Filomat 38:10 (2024), 3295–3312
https://doi.org/10.2298/FIL2410295M

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generalizations of Simpson type inequality for (α, m)-convex functions

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Abstract. Several scholars are interested in fractional operators with integral inequalities. Due to its characteristics and wide range of applications in science, engineering fields, artificial intelligence and fractional inequalities should be employed in mathematical investigations. In this paper, we establish the new identity for the Caputo-Fabrizio fractional integral operator. By utilizing this identity, the generalization of Simpson type inequality for (α, m)-convex functions via the Caputo-Fabrizio fractional integral operator. Furthermore, we also include the applications to special means, q-digamma functions, Simpson formula, Matrix inequalities, Modified Bessel function, and mind-point formula. These applications have given a new dimension to scholars.

1. Introduction

Convex functions have a known history in science and have been a focus of research for over a century. Numerous scholars are interested in integral inequalities due to the immediately establish of convexity theory and applications of fractional calculus. For many years, convex functions have been utilized to investigate inequalities of the Hermite-Hadamard type, Simpson type, Hermite-Hadamard-Mercer type, Ostrowski type, and other types. One of these integral inequalities, the Hermite-Hadamard inequality discussed in [1] has attracted the interest of a wide range of scholars. Several inequalities of the trapezoidal type were provided by Dragomir [2], Kirmaci [3], and give applications to special means. A few instances of recent fractional integral operators that have been utilized to analyze, lots of mathematicians established new refinements of the Hermite-Hadamard type inequality for various classes of convex functions and mappings, such as harmonically convex functions [4], quasi convex functions [5], convex functions [6], m-convex functions [7], s-convex functions of Raina type [8], and Riemann-Liouville [9], Proportional fractional [10], k-Riemann-Liouville [11], Caputo-Fabrizio [12], and generalized Atangana-Baleanu operator [13]. It is crucial we note that Leibniz and L'Hospital are with establish the idea of fractional calculus (1695). Other mathematicians have a lot contributed to the topic of fractional calculus and its wide range of applications, particularly Riemann, Liouville, Letnikov, Erdeli, Grunwald, and Kober. Fractional calculus is of interest due to its nature how to deal with a range of real-world problems to many physical and engineering fields. The investigation of fractional order integral and derivative functions over real and complex domains as

2020 *Mathematics Subject Classification*. 26D07; 26D10; 26D15.
Keywords. Simpson type inequalities; (α, m)-convex function; Fractional integrals; Hölder′s inequality; Power-mean inequality.
Received: 16 July 2023; Revised: 11 September 2023; Accepted: 12 September 2023
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well as its applications has become the focus of fractional calculus. Fractional calculus requires the use of arithmetic from classical calculus to follow more particular results. Differential equations of fractional order can be used to solve a wide range of mathematical models. Because they are particular cases of fractional order mathematical models, fractional mathematical models have more certain and precise results than classical mathematical models. Mathematical modelling can be utilized to define the endemics' unique transmission dynamics and obtain knowledge of how infection affects a new population. Non-integer order fractional differential equations are utilized to enhance actual phenomena' precision and accuracy. Furthermore these articles [14–21] gives more information and application of fractional calculus. However, we can take consider lots of orders and create much more significant result of fractional computation. Fractional integral operators to demonstrate well-known inequalities in recent years mathematicians have become more interested in employing a range of innovative theories. In fractional calculus, there exist many different kinds of integral operators. These operators have application for crucial and well-known integral inequalities.

In all of this years, Thomas Simpson established fundamental methods for numerical integration and estimate of definite integrals that are now known as Simpson's law. (1710-1761). But J. Kepler utilized an identical approximation over a century before, which is because it is often referred to as Kepler's law. Estimates based just on a three-step quadratic kernel are often referred to as Newton-type results as Simpson's method utilizes the three-point Newton-Cotes quadrature rule.

Simpson quadrature formula (Simpson′s 1/3):

$$\int_{\omega_1}^{\omega_2} f(x)\,dx \approx \frac{\omega_2-\omega_1}{6}\left[f(\omega_1)+4f\left(\frac{\omega_1+\omega_2}{2}\right)+f(\omega_2)\right]. \tag{1}$$

Simpson second formula or Newton-Cotes quadrature formula (Simpson′s 3/8):

$$\int_{\omega_1}^{\omega_2} f(x)\,dx \approx \frac{\omega_2-\omega_1}{8}\left[f(\omega_1)+3f\left(\frac{2\omega_1+\omega_2}{2}\right)+3f\left(\frac{\omega_1+2\omega_2}{2}\right)+f(\omega_2)\right]. \tag{2}$$

The following estimation known as Simpson's type inequality is one of many that are connected with specific quadrature laws:

Theorem 1.1. *[22] Let $f:[\omega_1,\omega_2]\to\mathbb{R}$ be a four times continuously differentiable mapping on (ω_1,ω_2) and $\left\|f^{(4)}\right\|_\infty=\sup_{x\in(\omega_1,\omega_2)}\left|f^{(4)}\right|<\infty$, then following inequalities holds:*

$$\left|\left[\frac{f(\omega_1)+f(\omega_2)}{6}+\frac{2}{3}f\left(\frac{\omega_1+\omega_2}{2}\right)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} f(x)\,dx\right|\le\frac{1}{2880}\left\|f^{(4)}\right\|_\infty(\omega_2-\omega_1)^4.$$

In recent years, numerous authors have focused on the generalization of Simpson's type inequality in various kinds of mappings. Some mathematicians have focused on the results of Simpson's, and Newton's type in order to obtain a convex mapping since convexity theory is an efficient and quick method for solving an immense kinds of problems from numerous areas of pure and applied mathematics. In particular, Dragomir et al. [22] introduced the most recent Simpson's inequalities and their applications in quadrature formulas. Additionally, Alomari et al. [23] established some of Simpson's type inequalities for s-convex functions. Sarikaya et al. [24] then identified the significance of the dependence of the variance of the Simpson's type inequality on convexity. For harmonic convex, and p-harmonic convex maps the authors established Newton's type inequality in [25]. A novel generalized, convex Newton-type inequality for functions with the local fractional derivative was described by Iftikhar et al. [26].

Theorem 1.2. *Let $f:[\omega_1,\omega_2]\to\mathbb{R}$ be a differentiable mapping whose derivative is continuous on (ω_1,ω_2) and $f'\in L[\omega_1,\omega_2]$, then the following inequality holds:*

$$\left|\left[\frac{f(\omega_1)+f(\omega_2)}{6}+\frac{2}{3}f\left(\frac{\omega_1+\omega_2}{2}\right)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}f(x)\,dx\right|\leq\frac{\omega_2-\omega_1}{3}\left\|f'\right\|_1. \tag{3}$$

where $\left\|f'\right\|_1=\int_{\omega_1}^{\omega_2}\left|f'(x)\right|dx$. The bound (3) for L-Lipschitzian mapping was give in [22] by $\frac{5}{36}L(\omega_2-\omega_1)$. Additionally, [22] provided the Simpson type inequality that is given below.

