is not a tree.

Lemma 2.5. *Let G be an n-vertex connected graph with domination number $\gamma \leq \frac{n}{3}$. Then*

$$\gamma(G) - ecc(G) \geq \begin{cases} \frac{n}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{n}{2} \rfloor & if\ n \equiv 0(mod\ 3); \\ \frac{n-1}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - n - \frac{3}{4} \rfloor & if\ n \equiv 1(mod\ 3); \\ \frac{n-2}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{3n}{2} - 2 \rfloor & if\ n \equiv 2(mod\ 3), \end{cases}$$

and this bound is sharp. Besides, the equality holds if and only if $G \cong P_n$ when $n \equiv 0(mod\ 3)$, or $G \in \{G_1, G_2\}$ when $n \equiv 1(mod\ 3)$.

Proof. Let

$$\begin{aligned} f(\gamma) &= \gamma - \frac{1}{n}\lfloor 3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma \rfloor \\ &= \gamma - \frac{1}{n}(3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma - C) \\ &= \frac{9}{4n}\gamma^2 - (\frac{3}{2n} + 2)\gamma + 1 + \frac{C}{n}, \end{aligned}$$

where C is a constant and $0 \le C < 1$. Then the symmetry axis of $f(\gamma)$ is $\gamma = \frac{4}{9}n + \frac{1}{3}$. In view of $\gamma \le \frac{n}{3} < \frac{4}{9}n + \frac{1}{3}$, then $f(\gamma)$ is a strictly decreasing function on γ when $\gamma \le \frac{n}{3}$. Therefore, $f(\gamma) \ge f(\lfloor \frac{n}{3} \rfloor)$ with equality if and only if $\gamma = \lfloor \frac{n}{3} \rfloor$. By Lemma 2.4, one can get

$$\gamma(G) - ecc(G) \ge f(\gamma) \ge f(\lfloor \frac{n}{3} \rfloor) = \begin{cases} \frac{n}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{n}{2} \rfloor & if\ n \equiv 0(mod\ 3); \\ \frac{n-1}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - n - \frac{3}{4} \rfloor & if\ n \equiv 1(mod\ 3); \\ \frac{n-2}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{3}{2}n - 2 \rfloor & if\ n \equiv 2(mod\ 3). \end{cases}$$

Equality in the above bound is attained for P_n, G_i, $i = 1, 2$, and the graph obtained from a path $P_{n-2} = v_1 \ldots v_{n-2}$ by attaching two pendent vertices to v_2 when $n \equiv 0(mod\ 3)$, $n \equiv 1(mod\ 3)$ and $n \equiv 2(mod\ 3)$, respectively. Hence, this bound is sharp.

Suppose that $\gamma(G) - ecc(G) = f(\lfloor \frac{n}{3} \rfloor)$, then $\zeta(G) = \lfloor 3n\gamma - n - \frac{9}{4}\gamma^2 + \frac{3}{2}\gamma \rfloor$ and $\gamma = \lfloor \frac{n}{3} \rfloor$.

If G is a tree, then $diam(G) = 3\gamma - 1$ by the proof of Lemma 2.4. When $n \equiv 0(mod\ 3)$, we get $diam(G) = n-1$, which follows that $G \cong P_n$ directly. When $n \equiv 1(mod\ 3)$, we have $\gamma = \frac{n-1}{3}$ and $diam(G) = n-2$. Assume that $V(G) = \{v_1, \ldots, v_n\}$ and $P = v_1 \cdots v_{n-1}$ is one of the longest paths in G. Recall that G_i is the graph obtained from a path $P = v_1v_2 \cdots v_{n-1}$ by attaching a pendent vertex to v_i, where $i \in \{2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Then $G \in \{G_2, \ldots, G_{\lfloor \frac{n}{2} \rfloor}\}$. If $G \in \{G_i \mid 2 \le i \le \lfloor \frac{n}{2} \rfloor,\ i \not\equiv 2(mod\ 3)\}$, then

$$\begin{aligned} \gamma(G) &= \lceil \frac{i-2}{3} \rceil + 1 + \lceil \frac{n-i-2}{3} \rceil \\ &> \frac{i-2}{3} + 1 + \frac{n-i-2}{3} \\ &= \frac{n-1}{3}, \end{aligned}$$

a contradiction to $\gamma = \frac{n-1}{3}$. Together with $\gamma(G_i) = \frac{n-1}{3}$, where $2 \le i \le \lfloor \frac{n}{2} \rfloor$ and $i \equiv 2(mod\ 3)$, we conclude that $G \in \{G_i \mid 2 \le i \le \lfloor \frac{n}{2} \rfloor, i \equiv 2(mod\ 3)\}$. Bearing in mind $ecc(G_2) > ecc(G_i)$ for $2 < i \le \lfloor \frac{n}{2} \rfloor$ and $i \equiv 2(mod\ 3)$, hence

$$f(\lfloor \frac{n}{3} \rfloor) = \gamma(G_2) - ecc(G_2) < \gamma(G_i) - ecc(G_i)$$

for $n \equiv 1(mod\ 3)$. Therefore, $G \cong G_2$.

If G is not a tree, then every spanning tree T of G with the same domination number such that $T \cong P_n$ when $n \equiv 0(mod\ 3)$, or $T \cong G_2$ when $n \equiv 1(mod\ 3)$ by the proof of Lemma 2.4 and the arguments in the above paragraph. Hence, $ecc(G) < ecc(P_n)$ when $n \equiv 0(mod\ 3)$, and $ecc(G) \le ecc(G_2)$ with equality holds if and only if $G \cong G_1$ when $n \equiv 1(mod\ 3)$ by Lemma 2.3. In addition, $\frac{n}{3} = \gamma(G) = \gamma(P_n)$ when $n \equiv 0(mod\ 3)$, and $\frac{n-1}{3} = \gamma(G) = \gamma(G_1) = \gamma(G_2)$ when $n \equiv 1(mod\ 3)$. Therefore,

$$\gamma(G) - ecc(G) > \gamma(P_n) - ecc(P_n) = f(\lfloor \frac{n}{3} \rfloor)$$

when $n \equiv 0(mod\ 3)$, a contradiction to the choice of G. And,

$$\gamma(G) - ecc(G) \ge \gamma(G_2) - ecc(G_2) = f(\lfloor \frac{n}{3} \rfloor)$$

with equality if and only if $G \cong G_1$.

