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A novel extension of Darbo's fixed point theorem and its application to a class of differential equations involving (k, ψ)-Hilfer fractional derivative

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Abstract. In this article, a new generalised class of operators called as $\mathcal{A}$-condensing operators is introduced. The fixed point as well as coupled fixed point results are established for the newly defined class of mappings. The rich theory of measure of noncompactness is utilized in this purpose. The new results extend some of the famous works in the literature. Finally, an application to (k, ψ)-Hilfer fractional differential equation of order $2 < p < 3$ and type $0 \leq q \leq 1$ is presented.

1. Introduction

For ease, we first provide a list of notations which will be perceived throughout as defined, unless otherwise stated. Suppose X is a Banach space, C be a nonvoid, convex, closed and bounded subset in X, $T : \mathrm{dom}(T) \subseteq X \to X$ is a map, B_X is the collection of all nonvoid and bounded subsets of X, f_T is the set of all fixed points of T in $\mathrm{dom}(T)$, η an arbitrary measure of noncompactness (MNC) (Definition 1.4) and $k \in (0, 1)$.

In 1910, the subsequent theorem [21] obtained recognition as Brouwer fixed point theorem (FPT).

Theorem 1.1. Suppose $F \subset \mathbb{R}^n$ be nonempty and convex. If F is compact then the continuous operator $T : F \to F$ possesses a fixed point.

It discusses about the existence of fixed point(s) for finite dimensional Banach spaces. Schauder [21] presented its generalization to infinite dimensional Banach spaces by defining the concept of compact operators.

Definition 1.2. An operator $T : X \to X$ is called compact if $T(V)$ is precompact whenever $V \in B_X$.

Theorem 1.3. Suppose X is infinite dimensional and $T : C \to C$ is continuous. Then T possesses a fixed point if T is compact.

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On the other hand, Kuratowski [3, 5] proposed the first MNC as a real valued function $\mathcal{K} : B_X \to [0, \infty)$ such that

$$\mathcal{K}(V) = \inf\{\varepsilon > 0 : V \subseteq \bigcup_{\alpha=1}^{n} Q_\alpha, Q_\alpha \subset X, \text{diam}(Q_\alpha) < \varepsilon \ (\alpha = 1, 2, \cdots, n)\}$$

for all $V \in B_X$. However, later axiomatic approach stood to be a more convenient form when dealing with MNC and has the following interpretation [3].

Definition 1.4. A map $\eta : B_X \to [0, \infty)$ is an MNC if

(η1) The family $\text{Ker}(\eta) = \{V \in B_X : \eta(V) = 0\}$ is nonvoid and $\text{Ker}(\eta) \subseteq P_X$ where P_X denotes the family of all precompact sets in X.

(η2) $V_1 \subset V_2$ implies $\eta(V_1) \leq \eta(V_2)$ (monotonicity).

(η3) $\eta(V) = \eta(\overline{V})$ (invariance under closure).

(η4) $\eta(\text{conv}(V)) = \eta(V)$ (invariance under passage to the convex hull).

(η5) $\eta(\beta V_1 + (1 - \beta)V_2) \leq \beta\eta(V_1) + (1 - \beta)\eta(V_2)$ where $\beta \in [0, 1]$.

(η6) If the sequence $\{V_n\}_{n=1}^{\infty}$ is decreasing in nature, where each $V_n \in B_X$ is closed in X and $\lim_{n\to\infty} \eta(V_n) = 0$ then $V_\infty = \bigcap_{n=1}^{\infty} V_n$ is nonvoid as well as compact (Generalized Cantor's intersection theorem).

Moreover, if $\eta(\lambda V) = |\lambda|\eta(V)$ for any scalar λ then η is called as homogeneous whereas if $\eta(V_1 + V_2) \leq \eta(V_1) + \eta(V_2)$ then η is known as a subadditive measure. $\mathcal{K}$ is a suitable example of such an MNC.

Darbo [20] utilized the behaviour of MNC $\mathcal{K}$ in such a way that it worked in weakening the Schauder FPT hypothesis. For this, Darbo introduced an inequality involving $\mathcal{K}$ and proved the following FPT.

Theorem 1.5. Suppose $T : C \to C$ be continuous and $\mathcal{K}(TG) \leq k\mathcal{K}(G)$ for every nonvoid set $G \subseteq C$ then T admits a fixed point.

Aghajani et al. [1] further weakened Theoerm 1.5 by defining Meir-Keeler condensing (MKC) operators and stated Theorem 1.9. The authors even characterized MKC operators with L-functions.

Definition 1.6. Let F be a nonvoid set in X. The operator $T : F \to F$ is called as an MKC operator if $\forall\ \epsilon > 0$ $\exists$ a $\delta > 0$ such that for every nonvoid set G in F,

$$\epsilon \leq \eta(G) < \epsilon + \delta \implies \eta(TG) < \epsilon.$$

Definition 1.7. [1] A map $\varphi : [0, \infty) \to [0, \infty)$ is known to be an L-function whenever $\varphi(0) = 0$ with $\varphi(v) > 0$ for $v \in (0, \infty)$ and for any $v > 0$ $\exists$ a $\delta > 0$ so that $\varphi(u) \leq v$ provided $u \in [v, v + \delta]$.

Remark 1.8. An operator T is MKC if and only if one can find an L-function φ such that $\eta(TG) < \varphi(\eta(G))$ whenever $\eta(G) > 0$.

Theorem 1.9. If $T : C \to C$ is continuous and an MKC operator then T possesses a fixed point. Moverover, f_T is compact.

A very recent survey is the article [8], presenting the literature on fixed points of the condensing operators through MNC. It also covers results concerning best proximity point (pair) of cyclic (noncyclic) condensing operators in Banach spaces. Readers with a keen interest in this area are directed to [9, 12–15].

Moving towards the main motivation of this article, Shahzad et al. defined $\mathcal{A}$-contraction [18] using the concept of T-sequence which strictly submerges the class of all R-contractions, Meir-Keeler contractions, $\mathcal{Z}$-contractions and more. Keeping this in view, we define $\mathcal{A}$-condensing operators in terms of MNC η using the concept of T_η-sequence.

2. Fixed point results via $\mathcal{A}$-condensing operators

We now present our notions namely, the T_η-sequence and $\mathcal{A}$-condensing operators.