Theorem 1.3. *Let $f:[\omega_1,\omega_2]\to\mathbb{R}$ be an absolutely continuous mapping on (ω_1,ω_2) whose derivative belongs to $L_P[\omega_1,\omega_2]$, where $\frac{1}{p}+\frac{1}{q}=1$, then the following inequality holds:*

$$\left|\left[\frac{f(\omega_1)+f(\omega_2)}{6}+\frac{2}{3}f\left(\frac{\omega_1+\omega_2}{2}\right)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}f(x)\,dx\right|\leq\frac{1}{6}\left(\frac{1+2^{q+1}}{3(q+1)}\right)^{\frac{1}{q}}(\omega_2-\omega_1)^{\frac{1}{q}}\left\|f'\right\|_P. \tag{4}$$

The following Hermite-Hadamard type inequalities for differentiable convex mappings that were stated in [27].

Theorem 1.4. *[27] Let $f:I\subset\mathbb{R}\to\mathbb{R}$ be a differentiable mapping on I^o where $\omega_1,\omega_2\in I$ with $\omega_1<\omega_2$. If $\left|f'\right|$ is convex on $[\omega_1,\omega_2]$, then the following inequality holds:*

$$\left|\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}f(x)\,dx-f\left(\frac{\omega_1+\omega_2}{2}\right)\right|\leq\frac{\omega_2-\omega_1}{4}\left[\frac{\left|f'(\omega_1)\right|+\left|f'(\omega_2)\right|}{2}\right]. \tag{5}$$

Definition 1.5. *[28] Let I be a convex set on $\mathbb{R}$. If $f:I\to\mathbb{R}$ is called convex on I for all $(\omega_1,\omega_2)\in I$ and $\eta\in[0,1]$, then following inequality holds:*

$$f(\eta\omega_1+(1-\eta)\omega_2)\leq\eta f(\omega_1)+(1-\eta)f(\omega_2). \tag{6}$$

The mapping f is concave on I the inequality (6) holds in reversed direction for all $(\omega_1,\omega_2)\in I$ and $\eta\in[0,1]$.

Definition 1.6. *[29] A $f:[0,\omega_2)\to\mathbb{R}$ is said to be (α,m)-convex if $(\alpha,m)\in[0,1]^2$, for every $x,y\in[0,\omega_2]$ and $\eta\in[0,1]$, then following inequality holds:*

$$f(\eta x+m(1-\eta)y)\leq\eta^{\alpha}f(x)+m(1-\eta^{\alpha})f(y).$$

Several mathematicians established new fractional operators, they are different from one another, the locality and singularity are different. There are essentially two kind of nonlocal fractional derivatives, the Riemann-Liouville and Caputo with singular kernels and others with non-singular kernels. In order to give the fractional calculus motivation and bring the most effective operators to the discussion, the Riemann-Liouville and Caputo-Fabrizio fractional integral operators is one of these as follows.

Definition 1.7. *[30] Suppose $f\in L[\omega_1,\omega_2]$. The left and right-sided Riemann-Liouville fractional integrals of order $\alpha>0$ defined by:*

$$
\begin{aligned}
I^{\alpha}_{\omega_1^+} f(\eta) &= \frac{1}{\Gamma\alpha}\int_{\omega_1}^{x} (x-\eta)^{\alpha-1} f(\eta)\, d\eta,\ x > \omega_1 \\
I^{\alpha}_{\omega_2^-} f(\eta) &= \frac{1}{\Gamma\alpha}\int_{x}^{\omega_2} (x-\eta)^{\alpha-1} f(\eta)\, d\eta,\ x < \omega_2,
\end{aligned}
$$

where $\Gamma(.)$ is the gamma function and $I^{0}_{\omega_1^+} f(\eta) = I^{0}_{\omega_2^-} f(\eta) = f(\eta)$.

Definition 1.8. *[31] Let $f \in H^1(\omega_1, \omega_2)$, $\omega_1 < \omega_2$, for all $\alpha \in [0,1]$, where $\beta(\alpha) > 0$ is a normalizer satisfying $\beta(0) = \beta(1) = 1$, then the left and right fractional integrals are defined by:*

$$
\begin{aligned}
\left({}^{CF}_{\omega_1}I^{\alpha} f\right)(x) &= \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)}\int_{\omega_1}^{x} f(x)\, dx \\
\left({}^{CF}I^{\alpha}_{\omega_2} f\right)(x) &= \frac{1-\alpha}{\beta(\alpha)} f(x) + \frac{\alpha}{\beta(\alpha)}\int_{x}^{\omega_2} f(x)\, dx.
\end{aligned}
$$

The main goal in this article to established a new integral identity using the Caputo-Fabrizio fractional integral operator, which holds a unique place among fractional integral operators. By employ this identity to generalization the novel Simpson type inequality via (α, m)-convex function. We also include the applications to special means, q-digamma functions, Simpson formula, Matrix inequalities, Modified Bessel function, and mind-point formula, taking many special cases of the main findings is discuses in literature.

2. Main results

In this section we established the new identity via Caputo-Fabrizio fractional integral operator. By using this identity we obtain our main result.

Lemma 2.1. *Let $f : I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous mapping on I^o where $\omega_1, \omega_2 \in I$ with $\omega_1 < \omega_2$, then the following equality holds:*

$$
\begin{aligned}
& \left[\frac{f(\omega_1)}{6} + \frac{4}{6} f\left(\frac{\omega_1+\omega_2}{2}\right) + \frac{f(\omega_2)}{6}\right] \\
& - \frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\omega_2} f\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k) \\
= \ & (\omega_2-\omega_1)\int_0^1 K(\eta) f'(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta,
\end{aligned}
$$

where

$$
K(\eta) = \begin{cases} \eta - \frac{1}{6} & \eta \in \left[0, \frac{1}{2}\right), \\ \eta - \frac{5}{6} & \eta \in \left[\frac{1}{2}, 1\right]. \end{cases}
$$

Proof. Let

$$
I = \int_0^1 K(\eta) f'(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta
$$

$$\begin{aligned} &= \int_0^{\frac{1}{2}} \left(\eta - \frac{1}{6}\right) f'(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &\quad + \int_{\frac{1}{2}}^1 \left(\eta - \frac{5}{6}\right) f'(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= I_1 + I_2. \end{aligned}$$

Integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^{\frac{1}{2}} \left(\eta - \frac{1}{6}\right) f'(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= \frac{1}{\omega_2 - \omega_1}\left(\eta - \frac{1}{6}\right) f(\eta\omega_2 + (1-\eta)\omega_1)\Big|_0^{\frac{1}{2}} - \frac{1}{\omega_2 - \omega_1}\int_0^{\frac{1}{2}} f(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= \frac{1}{3(\omega_2 - \omega_1)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{1}{6(\omega_2 - \omega_1)} f(\omega_1) - \frac{1}{\omega_2 - \omega_1}\int_0^{\frac{1}{2}} f(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= \frac{1}{3(\omega_2 - \omega_1)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{1}{6(\omega_2 - \omega_1)} f(\omega_1) - \frac{1}{(\omega_2 - \omega_1)^2}\int_{\omega_1}^{\frac{\omega_1+\omega_2}{2}} f(u)\, du. \end{aligned} \tag{7}$$

