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Approximation by Stancu-type operators on a triangular domain with curved sides

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Abstract. In this article, some Stancu-type operators, their products and boolean sum operators are constructed on a triangular domain with curved sides. Their interpolation features have been described. Also remainders of approximation formulas have been discussed. Graphical representations have been given to demonstrate the theoretical findings.

1. Introduction

The study of mathematical approximation theory has recently given a considerable amount of attention to the examination of Dunkl analogues and their significance in diverse mathematical operations. In a recent publication [29], V.N. Mishra, M. Raiz, and N. Rao introduced the Dunkl Analogue of Szász Schurer-Beta Operators, providing insight into their approximative behavior. This research delves into the intricate relationship between Dunkl analogues and their practicality in computational mathematics.

The study of complex mathematical sequences is enhanced by the utilization of Szász-type operators that involve q-Appell polynomials. This approach is particularly intriguing, as demonstrated through the research conducted in [20]. This work serves as a critical resource for integrating specialized mathematical methods into the overall scope of approximation theory.

The Dunkl analogue of the Szász Schurer Beta bivariate operator has been the subject of a recent inquiry [30]. This investigation, completed in 2023, has added to our comprehension of the many uses of Dunkl analogues in operations involving multiple variables. These discoveries emphasize the dynamic and constantly evolving nature of mathematical approximation theory and its ever-growing range of pragmatic implementations.

Expanding upon the progress made in prior research, the objective of this present investigation is to expand upon the examination of specialized operators in a new, relevant domain: the application of Stancu operators on triangular domains with curved borders. While previous research has predominantly

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focused on the Dunkl analogues and their utility in multivariate operations, this study delves into the realm of approximating functions on complicated, curved domains. This provides valuable insights into the behavior of Stancu operators in intricate geometries. By amalgamating the discoveries from the works of Mishra, Raiz, and Rao, with the current research on Stancu operators, this study aims to make a contribution towards the developing landscape of approximation theory in mathematics and its practical implications in various fields.

In the field of practical application, we frequently encounter the issue of approximating a function that is defined on a planar domain with curvature. Situations may arise where we need to interpolate or fit data, or solve problems related to boundary values. A popular technique to tackle such problems is to utilize piecewise polynomial spline functions, which are defined on a triangulation that only partly encompasses the domain. Although it may appear more intuitive to work with piecewise polynomial splines defined directly on a curved triangulation of GT_w, research on such splines in the spline literature has been scarce; where $GT_w = \{(x_w, y_w) \in \mathbb{R} \times \mathbb{R} | x_w + y_w \leq w\}$. The functions that form the basis are characterized by their non-negative values and are united by a common trait. The notion of partition of unity requires that functions have local supports, which can be achieved through the use of a single curved triangle, the union of all curved triangles attached to a common vertex, or two adjoining curved triangles. Typically, closed NURBs curves define the curves that define the boundary of a curved domain known as GT_w. However, these curves could also be expressed in terms of piecewise elementary parametric curves, such as lines, circles, ovals, and the like. Assuming that the boundary curves are oriented, we can determine on which side of the curve the domain lies. Error bounds for approximating smooth functions with polynomial splines on ordinary are also relevant to our purposes. Classical spline theory places significant emphasis on the utilization of triangulations. Through the implementation of triangulations, we can obtain outcomes for polynomials in this context. The quasi-interpolation operator is utilized to implement splines on curved triangulations. Macro-element spline spaces typically rely on a regular refinement of a given triangulation. In the case of curved domains with well-behaved borders, inscribed triangulations with an ample amount of boundary points can facilitate the creation of associated curved triangulations. Several methods based on polynomial splines have been developed for ordinary triangulations to fit functions with various shape constraints. These methods can also be adapted to address concerns on curved domains by utilizing curved triangulations instead. Triangulations with polynomial splines are valuable in solving fitting problems, particularly when dealing with noisy data. All of the previously mentioned methods, inclusive of penalized least-squares methods, can be expanded to operate with curved triangulations. The process of triangulating curved domains involves the partitioning of such regions into smaller, simpler shapes using triangular elements.

Stancu operators are a type of approximation operators used in numerical analysis and approximation theory. They are commonly employed to approximate functions defined on a given domain, including triangular domains. The Stancu operators are defined by a combination of Bernstein polynomials and certain weight functions. These weight functions affect the contribution of each Bernstein polynomial to the approximation. With the paper of Barnhill and Gregory [1], [2], [3], [14] and with the right choice of parameters c, d, Stancu polynomials have potential to produce a better approximation at a point than the Bernstein polynomials. Interpolation operators of Bernstein type have been defined in ([4], [5]). The choice of Stancu operators and their parameters will depend on factors like the desired accuracy of the approximation, the specific properties of the function being approximated, and the geometric characteristics of the triangular domain.