As a result, $\gamma(G) - ecc(G) \ge f(\lfloor \frac{n}{3} \rfloor)$ with equality if and only if $G \cong P_n$ when $n \equiv 0(mod\ 3)$, or $G \in \{G_1, G_2\}$ when $n \equiv 1(mod\ 3)$. This completes the proof. □

Lemma 2.6([9]). *Let G be a connected graph of order n and domination number γ, where $\gamma \ge \frac{n}{3}$. Then $\zeta(G) \le \zeta(T_{n,\gamma})$, where $T_{n,\gamma}$ is the tree of order n obtained from a path $P = v_1v_2 \cdots v_{2n-3\gamma}$ by appending an pendent vertex to each of the first $\lceil \frac{3\gamma-n}{2} \rceil$ vertices and each of the last $\lfloor \frac{3\gamma-n}{2} \rfloor$ vertices.*

It is obvious that $\gamma(T_{n,\gamma}) = 3\gamma - n + \lceil \frac{3n-6\gamma-2}{3} \rceil = \gamma$. Let G be an n-vertex connected graph with domination number $\gamma \ge \frac{n}{3}$. Suppose that $\zeta(G) = \zeta(T_{n,\gamma})$. From the proof of Lemma 2.6 (Theorem 3.5 in [9]), it is easy to see that $G \cong T_{n,\gamma}$ if G is a tree, or every spanning tree of G with the same domination number as G is isomorphic to $T_{n,\gamma}$ if G is not a tree.

Lemma 2.7. *Let G be an n-vertex connected graph with domination number $\gamma > \frac{n}{3}$. Then*

$$\gamma - ecc(G) \geq \begin{cases} \gamma(T_{n,\frac{n+3}{3}}) - ecc(T_{n,\frac{n+3}{3}}) & if\ n \equiv 0(mod\ 3) \\ \gamma(P_n) - ecc(P_n) & if\ n \equiv 1,2(mod\ 3) \end{cases}$$

with equality if and only if $G \cong P_n$ when $n \equiv 2(mod\ 3)$.

Proof. Let $\mathcal{T} = \{T_{n,\gamma} \mid \gamma > \frac{n}{3}\}$, where $T_{n,\gamma}$ is the tree defined in Lemma 2.6. Suppose that $T_{n,\gamma^*} \in \mathcal{T}$ is the graph such that

$$\gamma(T_{n,\gamma^*}) - ecc(T_{n,\gamma^*}) = min\{\gamma(T_{n,\gamma}) - ecc(T_{n,\gamma}) \mid T_{n,\gamma} \in \mathcal{T}\}.$$

Let u_i be the pendent vertex adjacent to v_i in graph T_{n,γ^*}, $i = 1, \ldots, t, s, \ldots, 2n - 3\gamma^*$, where $t = \lceil \frac{3\gamma^* - n}{2} \rceil$ and $s = 2n - 3\gamma^* + 1 - \lfloor \frac{3\gamma^* - n}{2} \rfloor$. If $\lfloor \frac{3\gamma^* - n}{2} \rfloor \geq 2$, then $\lceil \frac{3\gamma^* - n}{2} \rceil \geq 2$ and $\gamma^* \geq \frac{n+4}{3}$. Define

$$\widetilde{T} = T_{n,\gamma^*} - \{v_{t-1}v_t, v_s v_{s+1}\} + \{v_{t-1}u_t, v_{s+1}u_s\},$$

then $\gamma(\widetilde{T}) = 3\gamma^* - n - 2 + \lceil \frac{3n - 6\gamma^* + 2}{3} \rceil = \gamma^* - 1 \geq \frac{n+1}{3}$, which implies that $\widetilde{T} \in \mathcal{T}$. Note that $ecc(\widetilde{T}) > ecc(T_{n,\gamma^*})$. Thus,

$$\begin{aligned} \gamma(\widetilde{T}) - ecc(\widetilde{T}) &< \gamma^* - 1 - ecc(T_{n,\gamma^*}) \\ &< \gamma(T_{n,\gamma^*}) - ecc(T_{n,\gamma^*}), \end{aligned}$$

which contradicts to the choice of T_{n,γ^*}. This gives $\lfloor \frac{3\gamma^* - n}{2} \rfloor \leq 1$. Then $\frac{n+1}{3} \leq \gamma^* \leq \frac{n+3}{3}$, which follows that $\gamma^* = \lfloor \frac{n}{3} \rfloor + 1$. Hence,

$$\gamma(T_{n,\gamma}) - ecc(T_{n,\gamma}) \geq \gamma(T_{n,\gamma^*}) - ecc(T_{n,\gamma^*})$$

with equality if and only if $\gamma = \lfloor \frac{n}{3} \rfloor + 1$. Since $T_{n,\gamma^*} \cong P_n$ when $n \equiv 1, 2(mod\ 3)$,

$$\gamma - ecc(G) \geq \gamma(T_{n,\gamma}) - ecc(T_{n,\gamma}) \geq \begin{cases} \gamma(T_{n,\frac{n+3}{3}}) - ecc(T_{n,\frac{n+3}{3}}) & if\ n \equiv 0(mod\ 3); \\ \gamma(P_n) - ecc(P_n) & if\ n \equiv 1,2(mod\ 3), \end{cases}$$

by Lemma 2.6.