Multiply by $\frac{\alpha(\omega_2-\omega_1)^2}{\beta(\alpha)}$ with above equality (7), we get

$$\frac{\alpha(\omega_2 - \omega_1)^2}{\beta(\alpha)} I_1 = \frac{\alpha(\omega_2 - \omega_1)}{3\beta(\alpha)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{\alpha(\omega_2 - \omega_1)}{6\beta(\alpha)} f(\omega_1) - \frac{\alpha}{\beta(\alpha)}\int_{\omega_1}^{\frac{\omega_1+\omega_2}{2}} f(u)\, du. \tag{8}$$

Similarly,

$$\begin{aligned} I_2 &= \int_{\frac{1}{2}}^1 \left(\eta - \frac{5}{6}\right) f'(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= \frac{1}{\omega_2 - \omega_1}\left(\eta - \frac{5}{6}\right) f(\eta\omega_2 + (1-\eta)\omega_1)\Big|_{\frac{1}{2}}^1 - \frac{1}{\omega_2 - \omega_1}\int_{\frac{1}{2}}^1 f(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= \frac{1}{3(\omega_2 - \omega_1)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{1}{6(\omega_2 - \omega_1)} f(\omega_2) - \frac{1}{\omega_2 - \omega_1}\int_0^{\frac{1}{2}} f(\eta\omega_2 + (1-\eta)\omega_1)\, d\eta \\ &= \frac{1}{3(\omega_2 - \omega_1)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{1}{6(\omega_2 - \omega_1)} f(\omega_2) - \frac{1}{(\omega_2 - \omega_1)^2}\int_{\frac{\omega_1+\omega_2}{2}}^{\omega_2} f(u)\, du. \end{aligned} \tag{9}$$

Multiply by $\frac{\alpha(\omega_2-\omega_1)^2}{\beta(\alpha)}$ with above equality (9), we have

$$\frac{\alpha(\omega_2 - \omega_1)^2}{\beta(\alpha)} I_2 = \frac{\alpha(\omega_2 - \omega_1)}{3\beta(\alpha)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{\alpha(\omega_2 - \omega_1)}{6\beta(\alpha)} f(\omega_2) - \frac{\alpha}{\beta(\alpha)}\int_{\frac{\omega_1+\omega_2}{2}}^{\omega_2} f(u)\, du. \tag{10}$$

Adding the equalities (8) and (10) and subtracting $\frac{2(1-\alpha)}{\beta(\alpha)} f(k)$, we obtain

$$\begin{aligned} &(I_1 + I_2)\frac{\alpha(\omega_2 - \omega_1)^2}{\beta(\alpha)} - \frac{2(1-\alpha)}{\beta(\alpha)} f(k) \\ = &\left(\frac{\alpha(\omega_2 - \omega_1)}{3\beta(\alpha)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{\alpha(\omega_2 - \omega_1)}{6\beta(\alpha)} f(\omega_1) - \frac{\alpha}{\beta(\alpha)}\int_{\omega_1}^{\frac{\omega_1+\omega_2}{2}} f(u)\, du \right. \\ &\left. + \frac{\alpha(\omega_2 - \omega_1)}{3\beta(\alpha)} f\left(\frac{\omega_1 + \omega_2}{2}\right) + \frac{\alpha(\omega_2 - \omega_1)}{6\beta(\alpha)} f(\omega_2) - \frac{\alpha}{\beta(\alpha)}\int_{\frac{\omega_1+\omega_2}{2}}^{\omega_2} f(u)\, du\right) - \frac{2(1-\alpha)}{\beta(\alpha)} f(k) \end{aligned}$$

$$
\begin{aligned}
&= \frac{2\alpha(\omega_2-\omega_1)}{3\beta(\alpha)} f\left(\frac{\omega_1+\omega_2}{2}\right) + (f(\omega_1)+f(\omega_2))\frac{\alpha(\omega_2-\omega_1)}{6\beta(\alpha)} \\
&\quad -\frac{\alpha}{\beta(\alpha)}\int_{\omega_1}^{\omega_2} f(u)\,du - \frac{2(1-\alpha)}{\beta(\alpha)} f(k) \\
&= \frac{2\alpha(\omega_2-\omega_1)}{3\beta(\alpha)} f\left(\frac{\omega_1+\omega_2}{2}\right) + (f(\omega_1)+f(\omega_2))\frac{\alpha(\omega_2-\omega_1)}{6\beta(\alpha)} \\
&\quad -\left(\frac{\alpha}{\beta(\alpha)}\int_{\omega_1}^{k} f(u)\,du - \frac{(1-\alpha)}{\beta(\alpha)} f(k) + \frac{\alpha}{\beta(\alpha)}\int_{k}^{\omega_2} f(u)\,du - \frac{(1-\alpha)}{\beta(\alpha)} f(k)\right).
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
&(\omega_2-\omega_1)\int_0^1 K(\eta) f'(\eta\omega_2+(1-\eta)\omega_1)\,d\eta \\
&= \left[\frac{f(\omega_1)}{6} + \frac{4}{6} f\left(\frac{\omega_1+\omega_2}{2}\right) + \frac{f(\omega_2)}{6}\right] \\
&\quad -\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\omega_2} f\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k).
\end{aligned}
$$

The proof of Lemma 2.1 is completed. □

Theorem 2.2. *Consider that the assumptions of Lemma 2.1 hold. If $|f'|$ is (α, m)-convex on $[\omega_1, \omega_2]$, then the following fractional inequality holds:*

$$
\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6} + \frac{4}{6} f\left(\frac{\omega_1+\omega_2}{2}\right) + \frac{f(\omega_2)}{6}\right]\right. \\
&\quad \left. -\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\omega_2} f\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k)\right| \\
&\le (\omega_2-\omega_1)\left[m\frac{2^{-1-\alpha}\times 3^{-2-\alpha}\left(1-2^{2+\alpha}\times 3^{1+\alpha}-3^{2+\alpha}+5^{2+\alpha}+2^{\alpha}\times 3^{1+\alpha}\alpha\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_2}{m}\right)\right|\right. \\
&\quad \left. +m\,\frac{6^{-2-\alpha}\left(2\left(-1+3^{2+\alpha}-5^{2+\alpha}+17\times 6^{\alpha}\right)+2^{\alpha}\times 3^{2+\alpha}\alpha+5\times 6^{\alpha}\alpha^2\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_1}{m}\right)\right|\right].
\end{aligned}
$$