D.D. Stancu [24] constructed sequence of approximating operators that are more generalized version of the Classical Bernstein operators. These operators can be used in the creation of surfaces that meet a set of predetermined conditions, in the solution of differential equation problems using the finite element method, and in the numerical integration of function.

Since many features of Bernstein operators can be extended to the Stancu operators, numerous researchers in mathematics have examined these outcomes from this angle. Boundary requirements can be met precisely using the Stancu-type operators established on domains with curved sides. Stancu operators are constructed on GT_w triangle ([9]), [22], [27]). Recently, Iliyas *et. al.* [10] studied Lupaş type Bernstein operators on triangle with one curve side; and Mansoori *et. al.* [13] studied q-Bernstein operators on triangular domain with all curved sides.

The objective of this manuscript is to furnish a preliminary understanding of Stancu-type operators on a triangle with either two or three curved sides. The present study serves as an expansion upon the latest research conducted by Cheregi[6], as it delves into the behavior of Stancu interpolation operators when used on a triangle with more than one curved side. This paper will explore both the accuracy of the operators derived and their ability to approximate. This adds a level of complexity compared to the traditional Stancu operators applied to simpler domains like intervals or triangles with straight sides. The shape and curvature of the sides of the triangle will impact the distribution of points and the weighting of the approximating functions.

2. Construction of operators on a triangular domain with two or three curved sides

Let us consider the triangle GT_w with the three curved sides Γ_1, Γ_2 (along the coordinate axis) and Γ_3 (opposite to the C) with vertices $A = (0, w), B = (w, 0), C = (0, 0)$. The curved side Γ_1 is defined by $(x_w, g_1(x_w))$ with $g_1(0) = g_1(w) = 0$, $g_1(x_w) \leq 0$, $\forall x_w \in [0, w]$. Γ_2 is characterized by $(h_2(y_w), y_w)$ with $h_2(0) = h_2(w) = 0$, $h_2(y_w) \leq 0$, $\forall y_w \in [0, w]$ and Γ_3 is characterized by the functions g_3 and h_3, where inverse function of g_3 is h_3, i.e., $y_w = g_3(x_w)$ and $x_w = h_3(y_w)$; $x_w, y_w \in [0; w]$ and $g_3(0) = h_3(0) = w$ (Figure 1). We express by e_{ij} the monomial functions $e_{ij}(x, y) = x^i y^j$, $i, j \in \mathbb{N}$. In the following figure 1, x_w and y_w are represented by x and y respectively.

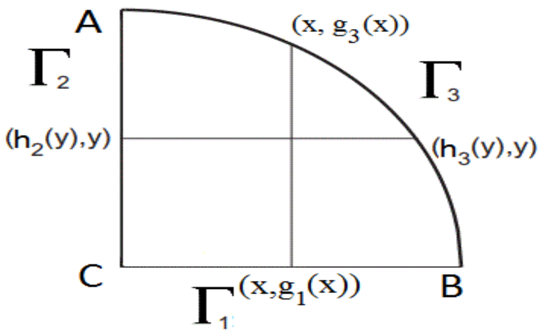

Figure 1: GT_w triangle

Let G be a function determined by $GT_w = \{(x_w, y_w) \in \mathbb{R} \times \mathbb{R} | x_w + y_w \leq w\}$, $x_w, y_w, w \in \mathbb{R}_+$, $(x_w, 0), (0, y_w)$ and $(h(y_w), y_w)$ are intersection points of curved sides Γ_i, $i = \overline{1, 3}$. [26]

D.D. Stancu[25] constructed sequence of positive linear positive operator based on two parameters c and d , $0 \leq c \leq d$. Suppose that F_G is a function defined on the set GT_w. The operators referred to as the Stancu-type operators $G^{(c,d)}_{x_w,m}$ and $G^{(c,d)}_{y_w,n}$ [23], $x_w, y_w \in [0, 1]$ are defined as follows

$$\left(G^{(c,d)}_{x_w,m} F_G\right)(x_w) = \sum_{i=0}^{m} s_{m,i}(x_w, y_w) F_G\left(h_2(y_w) + \frac{i + c}{m + d}(h_3(y_w) - h_2(y_w)), y_w\right)$$

$$\left(G^{(c,d)}_{y_w,n} F_G\right)(x_w) = \sum_{j=0}^{n} s_{n,j}(x_w, y_w) F_G\left(x_w, g_1(x_w) + \frac{j + c}{n + d}(g_3(x_w) - g_1(x_w))\right),$$

with

$$s_{m,i} = \binom{m}{i} \left(\frac{x_w - h_s(y_w)}{h_3(y_w) - h_2(x_w)}\right)^i \left(1 - \frac{x_w}{h(y_w)}\right)^{m-i}, \quad 0 \leq x_w + y_w \leq h(y_w) \quad \text{and}$$