Definition 2.1. Let $\{p_n\}$ and $\{q_n\}$ be two real sequences. We say that $\{(p_n, q_n)\}$ is a T_η-sequence if $\exists$ a sequence $\{G_n\}$ of nonvoid subsets of C such that $\forall\ n \in \mathbb{N}$,

$$p_n = \eta(TG_n) > 0 \quad \text{and} \quad q_n = \eta(G_n) > 0.$$

Definition 2.2. An operator $T : C \to C$ is $\mathcal{A}$-condensing if one can find a function $\rho : A \times A \to \mathbb{R}$ satisfying the subsequent conditions together with T as:

($A1$) $\operatorname{ran}(\eta) \subseteq A \subseteq \mathbb{R}$.

($A2$) If $\{(p_n, q_n)\} \subseteq A \times A$ is a T_η-sequence such that both $p_n, q_n \to L$ with $L \geq 0$ and verifying $L < p_n$ along with $\rho(p_n, q_n) > 0$ for every $n \in \mathbb{N}$ then $L = 0$.

($A3$) $\rho(\eta(TG), \eta(G)) > 0$ provided $\eta(G) > 0$ and $\eta(TG) > 0$ for every $\emptyset \neq G \subset C$.

It is proved in [18] that not every $\mathcal{A}$-contraction is Meir-Keeler contraction but the converse is always true. On the same line we have, not every $\mathcal{A}$-condensing operator is MKC but the converse is always true. See the following example.

Example 2.3. Let C be a closed ball in X, centered at α with radius $r > 0$ as

$$C = \{x \in X : \|x - \alpha\| \leq r\}.$$

Suppose $a, b \in C$ be arbitrary. Define $T : C \to C$ by

$$Tx = \begin{cases} a & \text{if} \quad x = b, \\ b & \text{if} \quad x = a, \\ \frac{a+b}{2} & \text{elsewhere,} \end{cases}$$

then $\eta(TG) = \eta(G)$ [i.e. $\eta(TG) \nless \eta(G)$] where $\eta(X) = \operatorname{diam}(X)$ and $G = \{a, b\}$. Hence, T is not an MKC operator. However, for any nonzero constant function ρ, ($A3$) is fulfilled. On the other hand, for ($A2$), let us assume that $\{(p_n, q_n)\}$ is a T_η-sequence converging to L satisfying $L < p_n$ and $\rho(p_n, q_n) > 0$ for each $n \in \mathbb{N}$. Then

$$p_n = \eta(TG_n) > 0 \text{ and } q_n = \eta(G_n) > 0.$$

Assume contrary that $L > 0$. Note that, for all $n \in \mathbb{N}$, $G_n \subseteq G \cup G^c$ where G^c is the complement of G, so that for any $n \in \mathbb{N}$,

$$p_n = 0 \text{ or } \|a - b\| \text{ or } \|\frac{a-b}{2}\|.$$

This shows that $\{p_n\}$ is an eventually constant sequence satisfying $L = p_n$ for infinitely many n, a contradiction. Therefore, ($A2$) is satisfied and so T is $\mathcal{A}$-condensing.

We now state the fixed point theorem via $\mathcal{A}$-condensing operators.

Theorem 2.4. Suppose $T : C \to C$ is continuous and an $\mathcal{A}$-condensing operator such that $\rho(s, t) \leq t - s$ for every $s, t \in A \cap (0, \infty)$ then T possesses a fixed point in C. Moreover, f_T is compact.

Proof. Define a sequence $\{G_n\}$ in C as $G_n = \overline{\operatorname{conv}}(T(G_{n-1}))$ for every $n \in \mathbb{N}$ where $G_0 = C$. We intend to show by mathematical induction that $\{G_n\}$ is a decreasing sequence. For $n = 1$, $G_1 = \overline{\operatorname{conv}}(T(G_0)) \subset G_0$. Let us assume that $G_n \subset G_{n-1}$. Consider, $G_{n+1} = \overline{\operatorname{conv}}(T(G_n)) \subset \overline{\operatorname{conv}}(T(G_{n-1})) = G_n$. Hence, we write

$$\cdots \subseteq G_{n+1} \subseteq G_n \subseteq G_{n-1} \subseteq \cdots \subseteq G_1 \subseteq G_0.$$

If $\eta(G_{n_0}) = 0$ for some $n_0 \in \mathbb{N}$ then G_{n_0} is precompact. Also, $T(G_{n_0}) \subseteq \overline{\text{conv}}(T(G_{n_0})) = G_{n_0+1} \subseteq G_{n_0}$. This means that, by Schauder FPT there exists a fixed point of T. So, we now assume that $\eta(G_n) \neq 0$ for every $n \in \mathbb{N}$. Set

$$p_n = \eta(T(G_n)) > 0 \quad \text{and} \quad q_n = \eta(G_n) > 0.$$

Then, $\{(p_n, q_n)\}$ is a T_η-sequence such that $p_n, q_n \to L$ and $0 \leq L \leq p_n \leq q_n$. If there exist some $k_0 \in \mathbb{N}$ such that $L = p_{k_0}$ then $p_n = q_n$ implies $0 \leq \rho(p_n, q_n) < 0$ for all $n > k_0$. This is a contradiction and hence, by the definition of $\mathcal{A}$-condensing operators we have $L = 0$ so that

$$\lim_{n\to\infty} \eta(G_n) = 0.$$

Set $G_\infty := \bigcap_{n=1}^{\infty} G_n$ then this set is nonvoid, convex and compact, implies $G_\infty \in \text{Ker}(\eta)$. Therefore, by Schauder FPT, the operator T possesses a fixed point. If $\eta(f_T) > 0$ then from $(A3)$ we write

$$0 < \rho(\eta(Tf_T), \eta(f_T)) \leq \eta(f_T) - \eta(Tf_T) = 0.$$

This cannot be possible and hence $\eta(f_T) = 0$, together with continuity of T implies f_T is compact. □

Apart from Theorem 1.9, we have the subsequent corollaries as a consequence of the above theorem. Note that, $(A1)$ is easy to satisfy by setting $\text{ran}(\eta) = A$.