Proof. By using the Lemma 2.1, since $|f'|$ is (α, m)-convex, we have

$$
\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6} + \frac{4}{6} f\left(\frac{\omega_1+\omega_2}{2}\right) + \frac{f(\omega_2)}{6}\right]\right. \\
&\quad \left. -\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha} f\right)(k) + \left({}^{CF}I^{\alpha}_{\omega_2} f\right)(k)\right] + \frac{2(1-\alpha)}{\beta(\alpha)} f(k)\right| \\
&\le (\omega_2-\omega_1)\int_0^1 |K(\eta)|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right| d\eta \\
&\le (\omega_2-\omega_1)\left[\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right| d\eta\right. \\
&\quad \left. +\int_{\frac{1}{2}}^{1}\left|\eta-\frac{5}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right| d\eta\right]
\end{aligned}
$$

$$
\begin{aligned}
&\leq (\omega_2-\omega_1)\left[\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left(\eta^\alpha\left|f'(\omega_2)\right|+m(1-\eta^\alpha)\left|f'\left(\frac{\omega_1}{m}\right)\right|\right)d\eta\right.\\
&\quad\left.+\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left(m\eta^\alpha\left|f'\left(\frac{\omega_2}{m}\right)\right|+(1-\eta^\alpha)\left|f'(\omega_1)\right|\right)d\eta\right]\\
&= (\omega_2-\omega_1)\left[m\frac{2^{-1-\alpha}\times 3^{-2-\alpha}\left(1-2^{2+\alpha}\times 3^{1+\alpha}-3^{2+\alpha}+5^{2+\alpha}+2^\alpha\times 3^{1+\alpha}\alpha\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_2}{m}\right)\right|\right.\\
&\quad\left.+m\,\frac{6^{-2-\alpha}\left(2\left(-1+3^{2+\alpha}-5^{2+\alpha}+17\times 6^\alpha\right)+2^\alpha\times 3^{2+\alpha}\alpha+5\times 6^\alpha\alpha^2\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_1}{m}\right)\right|\right].
\end{aligned}
$$

This completes the proof. □

Therefore, the following results can be deduce for convexity.

Corollary 2.3. *If we choose $m=\alpha=1$ in Theorem 2.2, we get*

$$
\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^\alpha f\right)(k)+\left({}^{CF}I_{\omega_2}^\alpha f\right)(k)\right]\right|\\
&\leq \frac{5(\omega_2-\omega_1)}{72}\left[\left|f'(\omega_2)\right|+\left|f'(\omega_1)\right|\right]. \qquad (11)
\end{aligned}
$$

Corollary 2.4. *If we choose $f(\omega_1)=f\left(\frac{\omega_1+\omega_2}{2}\right)=f(\omega_2)$ in Theorem 2.2, we get*

$$
\begin{aligned}
&\left|f\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^\alpha f\right)(k)+\left({}^{CF}I_{\omega_2}^\alpha f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
&\leq (\omega_2-\omega_1)\left[m\frac{2^{-1-\alpha}\times 3^{-2-\alpha}\left(1-2^{2+\alpha}\times 3^{1+\alpha}-3^{2+\alpha}+5^{2+\alpha}+2^\alpha\times 3^{1+\alpha}\alpha\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_2}{m}\right)\right|\right.\\
&\quad\left.+m\,\frac{6^{-2-\alpha}\left(2\left(-1+3^{2+\alpha}-5^{2+\alpha}+17\times 6^\alpha\right)+2^\alpha\times 3^{2+\alpha}\alpha+5\times 6^\alpha\alpha^2\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_1}{m}\right)\right|\right].
\end{aligned}
$$

Corollary 2.5. *If we choose $m=\alpha=1$ in Corollary 2.4, we get*

$$
\begin{aligned}
&\left|f\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^\alpha f\right)(k)+\left({}^{CF}I_{\omega_2}^\alpha f\right)(k)\right]\right|\\
&\leq \frac{5(\omega_2-\omega_1)}{72}\left[\left|f'(\omega_2)\right|+\left|f'(\omega_1)\right|\right]. \qquad (12)
\end{aligned}
$$

Remark 2.6. *If we choose $m=\alpha=1$ and $\beta(0)=\beta(1)=1$ in Theorem 2.2, we get*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[f(\omega_1)+4f\left(\frac{\omega_1+\omega_2}{2}\right)+f(\omega_2)\right]-\frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}f(x)\,dx\right|\\
&\leq \frac{5(\omega_2-\omega_1)}{72}\left[\left|f'(\omega_2)\right|+\left|f'(\omega_1)\right|\right].
\end{aligned}
$$

Remark 2.6 was proved by Sarikaya et. al in [32, Corollary 1].

Theorem 2.7. *Consider that the assumptions of Lemma 2.1 hold. If $\left|f'\right|^q$ is (α,m)-convex on $[\omega_1,\omega_2]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{\omega_2}^{\alpha}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\ &(\omega_2-\omega_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\left(\frac{2^{-1-\alpha}}{\alpha+1}\right)\left|f'(\omega_2)\right|^q+m\left(\frac{1}{2}-\frac{2^{-1-\alpha}}{\alpha+1}\right)\left|f'\left(\frac{\omega_1}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(m\left(\frac{1-2^{-1-\alpha}}{\alpha+1}\right)\left|f'\left(\frac{\omega_2}{m}\right)\right|^q+\left(\frac{1}{2}-\frac{1-2^{-1-\alpha}}{\alpha+1}\right)\left|f'(\omega_1)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

Proof. By using the Lemma 2.1, with the help of Hölder inequality and (α, m)-convexity of $\left|f'\right|^q$, we have

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{\omega_2}^{\alpha}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\ &(\omega_2-\omega_1)\int_0^1\left|K(\eta)\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\\
\leq\ &(\omega_2-\omega_1)\left[\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right]\\
\leq\ &(\omega_2-\omega_1)\left[\left(\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|^q d\eta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|^q d\eta\right)^{\frac{1}{q}}\right]\\
\leq\ &(\omega_2-\omega_1)\left[\left(\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_0^{\frac{1}{2}}\left(\eta^{\alpha}\left|f'(\omega_2)\right|^q+m(1-\eta^{\alpha})\left|f'\left(\frac{\omega_1}{m}\right)\right|^q\right)d\eta\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|^p d\eta\right)^{\frac{1}{p}}\left(\int_{\frac{1}{2}}^1\left(m\eta^{\alpha}\left|f'\left(\frac{\omega_2}{m}\right)\right|^q+(1-\eta^{\alpha})\left|f'(\omega_1)\right|^q\right)d\eta\right)^{\frac{1}{q}}\right]\\
\leq\ &(\omega_2-\omega_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\left(\frac{2^{-1-\alpha}}{\alpha+1}\right)\left|f'(\omega_2)\right|^q+m\left(\frac{1}{2}-\frac{2^{-1-\alpha}}{\alpha+1}\right)\left|f'\left(\frac{\omega_1}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.\\
&\left.+\left(m\left(\frac{1-2^{-1-\alpha}}{\alpha+1}\right)\left|f'\left(\frac{\omega_2}{m}\right)\right|^q+\left(\frac{1}{2}-\frac{1-2^{-1-\alpha}}{\alpha+1}\right)\left|f'(\omega_1)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.8. *If we choose $f(\omega_1)=f\left(\frac{\omega_1+\omega_2}{2}\right)=f(\omega_2)$ in Theorem 2.7, we get*

$$\left|f\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{\omega_2}^{\alpha}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$

$$\leq (\omega_2-\omega_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\left(\frac{2^{-1-\alpha}}{\alpha+1}\right)\left|f'(\omega_2)\right|^q+m\left(\frac{1}{2}-\frac{2^{-1-\alpha}}{\alpha+1}\right)\left|f'\left(\frac{\omega_1}{m}\right)\right|^q\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(m\left(\frac{1-2^{-1-\alpha}}{\alpha+1}\right)\left|f'\left(\frac{\omega_2}{m}\right)\right|^q+\left(\frac{1}{2}-\frac{1-2^{-1-\alpha}}{\alpha+1}\right)\left|f'(\omega_1)\right|^q\right)^{\frac{1}{q}}\right].$$