Assume that $n \equiv 2(mod\ 3)$. If G is a tree, then the above equality holds if and only if $G \cong P_n$ by Lemma 2.6. If G is not a tree and $\gamma - ecc(G) = \gamma(P_n) - ecc(P_n)$, then every spanning tree of G with the same domination number as G is isomorphic to P_n. Thus $ecc(G) < ecc(P_n)$ by Lemma 2.3, which follows that $\gamma - ecc(G) > \gamma(P_n) - ecc(P_n)$, a contradiction. Hence, $\gamma - ecc(G) \geq \gamma(P_n) - ecc(P_n)$ with equality if and only if $G \cong P_n$ when $n \equiv 2(mod\ 3)$. We have completed the proof. □

In what follows, we present a proof of theorem 2.2.

Proof of Theorem 2.2. We proceed by considering the following three cases.

Case 1. $n \equiv 0(mod\ 3)$.

When $\gamma(G) \leq \frac{n}{3}$, $\gamma(G) - ecc(G) \geq \frac{n}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \rfloor$ with equality if and only if $G \cong P_n$ by Lemma 2.5. In order to characterize the graph with minimal value $\gamma(G) - ecc(G)$, it suffices to compare $\gamma(P_n) - ecc(P_n)$ with $\gamma(T_{n,\frac{n+3}{3}}) - ecc(T_{n,\frac{n+3}{3}})$ by Lemmas 2.5 and 2.7. Since $\gamma(P_n) = \gamma(T_{n,\frac{n+3}{3}}) - 1$ and $ecc(P_n) > ecc(T_{n,\frac{n+3}{3}})$, $\gamma(P_n) - ecc(P_n) < \gamma(T_{n,\frac{n+3}{3}}) - ecc(T_{n,\frac{n+3}{3}})$. It implies that $\gamma(G) - ecc(G) \geq \frac{n}{3} - \frac{1}{n}\lfloor \frac{3}{4}n^2 - \frac{1}{2}n \rfloor$ with equality if and only if $G \cong P_n$.

Case 2. $n \equiv 1(mod\ 3)$.

By Lemmas 2.5 and 2.7, we only need to compare $\gamma(G_i) - ecc(G_i)$ with $\gamma(P_n) - ecc(P_n)$ for $i = 1, 2$. Since

$$\begin{aligned}
& \gamma(G_i) - ecc(G_i) - (\gamma(P_n) - ecc(P_n)) \\
= & \ \Big(\frac{n-1}{3} - \frac{1}{n}\Big\lfloor\frac{3}{4}n^2 - n - \frac{3}{4}\Big\rfloor\Big) - \Big(\frac{n+2}{3} - \frac{1}{n}\Big\lfloor\frac{3}{4}n^2 - \frac{1}{2}n\Big\rfloor\Big) \\
= & \begin{cases} \frac{1}{n} - \frac{1}{2} & if\ n\ is\ even \\ \frac{1}{2n} - \frac{1}{2} & if\ n\ is\ odd \end{cases} \\
< & \ 0,
\end{aligned}$$

for $i = 1, 2$, $\gamma(G) - ecc(G) \geq \frac{n-1}{3} - \frac{1}{n}\lfloor\frac{3}{4}n^2 - n - \frac{3}{4}\rfloor$ with equality if and only if $G \in \{G_1, G_2\}$ by Lemma 2.5.
Case 3. $n \equiv 2(mod\ 3)$.

Note that $\gamma(G) - ecc(G) \geq \frac{n-2}{3} - \frac{1}{n}\lfloor\frac{3}{4}n^2 - \frac{3}{2}n - 2\rfloor$ when $\gamma(G) \leq \frac{n}{3}$, and $\gamma(G) - ecc(G) \geq \gamma(P_n) - ecc(P_n)$ when $\gamma(G) > \frac{n}{3}$ by Lemmas 2.5 and 2.7. In addition,

$$\Big(\frac{n-2}{3} - \frac{1}{n}\Big\lfloor\frac{3}{4}n^2 - \frac{3}{2}n - 2\Big\rfloor\Big) - \Big(\frac{n+1}{3} - \frac{1}{n}\Big\lfloor\frac{3}{4}n^2 - \frac{1}{2}n\Big\rfloor\Big) = \frac{2}{n} > 0,$$

so $\gamma(G) - ecc(G) \geq \frac{n+1}{3} - \frac{1}{n}\lfloor\frac{3}{4}n^2 - \frac{1}{2}n\rfloor$ with equality if and only if $G \cong P_n$ by Lemma 2.7. This completes the proof. □

3. Conjecture A.481-U

Denote by $K_{a,b}$ the graph of order n obtained from a complete graph K_a by attaching a pendent vertex to each of the b vertices of K_a, where $a + b = n$ and $0 \leq b \leq a$.
Conjecture 3.1 (Conjecture A.481-U) ([15]).

$$\gamma(G) - ecc(G) \leq \begin{cases} \frac{n-6}{2} + \frac{1}{2n} & if\ n\ is\ odd \\ \frac{n-5}{2} & if\ n\ is\ even \end{cases}$$

with equality if and only if $rad(G) = 2$ *and* $\gamma(G) = \lfloor\frac{n}{2}\rfloor$, *where* $rad(G) = min\{ecc(v) \mid v \in V(G)\}$. *In addition, the equality is attained for the graph* $K_{\lceil\frac{n}{2}\rceil,\lfloor\frac{n}{2}\rfloor}$.
Lemma 3.2. *Let* G *be a connected graph of order* 4. *Then* $\gamma(G) - ecc(G) \leq 0$ *with equality if and only if* $G \in \{K_4, C_4\}$.