Corollary 2.5. Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. For an arbitrary MNC η, let $\eta(TV) \leq \Psi(\eta(V))\eta(V)$ where $\Psi : [0, \infty) \to [0, 1)$ is a map satisfying, $\Psi(s_n) \to 1 \implies s_n \to 0$, then T admits a fixed point.

Proof. Set $\widetilde{\Psi}(t) = \dfrac{1 + \Psi(t)}{2}$, $\forall\ t \in [0, \infty)$. Then $\widetilde{\Psi}(t_n) \to 1$ implies $t_n \to 0$. Moreover, $\Psi(t) < \widetilde{\Psi}(t) < 1\ \forall$ $t \in [0, \infty)$ so that

$$\eta(TV) \leq \Psi(\eta(V))\eta(V) < \widetilde{\Psi}(\eta(V)\eta(V).$$

Define the function ρ as

$$0 < \rho(\eta(TV), \eta(V)) := \widetilde{\Psi}(\eta(V))\eta(V) - \eta(TV) \leq \eta(V) - \eta(TV),$$

then $(A3)$ holds. For $(A2)$, let $\{(p_n, q_n)\}$ be a T_η-sequence satisfying $p_n \to L, q_n \to L, 0 \leq L < p_n$ and $\rho(p_n, q_n) > 0$ for every $n \in \mathbb{N}$. Hence,

$$p_n := \eta(TG_n) < \widetilde{\Psi}(\eta(G_n))\eta(G_n) < \eta(G_n) := q_n.$$

Applying $n \to \infty$, we get $\widetilde{\Psi}(\eta(G_n)) \to 1$ and therefore, $\eta(G_n) \to 0$. Thus, T is $\mathcal{A}$-condensing and so Theorem 2.4 concludes the rest. □

The above corollary corresponds to the Geraghty type condensing operators. The proof of the remaining corollaries can be similarly obtained. However, for more details, one can see [17, 18].

Corollary 2.6. [4] Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. If there exists a simulation function ζ such that $\zeta(\eta(TV), \eta(V)) \geq 0$ then T admits a fixed point.

Corollary 2.7. Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. If $\eta(TV) \leq \dfrac{\eta(V)}{1 + \mu\eta(V)}$, where $\mu > 0$, then T admits a fixed point.

Proof. Choose $\rho(s,t) = \dfrac{t}{1+k\mu t} - s$ then $0 < \rho(s,t) \leq t - s$ for all $s,t \in (0,\infty)$ so that $(A3)$ is fulfilled. For $(A2)$, let $p_n = \eta(TG_n) > 0, q_n = \eta(G_n) > 0$ converges to $L, 0 \leq L < p_n$ and $\rho(p_n, q_n) > 0$ for every $n \in \mathbb{N}$. Then $p_n < \dfrac{q_n}{1+k\mu q_n}$ so that as n approaches infinity, $L \leq \frac{L}{1+k\mu L} \implies L = 0$. Thus T is $\mathcal{A}$-condensing and so possesses a fixed point. $\square$

Corollary 2.8. Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. For an arbitrary MNC η, let $\eta(TV) \leq \nu\eta(V)$ where $\nu \in (0,1)$ then T admits a fixed point.

Corollary 2.9. Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. For an arbitrary MNC η, let $\eta(TV) \leq \Lambda(\eta(V))\eta(V)$ where Λ is a function from $[0,\infty)$ to $[0,\infty)$ satisfying $\limsup_{t\to a^+} \Lambda(t) < 1$, for each $t > 0$ then T admits a fixed point.

Corollary 2.10. Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. For an arbitrary MNC η, let $\eta(TV) \leq \eta(V) - \varphi(\eta(V))$ where $\varphi : [0,\infty) \to [0,\infty)$ is a lower semicontinuous map satisfying $\varphi^{-1}(\{0\}) = \{0\}$ then T admits a fixed point.

Corollary 2.11. Suppose $T : C \to C$ is continuous and $V \subseteq C$ is nonvoid. If $\eta(TV) < \dfrac{\eta(V)}{e^{\eta(TV)}}$, then T admits a fixed point.

3. Coupled fixed point results via $\mathcal{A}$-condensing operators

We now proceed towards defining the coupled fixed point theorem via $\mathcal{A}$-condensing operators. For that, we first write some preliminaries of our concern.

Definition 3.1. An ordered pair (u,v) is a coupled fixed point of a function $T : X \times X \to X$ whenever $T(u,v) = u$ and $T(v,u) = v$.

Lemma 3.2. [2] If $\eta_1, \eta_2, \ldots, \eta_m$ are MNCs on the metric spaces $E_1, E_2, \ldots, E_m$ respectively then

$$\widetilde{\eta}(H) = \Phi(\eta_1(H_1), \eta_2(H_2), \ldots, \eta_m(H_m)),$$

is also an MNC on $E_1 \times E_2 \times \cdots E_m$, where H_α stands for the natural projection of H into E_α respectively for $\alpha = 1,2,\ldots,m$, provided $\Phi : [0,\infty)^m \to [0,\infty)$ is a convex function and $\Phi(a_1, a_2, \ldots, a_m) = 0$ if and only if $a_\alpha = 0$ for all $\alpha = 1,2,\ldots,m$.

Theorem 3.3. Suppose $T : C\times C \to C\times C$ is continuous and an $\mathcal{A}$-condensing operator such that $\rho(s,t) \leq t-s$ for each $s,t \in A \cap (0,\infty)$ then T admits a fixed point in $C \times C$. Moreover, f_T is compact.