$$s_{n,j}=\binom{n}{j}\left(\frac{y_w-g_1(x_w)}{g_3(x_w)-g_1(x_w)}\right)^j\left(1-\frac{y_w-g_1(x_w)}{g_3(x_w)-g_1(x_w)}\right)^{n-j},\quad 0\le x_w+y_w\le f(x_w),$$

where uniform partitions of $[h_2(y_w),h_3(y_w)]$ and $[g_1(x_w),g_3(x_w)]$ are defined as follows

$$\Delta^{(c,d)}_{x_w,m}=\left\{h_2(y_w)+\frac{i+c}{m+d}(h_3(y_w)-h_2(y_w))|i=\overline{0,l}\right\},$$
$$\Delta^{(c,d)}_{y_w,n}=\left\{g_1(x_w)+\frac{j+c}{r+d}(g_3(x_w)-g_1(x_w))|j=\overline{0,n}\right\}.$$

Remark 2.1. *For $c=d=0$, $G^{(c,d)}_m$ change into classical Bernstein operator (B_m).*

Theorem 2.2. *If F_G is function on GT_w then*

(i) $G^{(c,d)}_{x_w,m}F_G=F_G$ *on* $\Gamma_2\cap\Gamma_3$,

(ii) $G^{(c,d)}_{y_w,n}F_G=F_G$ *on* $\Gamma_1\cap\Gamma_3$,

(iii) $(G^{(c,d)}_{x_w,m}e_{ij})(x_w,y_w)=\left[x^i_w+\frac{c-dx^i_w}{m+d}\right]y^j_w,\quad i,j\in\mathbb{N},$

$(G^{(c,d)}_{x_w,m}e_{2j})(x_w,y_w)=\left[x^2_w+\frac{m(x_w-h_2(y_w))(h_3(y_w)-x_w)+(c-dx_w)(2mx_w+dx_w+c)}{(m+d)^2}\right]y^j_w,\ j\in\mathbb{N},$

(iv) $(G^{(c,d)}_{y_w,n}e_{ij})(x_w,y_w)=x^i_w\left[y^j_w+\frac{c-dy^j_w}{n+d}\right],\quad i,j\in\mathbb{N},$

$(G^{(c,d)}_{y_w,n}e_{i2})(x_w,y_w)=x^i_w\left[y^2_w+\frac{n(y_w-g_1(x_w))(g_3(x_w)-y_w)+(c-dy_w)(2ny_w+dy_w+c)}{(n+d)^2}\right],\ i\in\mathbb{N}.$

Proof. One can prove properties (i) and (ii) easily from the following relations:

$$s_{m,i}(h_2(y_w),y_w)=\begin{cases}1, & \text{for } i=0\\ 0, & \text{for } i>0\end{cases},\quad s_{m,i}(h_3(y_w),y_w)=\begin{cases}0, & \text{for } i<m\\ 1, & \text{for } i=m\end{cases}$$

respectively by

$$s_{n,j}(x_w,g_1(x_w))=\begin{cases}1, & \text{for } j=0\\ 0, & \text{for } j>0\end{cases},\quad s_{n,j}(x_w,g_3(y_w))=\begin{cases}0, & \text{for } j<n\\ 1, & \text{for } j=n\end{cases}.$$

$$(G^{(c,d)}_{x_w,m}F_G)(h_2(y_w),y_w)=F_G(h_2(y_w),y_w),\quad (G^{(c,d)}_{x_w,m}F_G)(h_3(y_w),y_w)=F_G(h_3(y_w),y_w).$$

Regarding to the properties (iii) we obtain

$$(G^{(c,d)}_{x_w,m}e_{ij})(x_w,y_w)=y^j_w(G^{(c,d)}_{x_w,m}e_{i0})(x_w,y_w),\quad i,j\in\mathbb{N},$$

$$(G^{(c,d)}_{x_w,m}e_{00})(x_w,y_w)=(B_me_{00})(x_w,y_w)=1,$$

$$\begin{aligned}(G^{(c,d)}_{x_w,m}e_{10})(x_w,y_w)&=\frac{m}{m+d}(B_me_{10})(x_w,y_w)+\frac{c}{m+d}(B_me_{00})(x_w,y_w)=\frac{mx_w+c}{m+d}\\&=x_w+\frac{c-dx_w}{m+d},\end{aligned}$$