Proof. If $\Delta(G) = 3$, then $\gamma(G) = 1$. It is well known that $ecc(G) \geq ecc(K_4) = 1$ with equality if and only if $G \cong K_4$. In addition, $\gamma(K_4) = 1$, hence

$$\gamma(G) - ecc(G) \leq \gamma(K_4) - ecc(K_4) = 0,$$

and the equality holds if and only if $G \cong K_4$. If $\Delta(G) = 2$, then $G \in \{P_4, C_4\}$. Because

$$\gamma(P_4) - ecc(P_4) < \gamma(C_4) - ecc(C_4) = 0,$$

$\gamma(G) - ecc(G) \leq \gamma(C_4) - ecc(C_4) = 0$ with equality if and only if $G \cong C_4$. The result follows. □

In what follows, we introduce some graph sets, denoted by $\mathscr{G}_1, \ldots, \mathscr{G}_6$, defined in [3]. Let H be any graph with vertex set $\{v_1, \ldots, v_k\}$. Denote by $f(H)$ the graph obtained from H by adding new vertices $u_1, \ldots, u_k$ and the edges v_iu_i, $i = 1, \ldots, k$. Define

$$\mathscr{G}_1 = \{C_4\} \cup \{G \mid G = f(H)\ for\ some\ connected\ graph\ H\}.$$

Let $\mathscr{F} = \mathscr{A} \cup \mathscr{B}$ and

$$\mathscr{G}_2 = \mathscr{F} - \{C_4\},$$

where $\mathscr{A} = \{C_4, G(7, i) \mid i = 1, \ldots, 6\}$ and $\mathscr{B} = \{K_3, G(5, i) \mid i = 1, \ldots, 4\}$, as shown in Figure 1 and Figure 2, respectively.

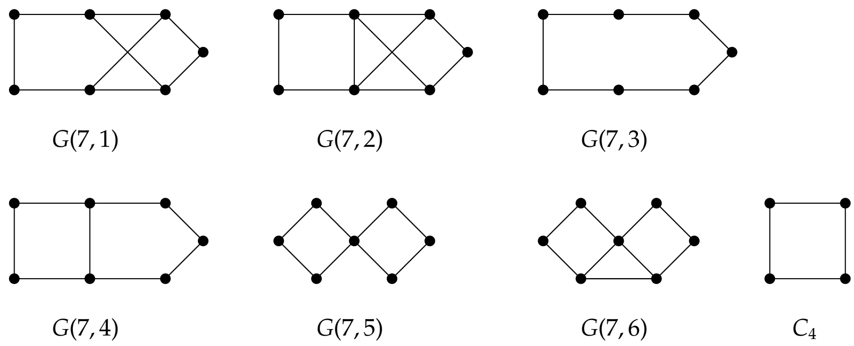

Figure 1. Graphs in family $\mathscr{A}$

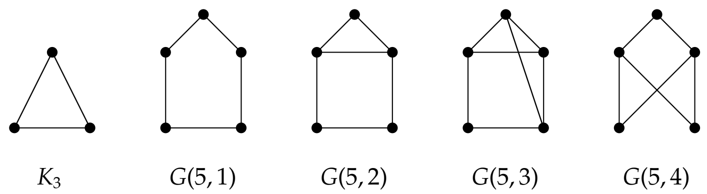

Figure 2. Graphs in family $\mathscr{B}$

For any graph H, let $\varphi(H)$ be the set of connected graphs, each of which can be formed from $f(H)$ by adding a new vertex x and edges joining x to one or more vertices of H. Then define

$$\mathscr{G}_3 = \{G \mid G = \varphi(H) \text{ } for\ some\ graph\ H\}.$$

Let $G \in \mathscr{G}_3$ and y be a vertex of a copy of C_4. Denote by $\theta(G)$ the graph obtained by joining G to C_4 with the single edge xy, where x is the new vertex added in forming G. Then define

$$\mathscr{G}_4 = \{G \mid G = \theta(H) \text{ } for\ some\ graph\ H \in \mathscr{G}_3\}.$$

Let u, v, w be the vertex sequence of a path P_3. For any graph H, let $\mathscr{P}(H)$ be the set of connected graphs which may be formed from $f(H)$ by joining each of u and w to one or more vertices of H. Then define

$$\mathscr{G}_5 = \{G \mid G = \mathscr{P}(H) \text{ } for\ some\ graph\ H\}.$$

For a graph $X \in \mathscr{B}$, let $U \subset V(X)$ be a set of vertices such that no fewer than $\gamma(X)$ vertices of X dominate $V(X) \setminus U$. Let $\mathscr{R}(H, X)$ be the set of connected graphs which may be formed from $f(H)$ by joining each vertex of U to one or more vertices of H for some set U as defined above and any graph H. Then define

$$\mathscr{G}_6 = \{G \mid G \in \mathscr{R}(H, X) \text{ } for\ some\ X \in \mathscr{B}\ and\ some\ H\}.$$

Lemma 3.3([3]). *A connected graph G of order n satisfies $\gamma(G) = \lfloor \frac{n}{2} \rfloor$ if and only if $G \in \mathscr{G} = \bigcup_{i=1}^{6} \mathscr{G}_i$, where $\mathscr{G}_i$, $i = 1, \ldots, 6$, is the set defined above.*

Let G be an n-vertex connected graph. If $G \in \mathscr{G}_1$, then n is even. If $G \in \bigcup_{i=2}^{6} \mathscr{G}_i$, then n is odd. Let $G' \in \mathscr{P}(K_{\frac{n-3}{2}}) \subseteq \mathscr{G}_5$ be the graph obtained from $f(K_{\frac{n-3}{2}})$ by joining each of u and w to every vertex of $K_{\frac{n-3}{2}}$, and $G'' \in \mathscr{R}(K_{\frac{n-3}{2}}, K_3) \subseteq \mathscr{G}_6$ be the graph obtained from $f(K_{\frac{n-3}{2}})$ by joining each vertex of $U = \{x, y\} \subseteq V(K_3)$ to every vertex of $K_{\frac{n-3}{2}}$. Then $ecc(G') = ecc(G'') = \frac{1}{n}(2 \cdot \frac{n+1}{2} + 3 \cdot \frac{n-1}{2}) = \frac{5n-1}{2n}$ for $n \geq 5$.