Proof. Define a sequence $\{G_n \times G_n\}$ in $C \times C$ as $G_n \times G_n = \overline{\text{conv}}(T(G_{n-1} \times G_{n-1}))$ for each $n \in \mathbb{N}$ where $G_0 \times G_0 = C \times C$. We intend to show by mathematical induction that $\{G_n \times G_n\}$ is a decreasing sequence. For $n = 1$, $G_1 \times G_1 = \overline{\text{conv}}(T(G_0 \times G_0)) \subset G_0 \times G_0$. Let us assume that $G_n \times G_n \subset G_{n-1} \times G_{n-1}$. Consider, $G_{n+1} \times G_{n+1} = \overline{\text{conv}}(T(G_n \times G_n)) \subset \overline{\text{conv}}(T(G_{n-1} \times G_{n-1})) = G_n \times G_n$. Hence, we write

$$\cdots \subseteq G_{n-1} \times G_{n-1} \subseteq G_n \times G_n \subseteq \cdots \subseteq G_1 \times G_1 \subseteq G_0 \times G_0.$$

From Lemma 3.2 we have $\widetilde{\eta}$ as an MNC on $X \times X$. If $\widetilde{\eta}(G_{n_0} \times G_{n_0}) = 0$ for some $n_0 \in \mathbb{N}$ then $G_{n_0} \times G_{n_0}$ is precompact. Also, $T(G_{n_0} \times G_{n_0}) \subseteq \overline{\text{conv}}(T(G_{n_0} \times G_{n_0})) = G_{n_0+1} \times G_{n_0+1} \subseteq G_{n_0} \times G_{n_0}$. This means that, by Schauder FPT there exists a fixed point of T. So, we now assume that $\widetilde{\eta}(G_n \times G_n) \neq 0$ for every $n \in \mathbb{N}$. Set

$$p_n = \widetilde{\eta}(T(G_n \times G_n)) > 0 \quad \text{and} \quad q_n = \widetilde{\eta}(G_n \times G_n) > 0.$$

Then, $\{(p_n, q_n)\}$ is a $T_{\widetilde{\eta}}$-sequence such that $p_n, q_n \to L$ and $0 \le L \le p_n \le q_n$. If there exist some $k_0 \in \mathbb{N}$ such that $L = p_{k_0}$ then $p_n = q_n$ implies $0 \le \rho(p_n, q_n) < 0$ for all $n > k_0$. This is a contradiction and hence, by the definition of $\widetilde{\mathcal{A}}$-condensing operators we have $L = 0$ so that

$$\lim_{n\to\infty} \widetilde{\eta}(G_n \times G_n) = 0.$$

Set $G_\infty \times G_\infty := \bigcap_{n=1}^{\infty} G_n \times G_n$ then this set is nonvoid, convex and compact so that $G_\infty \times G_\infty \in \mathrm{Ker}(\widetilde{\eta})$. Therefore, by Schauder FPT, the operator T admits a fixed point. □

By $\mathcal{F}$ we denote the class of all functions $f : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying

($f1$) $f(a_1 + a_2, b_1 + b_2) \le f(a_1, b_1) + f(a_2, b_2)$.

($f2$) $f(a, b) = 0 \Longleftrightarrow a = b = 0$.

($f3$) f is lower semicontinuous on $[0, \infty) \times [0, \infty)$. That is, for any two sequences $\{a_n\}$ and $\{b_n\}$ in $[0, \infty)$,

$$f(\liminf_{n\to\infty} a_n, \liminf_{n\to\infty} b_n) \le \liminf_{n\to\infty} f(a_n, b_n).$$

Corollary 3.4. [16] Suppose a continuous function $T : C \times C \to C \times C$ satisfies

$$\widetilde{\eta}(TV) \le \widetilde{\eta}(V) - f(\widetilde{\eta}(V), \widetilde{\eta}(V)),$$

for all nonvoid sets V in $C \times C$ and $f \in \mathcal{F}$ then T possesses a fixed point.

Proof. For any $r \in (0, 1)$ we write

$$\rho(\widetilde{\eta}(TV), \widetilde{\eta}(V)) = \widetilde{\eta}(V) - \widetilde{\eta}(TV) - rf(\widetilde{\eta}(V), \widetilde{\eta}(V)),$$

so that ($A3$) is fulfilled. Suppose $\{(p_n, q_n)\}$ is a $T_{\widetilde{\eta}}$-sequence such that $p_n, q_n \to L, 0 \le L < p_n$ and $\rho(p_n, q_n) > 0$ for every $n \in \mathbb{N}$. Then we have a sequence $\{G_n \times G_n\}$ of nonvoid sets in $C \times C$ such that

$$p_n = \widetilde{\eta}(T(G_n \times G_n)) > 0 \quad \text{and} \quad q_n = \widetilde{\eta}(G_n \times G_n) > 0.$$

Applying $\liminf_{n\to\infty}$, the inequality

$$0 < p_n < q_n - rf(q_n, q_n) \implies L \le L - rf(L, L),$$

so that $L = 0$. Thus, T is $\mathcal{A}$-condensing. □

By $\mathcal{G}$ we denote the class of all functions $g : [0, \infty) \times [0, \infty) \to [0, \infty)$ satisfying:

($g1$) g is continuous and nondecreasing function on $[0, \infty) \times [0, \infty)$ with respect to lexicographic order.

($g2$) $g(a, a) < a$ for all $a > 0$.

($g3$) $g(a_1, b_1) + g(a_2, b_2) \le 2g(\frac{a_1+a_2}{2}, \frac{b_1+b_2}{2})\ \forall\ a_1, a_2, b_1, b_2 \in [0, \infty)$.

Corollary 3.5. [16] Suppose a continuous function $T : C \times C \to C \times C$ satisfies

$$\widetilde{\eta}(TV) \le g(\widetilde{\eta}(V), \widetilde{\eta}(V)),$$

for all nonvoid sets V in $C \times C$ and $g \in \mathcal{G}$ then T possesses a fixed point in $C \times C$.

Proof. Suppose $\widetilde{\eta}(V)$ and $\widetilde{\eta}(TV)$ are positive then we set ρ as

$$\rho(\widetilde{\eta}(TV),\widetilde{\eta}(V)) = \frac{g(\widetilde{\eta}(V),\widetilde{\eta}(V)) + \widetilde{\eta}(V)}{2} - \widetilde{\eta}(TV),$$

so that $(A3)$ is fulfilled. For $(A2)$, let $\{(p_n, q_n)\}$ be a $T_{\widetilde{\eta}}$-sequence such that $p_n, q_n \to L, 0 \leq L < p_n$ and $\rho(p_n, q_n) > 0$ for each $n \in \mathbb{N}$. Then we have a sequence $\{G_n \times G_n\}$ of nonvoid sets in $C \times C$ such that

$$p_n = \widetilde{\eta}(T(G_n \times G_n)) > 0 \quad \text{and} \quad q_n = \widetilde{\eta}(G_n \times G_n) > 0.$$

Suppose $L > 0$ then applying $\lim\limits_{n\to\infty}$ to the inequality

$$p_n < \frac{g(q_n, q_n) + q_n}{2} \implies L \leq \frac{g(L, L) + L}{2},$$

so that $L \leq g(L, L)$, a contradiction. Thus, T is $\mathcal{A}$-condensing and hence, from Theorem 3.3, T has a fixed point. $\square$

In order to achieve the existence of coupled fixed point for a map $T : C \times C \to C$, we define T_Φ-sequence.