Corollary 2.9. *If we choose $m=\alpha=1$ in Theorem 2.7, we get*

$$\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]\right|$$
$$\leq (\omega_2-\omega_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}2^{\frac{-1}{q}}\left[\left|f'(\omega_2)\right|+\left|f'(\omega_1)\right|\right]. \tag{13}$$

Theorem 2.10. *Consider that the assumptions of Lemma 2.1 hold. If $\left|f'\right|^q$ is (α,m)-convex on $[\omega_1,\omega_2]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq (\omega_2-\omega_1)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\left[m\frac{2^{-1-\alpha}\times 3^{-2-\alpha}\left(1-2^{2+\alpha}\times 3^{1+\alpha}-3^{2+\alpha}+5^{2+\alpha}+2^{\alpha}\times 3^{1+\alpha}\alpha\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_2}{m}\right)\right|^q\right.$$
$$\left.+m\,\frac{6^{-2-\alpha}\left(2\left(-1+3^{2+\alpha}-5^{2+\alpha}+17\times 6^{\alpha}\right)+2^{\alpha}\times 3^{2+\alpha}\alpha+5\times 6^{\alpha}\alpha^2\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_1}{m}\right)\right|^q\right]^{\frac{1}{q}}.$$

Proof. By using the Lemma 2.1, with the help of power-mean inequality and (α,m)-convexity of $\left|f'\right|^q$, we have

$$\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq (\omega_2-\omega_1)\int_0^1\left|K(\eta)\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta$$
$$\leq (\omega_2-\omega_1)\left[\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right.$$
$$\left.+\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right]$$
$$\leq (\omega_2-\omega_1)\left[\left(\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|^q d\eta\right)^{\frac{1}{q}}\right.$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\eta-\frac{5}{6}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\eta-\frac{5}{6}\right|\left|f'\left(\eta\omega_2+(1-\eta)\,\omega_1\right)\right|^{q}d\eta\right)^{\frac{1}{q}}\Bigg]$$

$$\leq (\omega_2-\omega_1)\Bigg[\left(\int_{0}^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{0}^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left(\eta^{\alpha}\left|f'(\omega_2)\right|^{q}+m\left(1-\eta^{\alpha}\right)\left|f'\left(\frac{\omega_1}{m}\right)\right|^{q}\right)d\eta\right)^{\frac{1}{q}}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|\eta-\frac{5}{6}\right|d\eta\right)^{1-\frac{1}{q}}\left(\int_{\frac{1}{2}}^{1}\left|\eta-\frac{5}{6}\right|\left(m\eta^{\alpha}\left|f'\left(\frac{\omega_2}{m}\right)\right|^{q}+\left(1-\eta^{\alpha}\right)\left|f'(\omega_1)\right|^{q}\right)d\eta\right)^{\frac{1}{q}}\Bigg]$$

$$\leq (\omega_2-\omega_1)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\Bigg[m\frac{2^{-1-\alpha}\times 3^{-2-\alpha}\left(1-2^{2+\alpha}\times 3^{1+\alpha}-3^{2+\alpha}+5^{2+\alpha}+2^{\alpha}\times 3^{1+\alpha}\alpha\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_2}{m}\right)\right|^{q}$$

$$+m\,\frac{6^{-2-\alpha}\left(2\left(-1+3^{2+\alpha}-5^{2+\alpha}+17\times 6^{\alpha}\right)+2^{\alpha}\times 3^{2+\alpha}\alpha+5\times 6^{\alpha}\alpha^{2}\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_1}{m}\right)\right|^{q}\Bigg]^{\frac{1}{q}}.$$

This completes the proof. □

Corollary 2.11. *If we choose $\alpha = m = 1$ in Theorem 2.10, we get*

$$\left|\frac{1}{6}\left[f(\omega_1)+4f\left(\frac{\omega_1+\omega_2}{2}\right)+f(\omega_2)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{\omega_2}^{\alpha}f\right)(k)\right]\right|$$
$$\leq \frac{5}{72}(\omega_2-\omega_1)\left[\left|f'(\omega_2)\right|^{q}+\left|f'(\omega_1)\right|^{q}\right]^{\frac{1}{q}}.$$

Corollary 2.12. *If we choose $f(\omega_1)=f\left(\frac{\omega_1+\omega_2}{2}\right)=f(\omega_2)$ in Theorem 2.10, we get*

$$\left|f\left(\frac{\omega_1+\omega_2}{2}\right)-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{\omega_2}^{\alpha}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$

$$\leq (\omega_2-\omega_1)\left(\frac{5}{72}\right)^{1-\frac{1}{q}}\Bigg[m\frac{2^{-1-\alpha}\times 3^{-2-\alpha}\left(1-2^{2+\alpha}\times 3^{1+\alpha}-3^{2+\alpha}+5^{2+\alpha}+2^{\alpha}\times 3^{1+\alpha}\alpha\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_2}{m}\right)\right|^{q}$$

$$+m\,\frac{6^{-2-\alpha}\left(2\left(-1+3^{2+\alpha}-5^{2+\alpha}+17\times 6^{\alpha}\right)+2^{\alpha}\times 3^{2+\alpha}\alpha+5\times 6^{\alpha}\alpha^{2}\right)}{(\alpha+1)(\alpha+2)}\left|f'\left(\frac{\omega_1}{m}\right)\right|^{q}\Bigg]^{\frac{1}{q}}.$$

Remark 2.13. *If we choose $q = 1$ in Theorem 2.10, then Theorem 2.10 reduces to Theorem 2.2.*

Theorem 2.14. *Consider that the assumptions of Lemma 2.1 hold. If $\left|f'\right|^{q}$ is convex on $[\omega_1, \omega_2]$ and $q > 1$, then the following fractional inequality holds:*

$$\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.$$
$$\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}_{\omega_1}^{CF}I^{\alpha}f\right)(k)+\left({}^{CF}I_{\omega_2}^{\alpha}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|$$
$$\leq (\omega_2-\omega_1)\left[\frac{1}{p}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)+\frac{1}{2q}\left(\left|f'(\omega_2)\right|^{q}+\left|f'(\omega_1)\right|^{q}\right)\right].$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\;&(\omega_2-\omega_1)\int_0^1\left|K(\eta)\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\\
\leq\;&(\omega_2-\omega_1)\left[\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right].
\end{aligned}$$