Lemma 3.4. *Let G a connected graph of order $n(\geq 5)$ satisfies $\gamma(G) = \lfloor \frac{n}{2} \rfloor$. Then*

$$ecc(G) \geq \begin{cases} 2 & n = 5; \\ \frac{1}{n}\lfloor \frac{5n}{2} \rfloor & n \geq 6, \end{cases}$$

with equality if and only if $G \in \{G(5,1), G(5,2), G(5,3), G(5,4)\} \subseteq \mathscr{B}$ *when* $n = 5$*, see Figure 2, or* $G \in \{G(7,1), G(7,2), G', G'', K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}\}$ *when* $n \geq 6$*, where* $G(7,1), G(7,2) \in \mathscr{A}$*, as shown in Figure 1.*

Proof. By Lemma 3.3, we have $G \in \mathscr{G} = \bigcup_{i=1}^{6} \mathscr{G}_i$. Let H be any graph with vertex set $\{v_1, \ldots, v_{|V(H)|}\}$. Recall that $f(H)$ is the graph obtained from H by adding new vertices $u_1, \ldots, u_{|V(H)|}$ and the edges $v_iu_i, i = 1, \ldots, |V(H)|$. We prove this lemma by considering the following two cases.

Case 1. n is even.

In this case, $G \in \mathscr{G}_1$. In view of $n \geq 5$, then $G = f(H)$ for some connected graph H and $|V(H)| = \frac{n}{2}$. By the definition of $f(H)$, we have

$$\varepsilon_G(v_i) \geq d_G(v_i, u_j) = d_G(v_i, v_j) + d_G(v_j, u_j) \geq 2$$

and

$$\varepsilon_G(u_i) \geq d_G(u_i, u_j) = d_G(u_i, v_i) + d_G(v_i, v_j) + d_G(v_j, u_j) \geq 3$$

where $1 \leq i \neq j \leq \frac{n}{2}$. If $G \ncong f(K_{\frac{n}{2}})$, then $v_sv_t \notin E(H)$ for some $s, t \in \{1, \ldots, \frac{n}{2}\}$. It follows that

$$\varepsilon_G(v_s) \geq d_G(v_s, u_t) = d_G(v_s, v_t) + d_G(v_t, u_t) \geq 3.$$

Similarly, $\varepsilon_G(v_t) \geq 3$. Therefore,

$$\zeta(G) \geq 2\Big(\frac{n}{2} - 2\Big) + 3\Big(\frac{n}{2} + 2\Big) > \frac{5n}{2} = \zeta(f(K_{\frac{n}{2}})).$$

So $ecc(G) \geq \frac{5}{2}$ with equality if and only if $G \cong f(K_{\frac{n}{2}})$, which is equivalent to $G \cong K_{\frac{n}{2}, \frac{n}{2}}$.

Case 2. n is odd.

It suffices to prove that $ecc(G) \geq \frac{5n-1}{2n}$ for $n \geq 6$ and n is odd. Since n is odd, $G \in \bigcup_{i=2}^{6} \mathscr{G}_i$. If $G \in \mathscr{G}_2$, then

$$ecc(G) \geq \begin{cases} 2(< \frac{5n-1}{2n}) & if\ n = 5 \\ \frac{17}{7}(= \frac{5n-1}{2n}) & if\ n = 7 \end{cases}$$

by a direct calculation. The equality holds if and only if $G \in \{G(5,1), G(5,2), G(5,3), G(5,4)\}$ when $n = 5$, or $G \in \{G(7,1), G(7,2)\}$ when $n = 7$.

If $G \in \mathscr{G}_3$, then $G \in \varphi(H)$ for some graph H and $|V(H)| = \frac{n-1}{2}$. Note that $K_{\frac{n+1}{2}, \frac{n-1}{2}} \in \varphi(K_{\frac{n-1}{2}}) \subseteq \mathscr{G}_3$ and $\zeta\Big(K_{\frac{n+1}{2}, \frac{n-1}{2}}\Big) = \frac{5n-1}{2}$. If $G \ncong K_{\frac{n+1}{2}, \frac{n-1}{2}}$, then $v_sv_t \notin E(H)$ or $xv_s \notin E(G)$ for some $s, t \in \{1, \ldots, \frac{n-1}{2}\}$, where x is the new vertex added in forming G. If $v_sv_t \notin E(H)$ for some $s, t \in \{1, \ldots, \frac{n-1}{2}\}$, then $\varepsilon_G(v) \geq 3$ for $v \in \{v_s, v_t, u_1, \ldots, u_{\frac{n-1}{2}}\}$ and $\varepsilon_G(v) \geq 2$ for $v \in V(H) \setminus \{v_s, v_t\}$. In addition, $\varepsilon_G(x) \geq d_G(x, u_s) = d_G(x, v_s) + d_G(v_s, u_s) \geq 2$. Thus,

$$\begin{aligned} \zeta(G) &\geq 2\Big(\frac{n-1}{2} - 1\Big) + 3\Big(\frac{n-1}{2} + 2\Big) \\ &> \frac{5n-1}{2} = \zeta\Big(K_{\frac{n+1}{2}, \frac{n-1}{2}}\Big). \end{aligned}$$

If $xv_s \notin E(G)$ for some $s \in \{1, \ldots, \frac{n-1}{2}\}$, then $\varepsilon_G(x) \geq d_G(x, u_s) = d_G(x, v_s) + d_G(v_s, u_s) \geq 3$. Besides, $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_G(u_i) \geq 3$ for $i = 1, \ldots, \frac{n-1}{2}$. This gives

$$\zeta(G) \geq 2 \cdot \frac{n-1}{2} + 3\Big(\frac{n-1}{2} + 1\Big) > \zeta\Big(K_{\frac{n+1}{2}, \frac{n-1}{2}}\Big).$$

By the arguments above, $ecc(G) \geq \frac{5n-1}{2n}$, and the equality holds if and only if $G \cong K_{\frac{n+1}{2}, \frac{n-1}{2}}$.