Definition 3.6. Let $\{p_n\}$ and $\{q_n\}$ be two real sequences and Φ be as given in Lemma 3.2. We say that $\{(p_n, q_n)\}$ is a T_Φ-sequence if $\exists$ a sequence $\{G_n \times G'_n\}$ of nonempty subsets in $C \times C$ such that $\forall\ n \in \mathbb{N}$,

$$p_n = \Phi(\eta(T(G_n \times G'_n)), \eta(T(G'_n \times G_n))) > 0 \quad \text{and} \quad q_n = \Phi(\eta(G_n), \eta(G'_n)) > 0.$$

Definition 3.7. Let $\widetilde{\eta}$ be an MNC on $X \times X$, as given in Lemma 3.2. An operator $T : C \times C \to C$ is $\mathcal{A}_\Phi$-condensing if one can find a function $\rho : A \times A \to \mathbb{R}$ satisfying the subsequent conditions together with T as:

$(\widetilde{A1})$ $\text{ran}(\widetilde{\eta}) \subseteq A \subseteq \mathbb{R}$.

$(\widetilde{A2})$ If $\{(p_n, q_n)\} \subseteq A \times A$ is a T_Φ-sequence such that both $p_n, q_n \to L$ with $L \geq 0$ and verifying $L < p_n$ along with $\rho(p_n, q_n) > 0$ for every $n \in \mathbb{N}$ then $L = 0$.

$(\widetilde{A3})$ $\rho(\Phi(\eta(T(G_1 \times G_2)), \eta(T(G_2 \times G_1))), \Phi(\eta(G_1), \eta(G_2))) > 0$ provided $\Phi(\eta(T(G_1 \times G_2)), \eta(T(G_2 \times G_1))) > 0$ and $\Phi(\eta(G_1), \eta(G_2)) > 0$ for every nonempty set $G_1 \times G_2 \subset C \times C$.

Theorem 3.8. *Suppose $T : C \times C \to C$ be a continuous operator. If T is $\mathcal{A}_\Phi$-condensing such that $\rho(s, t) \leq t - s$, for every $s, t \in A \cap (0, \infty)$ then T possesses a coupled fixed point.*

Proof. Define $\widetilde{T} : C \times C \to C \times C$ as $\widetilde{T}(p, q) = (T(p, q), T(q, p))$ for all $(p, q) \in C \times C$ then $\widetilde{T}$ is continuous. We now show that $\widetilde{T}$ is $\mathcal{A}$-condensing also. The condition $(A1)$ trivially holds. Let, $\widetilde{\eta}(\widetilde{T}(G)) > 0$ and $\widetilde{\eta}(G) > 0$ then

$$\begin{aligned} \widetilde{\eta}(\widetilde{T}(G)) = \widetilde{\eta}(\widetilde{T}(G_1 \times G_2)) &= \widetilde{\eta}(T(G_1 \times G_2) \times T(G_2 \times G_1)) \\ &= \Phi(\eta(T(G_1 \times G_2)), \eta(T(G_2 \times G_1))) \end{aligned}$$

and

$$\widetilde{\eta}(G) = \widetilde{\eta}(G_1 \times G_2) = \Phi(\eta(G_1), \eta(G_2))$$

so that

$$\rho(\widetilde{\eta}(\widetilde{T}(G)), \widetilde{\eta}(G)) = \rho(\Phi(\eta(T(G_1 \times G_2)), \eta(T(G_2 \times G_1))), \Phi(\eta(G_1), \eta(G_2))) > 0$$

and hence, $\widetilde{T}$ satisfies $(A3)$. For $(A2)$, suppose $\{(p_n, q_n)\}$ is a $\widetilde{T}_{\widetilde{\eta}}$ sequence such that $p_n, q_n \to L$, $0 \leq L < p_n$ and $\rho(p_n, q_n) > 0$ for every $n \in \mathbb{N}$. Then we have a sequence $\{G_n \times G_n'\}$ of nonempty subsets in $C \times C$ such that

$$p_n = \widetilde{\eta}(\widetilde{T}(G_n \times G_n')) > 0 \quad \text{and} \quad q_n = \widetilde{\eta}(G_n \times G_n') > 0.$$

This means that

$$p_n = \Phi(\eta(T(G_n \times G_n')), \eta(T(G_n' \times G_n))) > 0 \quad \text{and} \quad q_n = \Phi(\eta(G_n), \eta(G_n')) > 0,$$

so that $\{(p_n, q_n)\}$ is a T_Φ sequence and so by $(\widetilde{A2})$, $L = 0$. As all the requirements of Theorem 3.3 are achieved, we affirm that T admits a coupled fixed point. □

Corollary 3.9. [16] Suppose a continuous function $T : C \times C \to C$ satisfies

$$\eta(T(V_1 \times V_2)) \leq \frac{1}{2}[\eta(V_1) + \eta(V_2)] - f(\eta(V_1), \eta(V_2)), \tag{1}$$

for all nonvoid sets V_1, V_2 in C and $f \in \mathcal{F}$ then T admits a coupled fixed point.

Proof. From Equation (1), we write

$$\begin{aligned}\eta(T(V_1 \times V_2)) + \eta(T(V_2 \times V_1)) &\leq \eta(V_1) + \eta(V_2) - f(\eta(V_1), \eta(V_2)) - f(\eta(V_2), \eta(V_1)) \\ &\leq \eta(V_1) + \eta(V_2) - f(\eta(V_1) + \eta(V_2), \eta(V_2) + \eta(V_1)).\end{aligned}$$

For every $a, b \in [0, \infty)$, choose $\Phi(a, b) = a + b$ and $\rho(a, b) = b - a - rf(b, b)$. Then T becomes $\mathcal{A}_\Phi$ condensing operator satisfying $\rho(s, t) \leq t - s$ and hence, the above theorem proves the rest. □

Corollary 3.10. [16] Suppose a continuous function $T : C \times C \to C$ satisfies

$$\eta(T(V_1 \times V_2)) \leq g(\eta(V_1), \eta(V_2)), \tag{2}$$

for all nonvoid sets V_1, V_2 in C and $g \in \mathcal{G}$ then T possesses a coupled fixed point.