By using the Young′s inequality

$$\omega_1\omega_2\leq\frac{1}{p}\omega_1^p+\frac{1}{q}\omega_2^q,$$

we obtain

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\;&(\omega_2-\omega_1)\left[\left(\frac{1}{p}\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|^p d\eta\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|^q d\eta\right)\right.\\
&\left.+\left(\frac{1}{p}\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|^p d\eta\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|^q d\eta\right)\right]\\
\leq\;&(\omega_2-\omega_1)\left[\left(\frac{1}{p}\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|^p d\eta\right)+\frac{1}{q}\left(\int_0^{\frac{1}{2}}\left(\eta\left|f'(\omega_2)\right|^q+(1-\eta)\left|f'(\omega_1)\right|^q\right)d\eta\right)\right.\\
&\left.+\left(\frac{1}{p}\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|^p d\eta\right)+\frac{1}{q}\left(\int_{\frac{1}{2}}^1\left(\eta\left|f'(\omega_2)\right|^q+(1-\eta)\left|f'(\omega_1)\right|^q\right)d\eta\right)\right]\\
\leq\;&(\omega_2-\omega_1)\left[\frac{1}{p}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)+\frac{1}{2q}\left(\left|f'(\omega_2)\right|^q+\left|f'(\omega_1)\right|^q\right)\right].
\end{aligned}$$

This completes the proof. □

Theorem 2.15. *Consider that the assumptions of Lemma 2.1 hold. If $\left|f'\right|^q$ is concave on $[\omega_1,\omega_2]$ and $q\geq 1$, then the following fractional inequality holds:*

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\;&\frac{5(\omega_2-\omega_1)}{72}\left[\left|f'\left(\frac{29\omega_2+61\omega_1}{90}\right)\right|+\left|f'\left(\frac{61\omega_2+29\omega_1}{90}\right)\right|\right].
\end{aligned}$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\ &(\omega_2-\omega_1)\int_0^1|K(\eta)|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta \qquad (14)\\
\leq\ &(\omega_2-\omega_1)\left[\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left|f'(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta\right].
\end{aligned}$$

By Jensen integral inequality, we obtain

$$\begin{aligned}
&\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|f'(\eta\omega_2+(1-\eta)\omega_1)\,d\eta\\
\leq\ &\left(\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|d\eta\right)\left|f'\left(\frac{\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|\left|(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta}{\int_0^{\frac{1}{2}}\left|\eta-\frac{1}{6}\right|d\eta}\right)\right|\\
=\ &\frac{5}{72}\left|f'\left(\frac{29\omega_2+61\omega_1}{90}\right)\right|, \qquad (15)
\end{aligned}$$

and

$$\begin{aligned}
&\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|f'(\eta\omega_2+(1-\eta)\omega_1)\,d\eta\\
\leq\ &\left(\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|d\eta\right)\left|f'\left(\frac{\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|\left|(\eta\omega_2+(1-\eta)\omega_1)\right|d\eta}{\int_{\frac{1}{2}}^1\left|\eta-\frac{5}{6}\right|d\eta}\right)\right|\\
=\ &\frac{5}{72}\left|f'\left(\frac{61\omega_2+29\omega_1}{90}\right)\right|. \qquad (16)
\end{aligned}$$

Using the inequalities (15) and (16) in (14), we have

$$\begin{aligned}
&\left|\left[\frac{f(\omega_1)}{6}+\frac{4}{6}f\left(\frac{\omega_1+\omega_2}{2}\right)+\frac{f(\omega_2)}{6}\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[\left({}^{CF}_{\omega_1}I^{\alpha}f\right)(k)+\left({}^{CF}I^{\alpha}_{\omega_2}f\right)(k)\right]+\frac{2(1-\alpha)}{\beta(\alpha)}f(k)\right|\\
\leq\ &\frac{5(\omega_2-\omega_1)}{72}\left[\left|f'\left(\frac{29\omega_2+61\omega_1}{90}\right)\right|+\left|f'\left(\frac{61\omega_2+29\omega_1}{90}\right)\right|\right].
\end{aligned}$$

This completes the proof. □

3. Application to special means

(*a*) The Arithmetic mean:

$$A=A(\omega_1,\omega_2):=\frac{\omega_1+\omega_2}{2},\ \omega_1,\omega_2\in\mathbb{R};$$

(b) The Logarithmic mean:

$$L = L(\omega_1, \omega_2) := \frac{\omega_2 - \omega_1}{\ln \omega_2 - \ln \omega_1}, \ \omega_1, \omega_2 \in \mathbb{R}, \ \omega_1 \neq \omega_2;$$

(c) The Generalized Logarithmic-mean:

$$L_r = L_r(\omega_1, \omega_2) := \left[\frac{\omega_2^{r+1} - \omega_1^{r+1}}{(r+1)(\omega_2 - \omega_1)} \right] \ r \in \mathbb{R} \backslash \{-1, 0\}, \ \omega_1, \omega_2 \in \mathbb{R}, \ \omega_1 \neq \omega_2.$$

Proposition 3.1. *Let $f : [\omega_1, \omega_2] \to \mathbb{R}$, $0 < \omega_1 < \omega_2$, $n \in \mathbb{N}$, $n \geq 2$ and $x \in [\omega_1, \omega_2]$, we have*

$$\left| \frac{1}{3} A\left(\omega_1^n, \omega_2^n\right) + \frac{2}{3} A^n(\omega_1, \omega_2) - L_n^n(\omega_1, \omega_2) \right| \leq \frac{5n(\omega_2 - \omega_1)}{72} \left[|\omega_1|^{n-1} + |\omega_2|^{n-1} \right].$$

Proof. The assertion follows from Theorem 2.2 $f(x) = x^n$, $\alpha = m = 1$ and $\beta(0) = \beta(1) = 1$. □

Proposition 3.2. *Let $f : [\omega_1, \omega_2] \to \mathbb{R}$, $0 < \omega_1 < \omega_2$, $n \in \mathbb{N}$, $n \geq 2$ and $x \in [\omega_1, \omega_2]$, we have*

$$\left| A^n(\omega_1, \omega_2) - L_n^n(\omega_1, \omega_2) \right| \leq \frac{5n(\omega_2 - \omega_1)}{72} \left[|\omega_1|^{n-1} + |\omega_2|^{n-1} \right].$$

Proof. The assertion follows from Corollary 2.5 $f(x) = x^n$, $\beta(0) = \beta(1) = 1$. □

Proposition 3.3. *Let $f : [\omega_1, \omega_2] \to \mathbb{R}$, $0 < \omega_1 < \omega_2$, $n \in \mathbb{N}$, $n \geq 2$ and $x \in [\omega_1, \omega_2]$, we have*

$$\left| \frac{1}{3} A\left(\omega_1^n, \omega_2^n\right) + \frac{2}{3} A^n(\omega_1, \omega_2) - L_n^n(\omega_1, \omega_2) \right| \leq n(\omega_2 - \omega_1) \left(\frac{1 + 2^{p+1}}{6^{p+1}(p+1)} \right)^{\frac{1}{p}} 2^{\frac{-1}{q}} \left[|\omega_1|^{n-1} + |\omega_2|^{n-1} \right]$$

Proof. The assertion follows from Theorem 2.7 $f(x) = x^n$, $\alpha = m = 1$ and $\beta(0) = \beta(1) = 1$. □

4. Application to Simpson Formula.

Let d is the partition of the interval $[\omega_1, \omega_2]$, $d : \omega_1 = x_0 < x_1 < x_2 < ..... < x_{n-1} < x_n = \omega_2$, $h_i = \frac{(x_{i+1} - x_i)}{2}$ and Let the Simpson formula.