If $G \in \mathscr{G}_4$, then $\varepsilon_G(x) \geq 3$ for all vertices $x \in V(G)$. Therefore,

$$ecc(G) \geq 3 > \frac{5n-1}{2n}.$$

If $G \in \mathscr{G}_5$, then $G \in \mathscr{P}(H)$ for some graph H and $|V(H)| = \frac{n-3}{2}$. Recall that $G' \in \mathscr{P}(K_{\frac{n-3}{2}}) \subseteq \mathscr{G}_5$, if $G \ncong G'$, then $v_sv_t \notin E(H)$ or $uv_s \notin E(G)$, where $s, t \in \{1, \ldots, \frac{n-3}{2}\}$ and u is a pendent vertex of P_3. If $v_sv_t \notin E(H)$ for some $s, t \in \{1, \ldots, \frac{n-3}{2}\}$, then $\varepsilon_G(v) \geq 3$ for $v \in \{v_s, v_t, u_1, \ldots, u_{\frac{n-3}{2}}\}$ and $\varepsilon_G(v) \geq 2$ for $v \in V(G) \setminus \{v_s, v_t, u_1, \ldots, u_{\frac{n-3}{2}}\}$. It follows that

$$\zeta(G) \geq 2 \cdot \frac{n-1}{2} + 3 \cdot \frac{n+1}{2} > \zeta(G').$$

If $uv_s \notin V(G)$ for some $s \in \{1, \ldots, \frac{n-3}{2}\}$, then

$$\varepsilon_G(u) \geq d_G(u, u_s) = d_G(u, v_s) + d_G(v_s, u_s) \geq 3.$$

For the inner vertex of P_3, say v, we have

$$\varepsilon_G(v) \geq d_G(v, u_s) = d_G(v, v_s) + d_G(v_s, u_s) \geq 3.$$

In addition, $\varepsilon_G(w) \geq 2$, $\varepsilon_G(v_i) \geq 2$ and $\varepsilon_G(u_i) \geq 3$ for $i = 1, \ldots, \frac{n-3}{2}$, where w is the pendent vertex of P_3 other than u. Hence,

$$\zeta(G) \geq 2 \cdot \frac{n-1}{2} + 3 \cdot \frac{n+1}{2} > \zeta(G').$$

So $ecc(G) \geq ecc(G') = \frac{5n-1}{2n}$ with equality if and only if $G \cong G'$.

If $G \in \mathscr{G}_6$, then $G \in \mathscr{R}(H, X)$ for some $X \in \mathscr{B} = \{K_3, G(5,1), G(5,2), G(5,3), G(5,4)\}$ and some graph H. Besides, $|V(H)| = \frac{n-3}{2}$ when $X = K_3$, and $|V(H)| = \frac{n-5}{2}$ when $X \in \mathscr{B} \setminus \{K_3\}$.

If the graph $X \in \mathscr{B} \setminus \{K_3\}$, then $|X| = 5$. It is easy to verify that $\gamma(X) = 2$. Let $\gamma(V(X) \setminus U)$ be the minimum number of the vertices of X dominate $V(X) \setminus U$. By the definition of U, we derive that $\gamma(V(X) \setminus U) \geq \gamma(X) = 2$. If $|U| \geq 3$, then $|V(X) \setminus U| \leq 2$. If $|V(X) \setminus U| = 1$, then $\gamma(V(X) \setminus U) = 1 < \gamma(X) = 2$, a contradiction. If $|V(X) \setminus U| = 2$, assume that $V(X) \setminus U = \{s_1, s_2\}$, then $s_1s_2 \in E(X)$ or $N_X(s_1) \cap N_X(s_2) \neq \emptyset$. Thus $\gamma(V(X) \setminus U) = 1 < \gamma(X) = 2$, a contradiction to the definition of U. Therefore, $|U| \leq 2$. For any vertex $x' \in V(X) \setminus U$ and some vertex $x \in U$, we have

$$d_G(x', u_i) = d_G(x', x) + d_G(x, v_i) + d_G(v_i, u_i) \geq 3,$$

where $i \in \{1, \ldots, \frac{n-5}{2}\}$. Hence, $\varepsilon_G(x') \geq 3$ and $\varepsilon_G(u_i) \geq 3$ for $i = 1, \ldots, \frac{n-5}{2}$. Bearing in mind $\varepsilon_G(v) \geq 2$ for $v \in V(H) \cup U$, one can easily get

$$\zeta(G) \geq 2 \cdot \frac{n-1}{2} + 3 \cdot \frac{n+1}{2} > \zeta(G'').$$

Suppose that $X = K_3$ and $V(K_3) = \{x, y, z\}$. Then $U = \{x\}$ or $U = \{x, y\}$. If $U = \{x\}$, then $d_G(u_i, y) = d_G(u_i, v_i) + d_G(v_i, x) + d_G(x, y) \geq 3$ and $d_G(u_i, z) \geq 3$, where $i \in \{1, \ldots, \frac{n-3}{2}\}$. Hence, $\varepsilon_G(y) \geq 3$, $\varepsilon_G(z) \geq 3$ and $\varepsilon_G(u_i) \geq 3$ for $i = 1, \ldots, \frac{n-3}{2}$. It is easy to see that $\varepsilon_G(v) \geq 2$ for $v \in V(H) \cup \{x\}$, so