Proof. From Equation (2), we write

$$\begin{aligned}\eta(T(V_1 \times V_2)) + \eta(T(V_2 \times V_1)) &\leq g(\eta(V_1), \eta(V_2)) + g(\eta(V_2), \eta(V_1)) \\ &\leq 2g\Big(\frac{\eta(V_1) + \eta(V_2)}{2}, \frac{\eta(V_1) + \eta(V_2)}{2}\Big).\end{aligned}$$

Suppose $s = \Phi(\eta(T(G_1 \times G_2)), \eta(T(G_2 \times G_1)))$ and $t = \Phi(\eta(G_1), \eta(G_2))$ be positive and for every $a, b \in [0, \infty)$, choose $\Phi(a, b) = \frac{a+b}{2}$ then we set ρ as

$$\rho(s, t) = \frac{g(t, t) + t}{2} - s,$$

so that $(\widetilde{A3})$ is fulfilled. For $(\widetilde{A2})$, let $\{(p_n, q_n)\}$ be a T_Φ-sequence such that $p_n, q_n \to L, 0 \leq L < p_n$ and $\rho(p_n, q_n) > 0$ for each $n \in \mathbb{N}$. Then we have a sequence $\{G_n \times G_n\}$ of nonvoid sets in $C \times C$ such that

$$p_n = \widetilde{\eta}(T(G_n \times G_n)) > 0 \quad \text{and} \quad q_n = \widetilde{\eta}(G_n \times G_n) > 0.$$

Suppose $L > 0$ then applying $\lim\limits_{n \to \infty}$ to the inequality

$$p_n < \frac{g(q_n, q_n) + q_n}{2} \implies L \leq \frac{g(L, L) + L}{2}$$

so that $L \leq g(L, L)$, a contradiction. Thus, T is $\mathcal{A}_\Phi$-condensing and hence, from Theorem 3.3, T has a fixed point. □

4. Application to (k,ψ)-Hilfer fractional differential equations

Fractional Calculus deals with the study of differentiation and integration of arbitrary order and thus generalises the classical structure. This generalisation grabbed focus due to its efficiency in providing more accurate description to the real world phenomenas. For a brief study, one can see [10]. Motivated by the definition of Riemann Liouville and Caputo deriavtives, the authors [7] proposed Hilfer derivative and solved an existence-uniqueness problem involving this deriavtive for the order lying between 0 to 1. Later to this, in 2018, Sousa et al. [19] initiated the discussion of ψ-Hilfer derivative, involving a continuously differentiable increasing function ψ. Various authors using renowned fixed point theorems showed up with the existence of solutions to more and more generalised form of such fractional order differential equations. In recent times, Kucche et al. in their paper [11] stated the most general form and defined (k,ψ)-Hilfer fractional differential operator of order $\alpha \in (0,\infty)$ and type $\beta \in [0,1]$ acting on a function $h \in C^n[a,b]$ with $n = \lceil \frac{\alpha}{k} \rceil \in \mathbb{N}$ as

$$^{k,H}D^{\alpha,\beta;\psi}h(t) = {}^kI^{\beta(nk-\alpha);\psi}\left(\frac{k}{\psi'(t)}\frac{d}{dt}\right)^n {}^kI^{(1-\beta)(nk-\alpha);\psi}h(t),$$

where $k \in (0,\infty)$, $\psi \in C^n[a,b]$ is an increasing function such that $\psi'(t) \neq 0 \; \forall \, t \in [a,b]$, ${}^kI^{\gamma;\psi}$ is the (k,ψ)-Riemann Liouville integral (RLI) of order $\gamma \in (0,\infty)$ as

$$^kI^{\gamma;\psi}h(t) = \frac{1}{k\Gamma_k(\gamma)}\int_a^t \psi'(s)[\psi(t)-\psi(s)]^{\frac{\gamma}{k}-1}h(s)ds,$$

and the notation Γ_k stands for the k-gamma function given by $\Gamma_k(\gamma) = \int_0^\infty t^{\gamma-1}e^{-\frac{t^k}{k}}dt$, which enjoys the following properties:

$$\Gamma_k(\gamma+k) = \gamma\Gamma_k(\gamma), \quad \Gamma_k(k) = 1 \quad \text{and} \quad \Gamma_k(\gamma) = k^{\frac{\gamma}{k}-1}\Gamma(\frac{\gamma}{k}).$$

Here, we consider (k,ψ)-Hilfer fractional differential equation (HFDE) of order $2 < p < 3$ and type $q \in [0,1]$ of the form:

$$^{k,H}D^{p,q;\psi}y(t) = G(t,y(t)), \tag{3}$$

$$y(a) = 0, y'(a) = 0, uy(b) + v\delta_\psi y(b) = w\,{}^kI^{\nu;\psi}g(\zeta, y(\zeta)), \tag{4}$$

satisfying the stated boundary conditions. The quantities a, b, u, v and w are suitable real scalars with $J = [a,b]$. The functions $g, G : J \times \mathbb{R} \to \mathbb{R}$ and $\psi : J \to \mathbb{R}$ are continuous such that $\psi'(t) > 0$ for all $t \in J$ with $\delta_\psi \equiv \frac{k}{\psi'(t)}\frac{d}{dt}$, $a < \zeta < b$ and ${}^kI^{\nu;\psi}$ is the (k,ψ)-RLI of order $\nu \in (0,\infty)$. The following lemma gives an equivalence between the differential equation (3)-(4) with the integral equation (5).

Lemma 4.1. *Let $a < b$, $2 < p < 3$, $q \in [0,1]$ and $\omega_k = p + q(3k - p)$ then the equivalent integral to the above differential equation is*

$$y(t) = {}^kI^{p;\psi}G(t,y(t)) + \frac{(\psi(t)-\psi(a))^{\frac{\omega_k}{k}-1}}{AL^{\frac{\omega_k}{k}-1}}\Big[w\,{}^kI^{\nu;\psi}g(\zeta,y(\zeta)) - u\,{}^kI^{p;\psi}G(b,y(b)) - v\,{}^kI^{p-k;\psi}G(b,y(b))\Big], \tag{5}$$

where, $L = \psi(b) - \psi(a)$ and $A = u + \dfrac{v(\omega_k - k)}{L} \neq 0$.