$$S(f, d) = \sum_{i=0}^{n-1} \frac{f(x_i) + 4f(x_i + h_i) + f(x_{i+1})}{6} (x_{i+1} - x_i).$$

If the mapping $f : [\omega_1, \omega_2] \to \mathbb{R}$ is a differentiable such that $f^{(4)}(x)$ exists on (ω_1, ω_2) and $M = \max_{x \in (\omega_1, \omega_2)} \left| f^{(4)}(x) \right| < \infty$, then

$$I = \int_{\omega_1}^{\omega_2} f(x)\, dx = S(f, d) + E_S(f, d),$$

where the approximation error $E_S(f, d)$ of the interval I by Simpson Formula $S(f, d)$ satisfies:

$$\left| E_S(f, d) \right| \leq \frac{k}{90} \sum_{i=0}^{n-1} (x_{i+1} - x_i)^5.$$

Proposition 4.1. *Under the assumption of Lemma 2.1, then in Corollary 2.5 for every division d of* $[\omega_1, \omega_2]$*, then the following inequality holds:*

$$\left|E_S(f,d)\right| \leq \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

Proof. Applying the Corollary 2.5 on the subinterval $[x_i, x_{i+1}]$, $(i = 0, 1, 2, 3....n-1)$ of the division d, we have □

$$\left|\frac{(x_{i+1}-x_i)}{6}\left(f(x_i) + 4f\left(\frac{x_i + x_{i+1}}{2}\right) + f(x_{i+1})\right) - \int_{x_i}^{x_{i+1}} f(x)\,dx\right|$$
$$\leq \frac{5(x_{i+1}-x_i)^2}{72}\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

Summing over i from 0 to $n-1$ and let that $\left|f'\right|$ is (α, m)-convex, we deduce by the triangle inequality, we have

$$\left|S(f,d) - \int_{\omega_1}^{\omega_2} f(x)\,dx\right| \leq \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

This completes the proof.

Proposition 4.2. *Under the assumption of Lemma 2.1, for every division d of* $[\omega_1, \omega_2]$ *, then the following inequality hold:*

$$\left|E_S(f,d)\right| \leq \frac{5}{72}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^2\left[\left(\left|f'\left(\frac{29x_{i+1}+61x_i}{90}\right)\right| + \left|f'\left(\frac{61x_{i+1}+29x_i}{90}\right)\right|\right)\right].$$

Proof. The proof is similar to the proposition 4.1, using the Theorem 2.15. □

5. Application to matrix:

Example: We denote by C^n the set of $n \times n$ complex matrices, M_n the algebra of $n \times n$ complex matrices, and by M_n^+ the strictly positive matrices in M_n. That is, $\omega_1 \in M_n^+$ if $\langle A_{\omega_1}, \omega_1\rangle > 0$ for all nonzero $\omega_1 \in C^n$. In [34] Sababheh proved that the function $\psi(\theta) = \left\|A^\theta XB^{1-\theta} + A^{1-\theta}XB^\theta\right\|$, $\omega_1, \omega_2 \in M_n^+$, $X \in M_n$ is convex for all $\theta \in [0, 1]$. Then by using the Corollary 2.5, we have

$$\left\|A^{\frac{\omega_1+\omega_2}{2}}XB^{1-\frac{\omega_1+\omega_2}{2}} + A^{1-\frac{\omega_1+\omega_2}{2}}XB^{\frac{\omega_1+\omega_2}{2}}\right\|$$
$$-\frac{\beta(\alpha)}{\alpha(\omega_2-\omega_1)}\left[{}^{CF}_{\omega_1}I^\alpha\left\|A^kXB^{1-k} + A^{1-k}XB^k\right\| + \right.$$
$$\left.{}^{CF}I^\alpha_{\omega_2}\left\|A^kXB^{1-k} + A^{1-k}XB^k\right\|\}\right]$$
$$+\frac{2(1-\alpha)}{\alpha(\omega_2-\omega_1)}\left\|A^kXB^{1-k} + A^{1-k}XB^k\right\| \leq \frac{5(\omega_2-\omega_1)}{72}$$
$$\left\|A^{\omega_1}XB^{1-\omega_1} + A^{1-\omega_1}XB^{\omega_1}\right\| + \left\|A^{\omega_2}XB^{1-\omega_2} + A^{1-\omega_2}XB^{\omega_2}\right\|.$$

6. Q-digamma Functions

Let $0 < \psi < 1$, the q-digamma(psi) functions φ_ψ , is the ψ- analogue of the digamma function ψ defined as [35].

$$\begin{aligned}\varphi_\psi &= -\ln(1-\psi) + \ln\psi \sum_{k=0}^{\infty} \frac{\psi^{k+x}}{1-\psi^{k+x}} \\ &= -\ln(1-\psi) + \ln\psi \sum_{k=0}^{\infty} \frac{\psi^{kx}}{1-\psi^{kx}}.\end{aligned}$$

For $\psi > 1$ and $x > 0$, the q-digamma functions φ_ψ defined as

$$\begin{aligned}\varphi_\psi &= -\ln(\psi-1) + \ln\psi \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\psi^{-(k+x)}}{1-\psi^{-(k+x)}}\right] \\ &= = -\ln(\psi-1) + \ln\psi \left[x - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\psi^{-kx}}{1-\psi^{-kx}}\right].\end{aligned}$$

Proposition 6.1. *Let $\omega_1, \omega_2 \in \mathbb{R}$, $0 < \omega_1 < \omega_2$, then we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[\varphi_\psi^{'}(\omega_1) + 4\varphi_\psi^{'}\left(\frac{\omega_1+\omega_2}{2}\right) + \varphi_\psi^{'}(\omega_2)\right]\right| - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi_\psi(u)\,du \\ \leq\ & \frac{5(\omega_2-\omega_1)}{72}\left[\left|\varphi_\psi^{'}(\omega_2)\right| + \left|\varphi_\psi^{'}(\omega_1)\right|\right].\end{aligned}$$

Proof. The assertion can be obtained immediately by using the Corollary 2.5 to $f(x) = \varphi_\psi(x)$ and $x > 0$, $\beta(0) = \beta(1) = 1$, $f'(x) = \varphi_\psi^{'}(x)$ is convex on $(0, +\infty)$. □

Proposition 6.2. *Let $\omega_1, \omega_2 \in \mathbb{R}$, $0 < \omega_1 < \omega_2$, and $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$,then we have*

$$\begin{aligned}&\left|\frac{1}{6}\left[\varphi_\psi^{'}(\omega_1) + 4\varphi_\psi^{'}\left(\frac{\omega_1+\omega_2}{2}\right) + \varphi_\psi^{'}(\omega_2)\right]\right| - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2} \varphi_\psi(u)\,du \\ \leq\ & (\omega_2-\omega_1)\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}} 2^{\frac{-1}{q}}\left[\left|\varphi_\psi^{'}(\omega_2)\right| + \left|\varphi_\psi^{'}(\omega_1)\right|\right].\end{aligned}$$

Proof. The assertion can be obtained immediately by using the Corollary 2.9 to $f(x) = \varphi_\psi(x)$ and $x > 0$, $\beta(0) = \beta(1) = 1$, $f'(x) = \varphi_\psi^{'}(x)$ is convex on $(0, +\infty)$. □

7. Modified Bessel function:

Let the first kind of Modified bessel function I_σ,which have the series representation ([35])

$$I_\sigma(x) = \sum_{m\geq 0} \frac{\left(\frac{x}{2}\right)^{\sigma+2m}}{m!\Gamma(\sigma+m+1)},$$

Where $x \in \mathbb{R}$ and $\sigma > -1$ second kind of Modified bessel function K_σ [35] is defined as:

$$K_\sigma(x) = \frac{\pi}{2}\frac{I_{-\sigma}(x) - I_\sigma(x)}{\sin\sigma\pi}.$$

let the function $\Psi_\sigma(x) : \mathbb{R} \to [1, \infty)$ defined by

$$\Psi_\sigma(x) = 2^\sigma\Gamma(\sigma+1)x^{-\sigma}I_\sigma(x)$$

.