$$\zeta(G) \geq 2 \cdot \frac{n-1}{2} + 3 \cdot \frac{n+1}{2} > ecc(G'').$$

If $U = \{x, y\}$, then $G'' \in \{G \mid G \in \mathscr{R}(H, K_3) \textit{ for some } H\}$. If $G \ncong G''$, then $v_sv_t \notin E(G)$ or $xv_s \notin E(G)$ for some $s, t \in \{1, \ldots, \frac{n-3}{2}\}$. If $v_sv_t \notin E(G)$, then $\varepsilon_G(v) \geq 3$ for $v \in \{v_s, v_t, u_1, \ldots, u_{\frac{n-3}{2}}\}$ and $\varepsilon_G(v) \geq 2$ for $v \in V(G) \setminus \{u_1, \ldots, u_{\frac{n-3}{2}}, v_s, v_t\}$, and thus

$$\zeta(G) > \zeta(G'').$$

If $xv_s \notin E(G)$, then $\varepsilon_G(v) \geq 3$ for $v \in \{u_1, \ldots, u_{\frac{n-3}{2}}, x, z\}$ and $\varepsilon_G(v) \geq 2$ for $v \in V(G) \setminus \{u_1, \ldots, u_{\frac{n-3}{2}}, x, z\}$. Hence,

$$\zeta(G) > \zeta(G'').$$

And thus $ecc(G) \geq \frac{5n-1}{2n}$ with equality if and only if $G \cong G''$.

In conclusion, if n is odd, then

$$ecc(G) \geq \begin{cases} 2 & n = 5 \\ \frac{5n-1}{2n} & n \geq 6 \end{cases}$$

with equality if and only if $G \in \{G(5,1), G(5,2), G(5,3), G(5,4)\}$ when $n = 5$, or $G \in \{K_{\frac{n+1}{2},\frac{n-1}{2}}, G(7,1), G(7,2), G', G''\}$ when $n \geq 6$. Together with Case 1, the result follows. □

With some tiny modification of Conjecture 3.1, we derive the following corrected version:

Theorem 3.5. *Let G be an n-vertex connected graph, where $n \geq 2$. Then*

$$\gamma(G) - ecc(G) \leq \begin{cases} 0 & if\ n \leq 5 \\ \lfloor \frac{n}{2} \rfloor - \frac{1}{n} \lfloor \frac{5n}{2} \rfloor & if\ n \geq 6 \end{cases}$$

with equality if and only if $G \in \{P_2, C_3, K_4, C_4, K_5, G(5,1), G(5,2), G(5,3), G(5,4)\}$ if $n \leq 5$, or $G \in \{K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}, G(7,1), G(7,2), G', G''\}$ when $n \geq 6$.

Proof. It is easy to verify that the statement holds for $2 \leq n \leq 3$. By Lemma 3.2, the result follows immediately when $n = 4$. In what follows, we consider $n \geq 5$.

If $n = 5$, then $1 \leq \gamma(G) \leq \lfloor \frac{n}{2} \rfloor = 2$. When $\gamma(G) = 1$, we have $\gamma(G) - ecc(G) \leq \gamma(K_5) - ecc(K_5) = 0$. When $\gamma(G) = 2$, $\gamma(G) - ecc(G) \leq 0$ with equality if and only if $G \in \{G(5,1), G(5,2), G(5,3), G(5,4)\}$ by Lemma 3.4.

If $n \geq 6$ and $\gamma(G) = \lfloor \frac{n}{2} \rfloor$, then $\gamma(G) - ecc(G) \leq \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{n} \left\lfloor \frac{5n}{2} \right\rfloor$, and the equality holds if and only if $G \in \{K_{\lceil \frac{n}{2} \rceil, \lfloor \frac{n}{2} \rfloor}, G(7,1), G(7,2), G', G''\}$ by Lemma 3.4. Suppose that $n \geq 6$ and $\gamma(G) < \lfloor \frac{n}{2} \rfloor$. If $\Delta(G) = n-1$, then $\gamma(G) = 1$, and thus $\gamma(G) - ecc(G) \leq \gamma(K_n) - ecc(K_n) = 0$. Note that

$$\begin{aligned} \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{n} \left\lfloor \frac{5n}{2} \right\rfloor &= \begin{cases} \frac{n}{2} - \frac{5}{2} & if\ n\ is\ even \\ \frac{n}{2} + \frac{1}{2n} - 3 & if\ n\ is\ odd \end{cases} \\ &> 0. \end{aligned}$$

Hence, $\gamma(G) - ecc(G) < \left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{n} \left\lfloor \frac{5n}{2} \right\rfloor$. If $\Delta(G) \leq n-2$, we get $ecc(G) \geq 2$ since $\varepsilon(v) \geq 2$ for every vertex $v \in V(G)$. Thus,

$$\begin{aligned} \gamma(G) - ecc(G) - \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{n} \left\lfloor \frac{5n}{2} \right\rfloor \right) &\leq \left\lfloor \frac{n}{2} \right\rfloor - 1 - 2 - \left(\left\lfloor \frac{n}{2} \right\rfloor - \frac{1}{n} \left\lfloor \frac{5n}{2} \right\rfloor \right) \\ &= \begin{cases} -\frac{1}{2} & if\ n\ is\ even \\ -\frac{1}{2} - \frac{1}{2n} & if\ n\ is\ odd \end{cases} \\ &< 0. \end{aligned}$$

Hence, the graph G can not be the graph with maximum value $\gamma(G) - ecc(G)$ if $\gamma(G) < \lfloor \frac{n}{2} \rfloor$. This completes the proof. □

4. Upper bound on $\gamma(T) - ecc(T)$

In this section, we present the upper bound on $\gamma(T) - ecc(T)$ among all n-vertex trees T, and characterize the extremal trees.