We are now about to show the existence of the solution of the system (3)-(4) for a more general setting. Let $S = C(J,E)$ be the Banach space consisting of all continuous functions defined from J in to a Banach space E together with the supremum norm. Choose a subset D of S as

$$D = \{y \in S : y \in C(J,B) \text{ and } y(a) = 0\},$$

where $B = B(q_0, \kappa)$ represents a closed ball centered at q_0 with radius κ in E. The functions $g, G : J \times B \to E$ are all continuous. Clearly, D is nonvoid, convex, closed and bounded set in S. Define $\mathcal{T}$ on D as

$$\mathcal{T} y(t) = {}^kI^{p;\psi}G(t, y(t)) + \frac{(\psi(t)-\psi(a))^{\frac{\omega_k}{k}-1}}{AL^{\frac{\omega_k}{k}-1}}\Big[w\,{}^kI^{\nu;\psi}g(\zeta, y(\zeta)) - u\,{}^kI^{p;\psi}G(b, y(b)) - v\,{}^kI^{p-k;\psi}G(b, y(b))\Big].$$

Lemma 4.2. If $N_h = \sup_{s\in[a,b]}\{\|h(s, y(s))\| : y \in D\}$ for $h = g, G$, and $B^{\frac{r}{k}} = \dfrac{L^{\frac{r}{k}}}{\Gamma_k(r+k)}$ for $r = k, p, \nu$ such that

$$\Big[1 + \frac{|u|}{|A|} + \frac{|v|}{|AB|}\Big]N_G B^{\frac{p}{k}} + \frac{|w|}{|A|}N_g B^{\frac{\nu}{k}} + \|q_0\| \le \kappa$$

then T is invariant in D.

Proof. Suppose $y \in D$ and consider

$$\begin{aligned}
\Big\|(\mathcal{T} y)(t)\Big\| &\le \Big\|{}^kI^{p;\psi}G(t, y(t))\Big\| + \left|\frac{(\psi(t)-\psi(a))^{\frac{\omega_k}{k}-1}}{AL^{\frac{\omega_k}{k}-1}}\right|\Big[\Big\|w\,{}^kI^{\nu;\psi}g(\zeta, y(\zeta))\| + \|u\,{}^kI^{p;\psi}G(b, y(b))\| \\
&\quad + \|v\,{}^kI^{p-k;\psi}G(b, y(b))\Big\|\Big] \\
&\le \frac{N_G}{k\Gamma_k p}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\frac{p}{k}-1}ds + \frac{1}{|A|}\Big[\frac{|w|N_g}{k\Gamma_k \nu}\int_a^\zeta \psi'(s)(\psi(\zeta)-\psi(s))^{\frac{\nu}{k}-1}ds \\
&\quad + \frac{|u|N_G}{k\Gamma_k p}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\frac{p}{k}-1}ds + \frac{|v|N_G}{k\Gamma_k(p-k)}\int_a^b \psi'(s)(\psi(b)-\psi(s))^{\frac{p-k}{k}-1}ds\Big] \\
&\le \frac{N_G(\psi(t)-\psi(a))^{\frac{p}{k}}}{p\Gamma_k p} + \frac{1}{|A|}\Big[\frac{|w|N_g}{\nu\Gamma_k\nu}(\psi(\zeta)-\psi(a))^{\frac{\nu}{k}} + \frac{|u|N_G}{p\Gamma_k p}(\psi(b)-\psi(a))^{\frac{p}{k}} \\
&\quad + \frac{|v|N_G}{(p-k)\Gamma_k(p-k)}(\psi(b)-\psi(a))^{\frac{p-k}{k}}\Big] \\
&\le \frac{N_G L^{\frac{p}{k}}}{p\Gamma_k p} + \frac{1}{|A|}\Big[\frac{|w|N_g L^{\frac{\nu}{k}}}{\nu\Gamma_k\nu} + \frac{|u|N_G L^{\frac{p}{k}}}{p\Gamma_k p} + \frac{|v|N_G L^{\frac{p-k}{k}}}{(p-k)\Gamma_k(p-k)}\Big] \\
&\le \kappa - \|q_0\|.
\end{aligned}$$

Therefore, we have $\mathcal{T} y \in D$ and hence, $\mathcal{T}$ is invariant in nature. □

We now state the Mean Value Theorem of Integral Calculus for (k, ψ)-Riemann Lioville integral.

Lemma 4.3. [6] If $\mu, k > 0$ and $\hat{h}$ is any continuous function then we can find $z \in (a, b)$ such that

$${}^kI^{\mu;\psi}\hat{h}(t) = \frac{1}{k\Gamma_k\mu}\int_a^t \psi'(s)(\psi(t)-\psi(s))^{\frac{\mu}{k}-1}\hat{h}(s)ds = \frac{(\psi(t)-\psi(a))^{\frac{\mu}{k}}}{\mu\Gamma_k\mu}\hat{h}(z).$$

Theorem 4.4. Along with the assumptions of Lemma 4.2, suppose there exists a positive real scalar λ such that for all $t \in [a, b]$ and $y \in D$ we have

$$\max\{\|g(t, y(t)) - g(t, 0)\|, \|G(t, y(t)) - G(t, 0)\|\} \le \lambda\|y(t) - 0\|,$$

where $\lambda \in (0, 1]$ satisfying $\lambda M < 1$ and $M = \frac{1}{|A|}\Big[\Big(|A| + |u| + \frac{|v|}{B}\Big)B^{\frac{p}{k}} + |w|B^{\frac{\nu}{k}}\Big]$. Then, the system (3)-(4) of (k, ψ)-HFDE has a solution.