Proposition 7.1. *Let $\sigma > -1$ and $0 < \omega_1 < \omega_2$, then we have*

$$\left|\frac{1}{6}\left[\Psi_\sigma(\omega_1) + 4\Psi_\sigma\left(\frac{\omega_1+\omega_2}{2}\right) + \Psi_\sigma(\omega_2)\right]\right| - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\Psi_\sigma(x)\,dx$$
$$\leq \frac{5(\omega_2-\omega_1)}{72(\sigma+1)}\left[|\Psi_{\sigma+1}(\omega_2)| + |\Psi_{\sigma+1}(\omega_1)|\right].$$

Proof. The assertion can be obtained immediately by using the Corollary 2.3 to $f(x) = \Psi_\sigma(x)$ and $x > 0$, $\beta(0) = \beta(1) = 1$, $\Psi'_\sigma(x) = \frac{x}{\sigma+1}\Psi_{\sigma+1}(x)$. □

Proposition 7.2. *Let $\sigma > -1$ and $0 < \omega_1 < \omega_2$, then we have*

$$\left|\Psi_\sigma\left(\frac{\omega_1+\omega_2}{2}\right) - \frac{1}{\omega_2-\omega_1}\int_{\omega_1}^{\omega_2}\Psi_\sigma(x)\,dx\right| \leq \frac{5(\omega_2-\omega_1)}{72(\sigma+1)}\left[|\Psi_{\sigma+1}(\omega_2)| + |\Psi_{\sigma+1}(\omega_1)|\right].$$

Proof. The assertion can be obtained immediately by using the Corollary 2.5 to $f(x) = \Psi_\sigma(x)$ and $x > 0$, $\beta(0) = \beta(1) = 1$, $\Psi'_\sigma(x) = \frac{x}{\sigma+1}\Psi_{\sigma+1}(x)$ □

Proposition 7.3. *Let $\sigma > -1$ and $0 < \omega_1 < \omega_2$, $\frac{1}{p} + \frac{1}{q} = 1$, $q > 1$then we have*

$$\left|\frac{1}{6}\left[\Psi_\sigma(\omega_1) + 4\Psi_\sigma\left(\frac{\omega_1+\omega_2}{2}\right) + \Psi_\sigma(\omega_2)\right]\right|$$
$$\leq \frac{(\omega_2-\omega_1)}{(\sigma+1)}\left(\frac{1+2^{p+1}}{6^{p+1}(p+1)}\right)^{\frac{1}{p}}\left[\left(\frac{1}{8}|\Psi_{\sigma+1}(\omega_2)|^q + \frac{3m}{8}|\Psi_{\sigma+1}(\omega_1)|^q\right)^{\frac{1}{q}}\right.$$
$$\left.\left(\frac{3}{8}|\Psi_{\sigma+1}(\omega_2)|^q + \frac{m}{8}|\Psi_{\sigma+1}(\omega_1)|^q\right)^{\frac{1}{q}}\right].$$

Proof. The assertion can be obtained immediately by using the Corollary 2.8 to $f(x) = \Psi_\sigma(x)$ and $x > 0$, $\beta(0) = \beta(1) = 1$, $\Psi'_\sigma(x) = \frac{x}{\sigma+1}\Psi_{\sigma+1}(x)$. □

8. Application to mind-point formula:

Let d is the partition of the interval $[\omega_1, \omega_2]$, $d : \omega_1 = x_0 < x_1 < x_2 < ..... < x_{n-1} < x_n = \omega_2$, $h_i = \frac{(x_{i+1}-x_i)}{2}$ and Let the mind-point formula

$$M(f,d) = \sum_{i=0}^{n-1}(x_{i+1}-x_i)\,f\left(\frac{x_i+x_{i+1}}{2}\right).$$

We know that the $f : [\omega_1, \omega_2] \to \mathbb{R}$ is a differentiable mapping such that $f''(x)$ on (ω_1, ω_2) and $k = \sup_{x\in[\omega_1,\omega_2]}\left|f''(x)\right| < \infty$, then

$$I = \int_{\omega_1}^{\omega_2} f(x)\,dx = M(f,d) + E_M(f,d),$$

where the error terms $E_M(f,d)$ on the interval I the mind-point formula $M(f,d)$ satisfied as:

$$\left|E_M(f,d)\right| \leq \frac{K}{24}\sum_{i=0}^{n-1}(x_{i+1}-x_i)^3.$$

The new assumptions we suggest in the following section for the remainder term $E_M(f,d)$ in terms of the first derivative better those of [33].

Proposition 8.1. *Under the assumption of Lemma 2.1, then in Corollary 2.5, for every division D of* $[\omega_1, \omega_2]$*, then the following inequality holds:*

$$\left|E_M(f,d)\right| \leq \frac{5}{72}\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right]$$

Proof. By using the Corollary 2.5 on the subinterval $[x_i, x_{i+1}]$, $i = 0, 1, 2.....n-1$ of division D, we have

$$\left|(x_{i+1}-x_i)f\left(\frac{x_i+x_{i+1}}{2}\right) - \int_{x_i}^{x_{i+1}} f(x)\,dx\right| \leq \frac{5(x_{i+1}-x_i)^2}{72}\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

By summing on i from 0 to $n-1$, and by triangle inequality, we have

$$\left|E_M(f,d)\right| \leq \frac{5}{72}\sum_{i=1}^{n-1}(x_{i+1}-x_i)^2\left[\left|f'(x_i)\right| + \left|f'(x_{i+1})\right|\right].$$

This completes the proof. □

9. Conclusion

It is obvious that recent establish in the field of inequalities have focused on finding new bounds, and generalized versions of some well-known inequalities utilizing a different kind of fractional integral operators. Scholars present novelty to this field by employing novel ideas, applications and operators. In this paper, we establish the new identity via Caputo-Fabrizio fractional integral operator. Employing this new identity the generalization of Simpson type inequality for (α, m)-convex functions. Moreover, we also include the applications to special means, q-digamma functions, Simpson formula, Matrix inequalities, Modified Bessel function, and mind-point formula. These applications have given a new dimension to the scholars. In the future scholars may work with modified Caputo-Fabrizio fractional operators and modified A-B fractional operators described in manuscripts (see [36], [37]).

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