Lemma 4.1([27]). *Let uv be a bridge of the graph G. Suppose that G_u and G_v are the components of $G - uv$ containing u and v, respectively. Construct the graph G' by identifying the vertices u and v (and call this vertex also u') with additional pendent edges $u'v'$. We say that $G' = \sigma(G, uv)$ is a σ-transform of G. Then $ecc(G') < ecc(G)$.*

Lemma 4.2([33]). *For a graph G, we have $\gamma(G) \leq m(G)$.*

Lemma 4.3([10]). *Let T be an n-vertex tree with matching number m, where $2 \leq m \leq \lfloor \frac{n}{2} \rfloor$. Then*

$$ecc(T) \geq \begin{cases} 3 - \frac{2}{n} & if\ m = 2 \\ 3 + \frac{m-2}{n} & if\ m \geq 3 \end{cases}$$

with equality if and only if $T = U_{(n,m)}$, *where* $U_{(n,m)}$ *is the tree obtained by attaching* $m-1$ *paths on two vertices to the center of the star* S_{n-2m+2}.

Lemma 4.4. *Let* $\mathscr{T}_{n,\gamma}$ *be the set of all n-vertex trees with domination number* $\gamma \geq 2$. *Assume that* T^* *is the tree having the minimum value* $ecc(T)$ *among* $\mathscr{T}_{n,\gamma}$. *Then* $\gamma(T^*) = m(T^*) = \gamma$.

Proof. By Lemma 4.2, we have $\gamma(T^*) \leq m(T^*)$. Suppose that $D = \{v_1, \ldots, v_\gamma\}$ is a minimum dominating set of T^*. Then there exist a matching $M = \{v_1u_1, \ldots, v_\gamma u_\gamma\}$ in T^*. If $\gamma(T^*) < m(T^*)$, then there exists an edge x_1x_2 which has no common vertex with each edge v_iu_i, $i = 1, \ldots, \gamma$. Assume that x_1 and x_2 are dominated by $v_i \in D$ and $v_j \in D$, respectively, $1 \leq i, j \leq \gamma$. If $i = j$, then the cycle $C_3 = x_1x_2v_i$ is a subgraph of T^*, a contradiction. Hence, $i \neq j$ and $\{x_1v_i, x_2v_j\} \subseteq E(T^*)$. Define $T' = \sigma(T^*, x_1v_i)$ and $T'' = \sigma(T', x_2v_j)$. Then we have $T'' \in \mathscr{T}_{n,\gamma}$. Moreover, $ecc(T'') < ecc(T^*)$ by Lemma 4.1, which leads to a contradiction. Hence, $\gamma(T^*) = m(T^*)$, as desired. □

Theorem 4.5. *Let T be an n-vertex tree, where* $n \geq 6$. *Then*

$$\gamma(T) - ecc(T) \leq \left(1 - \frac{1}{n}\right)\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{n} - 3,$$

and the equality holds if and only if $T \cong U_{(n,\lfloor \frac{n}{2} \rfloor)}$.

Proof. Assume that $\gamma(T) = \gamma$. If $\gamma = 1$, it is obvious that $\gamma(T) - ecc(T) \leq 1 - ecc(S_n) = \frac{1}{n} - 1$. Let $\mathscr{T}_{n,\gamma}$ be the set of all n-vertex trees with domination number γ. Assume that T^* is the tree having the minimum value $ecc(T)$ among $\mathscr{T}_{n,\gamma}$. Then $\gamma = \gamma(T) = \gamma(T^*) = m(T^*)$ by Lemma 4.4. If $\gamma \geq 2$, then

$$\begin{aligned} \gamma(T) - ecc(T) &\leq \gamma(T^*) - ecc(T^*) \\ &\leq \begin{cases} m(T^*) - 3 + \frac{n}{2} & if\ m(T^*) = 2 \\ m(T^*) - 3 - \frac{m(T^*)-2}{n} & if\ m(T^*) \geq 3 \end{cases} \\ &= \begin{cases} \frac{2}{n} - 1 & if\ \gamma = 2 \\ (1 - \frac{1}{n})\gamma + \frac{2}{n} - 3 & if\ \gamma \geq 3 \end{cases} \end{aligned}$$

with equality if and only if $T \cong T^* \cong U_{(n,\gamma)}$ by Lemma 4.3. For $\gamma \geq 3$, let $f(\gamma) = (1 - \frac{1}{n})\gamma + \frac{2}{n} - 3$. Then

$$-\frac{1}{n} = f(3) \leq f(\gamma) \leq f\left(\left\lfloor \frac{n}{2} \right\rfloor\right) = \left(1 - \frac{1}{n}\right)\left\lfloor \frac{n}{2} \right\rfloor + \frac{2}{n} - 3.$$

Note that $f(\gamma) - (\frac{2}{n} - 1) \geq -\frac{1}{n} - (\frac{2}{n} - 1) > 0$, where $n \geq 6$, and $\frac{2}{n} - 1 > \frac{1}{n} - 1$. Hence $\gamma(T) - ecc(T) \leq f(\gamma) \leq (1 - \frac{1}{n})\lfloor \frac{n}{2} \rfloor + \frac{2}{n} - 3$, and the equality holds if and only if $T \cong U_{(n,\gamma)}$ and $\gamma = \lfloor \frac{n}{2} \rfloor$, which is equivalent to $T \cong U_{(n,\lfloor \frac{n}{2} \rfloor)}$. We have completed the proof. □

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