Proof. For any nonvoid set V in D, consider

$$\begin{aligned}
\eta(\mathcal{T}V) &= \sup_{t\in I}\{\eta(\{\mathcal{T}y(t) : y \in V\})\\
&= \sup_{t\in I}\Big\{\eta\Big(\Big\{{}^{k}I^{p;\psi}G(t,y(t)) + \frac{(\psi(t)-\psi(a))^{\frac{\omega_k}{k}-1}}{AL^{\frac{\omega_k}{k}-1}}\Big[w\,{}^{k}I^{\nu;\psi}g(\zeta,y(\zeta)) - u\,{}^{k}I^{p;\psi}G(b,y(b))\\
&\quad - v\,{}^{k}I^{p-k;\psi}G(b,y(b))\Big] : y \in D\Big\}\Big)\Big\}\\
&= \sup_{t\in I}\Big\{\eta\Big(\Big\{\frac{(\psi(t)-\psi(a)^{\frac{p}{k}}G(z,x(z)))}{p\Gamma_k p} + \frac{(\psi(t)-\psi(a))^{\frac{\omega_k}{k}-1}}{AL^{\frac{\omega_k}{k}-1}}\Big[\frac{w(\psi(\zeta)-\psi(a))^{\frac{\nu}{k}}}{\nu\Gamma_k\nu}g(z,x(z))\\
&\quad - \frac{u(\psi(b)-\psi(a))^{\frac{p}{k}}}{p\Gamma_k p}G(z,x(z)) - \frac{v(\psi(b)-\psi(a))^{\frac{p-k}{k}}}{p\Gamma_k p}G(z,x(z))\Big] : \text{ for some } z \in [a,b]\Big\}\Big)\Big\}\\
&< \frac{L^{\frac{p}{k}}}{p\Gamma_k p}\eta\Big(\Big\{G(z,x(z)) + G(z,0) - G(z,0)\Big\}\Big) + \frac{1}{|A|}\Big[\frac{|w|L^{\frac{\nu}{k}}}{\nu\Gamma_k\nu}\eta\Big(\Big\{g(z,x(z)) + g(z,0) - g(z,0)\Big\}\Big)\\
&\quad + \frac{|u|L^{\frac{p}{k}}}{p\Gamma_k p}\eta\Big(\Big\{G(z,x(z)) + G(z,0) - G(z,0)\Big\}\Big) + \frac{|v|L^{\frac{p-k}{k}}}{(p-k)\Gamma_k(p-k)}\eta\Big(\Big\{G(z,x(z)) + G(z,0) - G(z,0)\Big\}\Big)\Big]\\
&\le \lambda M\eta(V).
\end{aligned}$$

By choosing $\rho(t,s) = \lambda Ms - t$ and considering η as $\mathcal{K}$, the operator $\mathcal{T}$ becomes $\mathcal{A}$-condensing. Hence, all the hypothesis of Theorem 2.4 are fulfilled and so, the fixed point of $\mathcal{T}$ is the solution of the system (3)-(4). □

We now give an example in support of the above theorem.

Example 4.5. Consider the (k,ψ)-HFDE of the form

$$\begin{aligned}
&{}^{\frac{3}{2},H}D^{\frac{5}{2},\frac{1}{4};e^t}y(t) = \frac{1}{2}\tan\Big(\frac{4y(t)}{9}\Big),\\
&y(0) = 0, y'(0) = 0, 3y(1) + 2\delta_\psi y(1) = 0\cdot {}^{\frac{3}{2}}I^{\nu;e^t}\Big|\tfrac{7y}{13} - \tfrac{1}{2}\Big|.
\end{aligned}$$

Upon comparison, we have the following values of the variables: $p = \frac{5}{2}, q = \frac{1}{4}, k = \frac{3}{2}, \psi(t) = e^t, u = 3, v = 2, w = 0, a = 0, b = 1$ and $g \equiv \big|\frac{7y}{13} - \frac{1}{2}\big|$. Therefore, $L = e-1, \omega_k = 3, A = \frac{3e}{e-1}, B = \frac{2(e-1)}{3}, B^{\frac{5}{3}} \approx 0.8336$ so that $M = 1.6672$. Consider, $y,z \in C([0,1],[-\kappa,\kappa])$ with $\kappa = \frac{19}{10}$ then

$$|G(t,y(t)) - G(t,z(t))| = \frac{1}{2}\Big|\tan\Big(\frac{4y}{9}\Big) - \tan\Big(\frac{4z}{9}\Big)\Big| \le 0.50|y-z|. \tag{6}$$

This inequality can be easliy seen through Figure 1, as the lower surface represents the LHS of (6) whereas the upper surface stands for the RHS of (6). Also, the inequality

$$|g(t,y(t)) - g(t,0)| = \Big|\Big|\frac{7y}{13} - \frac{1}{2}\Big| - \frac{1}{2}\Big| \le 0.57|y(t)|, \tag{7}$$

gets validated by Figure 2. As a result, $\lambda = 0.57$ and hence, $\lambda M \approx 0.95 < 1$. Moreover, $MN_G < \kappa$. Therefore, by Theorem 4.4, solution of the given system exists.

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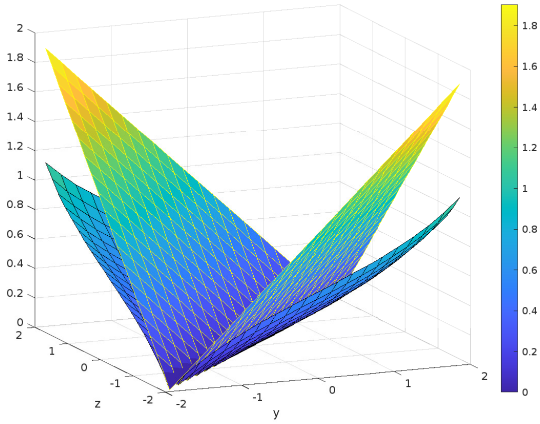

Figure 1: LHS and RHS of inequality (6)

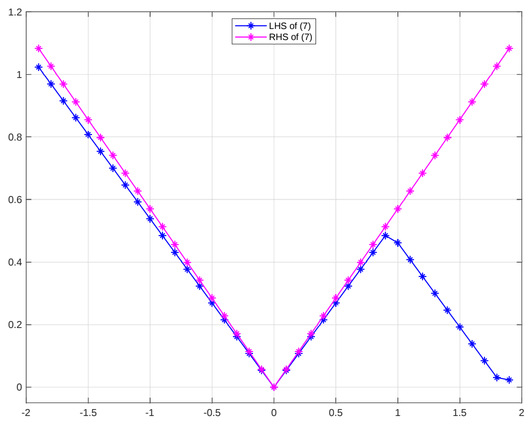

Figure 2: LHS and RHS of inequality (7)

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