$$
\begin{aligned}
(G^{(c,d)}_{x_w,m}e_{20})(x_w,y_w) =& \frac{m^2}{(m+d)^2}(B_m e_{20})(x_w,y_w) + \frac{2cm}{(m+d)^2}(B_m e_{10})(x_w,y_w) \\
&+ \frac{c^2}{(m+d)^2}(B_m e_{00})(x_w,y_w) \\
=& \frac{m^2}{(m+d)^2}\left[x_w^2 + \frac{(x_w - h_2(y_w))(h_3(y_w) - x_w)}{m}\right] + \frac{2cm}{(m+d)^2}x_w + \frac{c^2}{(m+d)^2} \\
=& x_w^2 + \frac{m(x_w - h_2(y_w))(h_3(y_w) - x_w) - x_w^2 d^2 - 2mdx_w^2 + 2cmx_w + c^2}{(m+d)^2} \\
=& x_w^2 + \frac{m(x_w - h_2(x_w))(h_3(y_w) - x_w) + 2mx_w(-dx_w + c) + c^2 - d^2x_w^2}{(m+d)^2} \\
=& x_w^2 + \frac{m(x_w - h_2(y_w))(h_3(y_w) - x_w) + (c - dx_w)(2mx_w + dx_w + c)}{(m+d)^2}.
\end{aligned}
$$

Properties (iv) is easy to follow in a similar way. □

We consider the approximation formula as follows

$$F_G = G^{(c,d)}_{x_w,m}F_G + R_{x_w,m}F_G,$$

where estimation error is denoted by $R_{x_w,l}F_G$.

Theorem 2.3. *If $F_G(\cdot, y_w) \in C\,[h_2(y_w), h_3(y_w)]$ then*

$$|(R_{x_w,m}F_G)(x_w,y_w)| \le \left(1 + \frac{(h_3(y_w) - h_2(y_w))\sqrt{m+4d^2}}{2\delta(m+d)}\right)\omega(F_G(\cdot,y_w);\delta), \quad y_w \in [0,w]$$

where $\omega(F_G(\cdot,y_w);\delta)$ is the usual modulus of continuity of the function F_G with regard to the variable x. Moreover, if $\delta = \frac{\sqrt{m+4d^2}}{m+d}$ then

$$|(R_{x_w,m}F_G)(x_w,y_w)| \le \left(1 + \frac{h_3(y_w) - h_2(y_w)}{2}\right)\omega\left(F_G(\cdot,y_w); \frac{\sqrt{m+4d^2}}{m+d}\right).$$

Proof. Regarding to $(B^{x_w}_m e_{00})(x_w,y_w) = 1 = (G^{(c,d)}_{x_w,m}e_{00})(x_w,y_w)$ proceed that

$$|(R_{x_w,m}F_G)(x_w,y_w)| \le \sum_{i=0}^{m} s_{m,i}(x_w,y_w)\left|F_G(x_w,y_w) - F_G\left(h_2(y_w) + \frac{i+c}{m+d}(h_3(y_w) - h_2(y_w)), y_w\right)\right|.$$

Using the inequality

$$
\begin{aligned}
&\left|F_G(x_w,y_w) - F_G\left(h_2(y_w) + \frac{i+c}{m+d}(h_3(y_w) - h_2(y_w)), y_w\right)\right| \\
&\le \left(\frac{1}{\delta}\left|x_w - \left(h_2(y_w) + \frac{i+c}{m+d}(h_3(y_w) - h_2(y_w))\right)\right| + 1\right)\cdot\omega(F_G(\cdot,y_w);\delta)
\end{aligned}
$$

one attains

$$
\begin{aligned}
|(R_{x_w,m}F_G)(x_w,y_w)| &\leq \sum_{i=0}^{m} s_{m,i}(x_w,y_w)\left(\frac{1}{\delta}\left|x_w-\left(h_2(y_w)+\frac{i+c}{m+d}(h_3(y_w)-h_2(y_w))\right)\right|+1\right) \\
&\times \omega(F_G(\cdot,y_w);\delta) \\
&\leq \left[1+\frac{1}{\delta}\left(\sum_{i=0}^{m} s_{m,i}(x_w,y_w)\left(x_w-\left(h_2(y_w)+\frac{i+c}{m+d}(h_3(y_w)-h_2(y_w))\right)\right)^2\right)^{1/2}\right] \\
&\times \omega(F_G(\cdot,y_w);\delta) \\
&= \left[1+\frac{1}{\delta}\sqrt{\frac{m(x_w-h_2(y_w))(h_3(y_w)-x_w)+(c-dx_w)(2mx_w+dx_w+c)}{(m+d)^2}}\right]\omega(F_G(\cdot,y_w);\delta) \\
&\leq \left[1+\frac{1}{\delta}\cdot\frac{1}{m+d}\sqrt{\frac{m(h_3(y_w)-h_2(y_w))^2}{4}+(c-dx)^2}\right]\omega(F_G(\cdot,y_w);\delta),
\end{aligned}
$$

where $0 \leq c \leq d$, with $\max\limits_{h_2(y_w)\leq x_w\leq h_3(y)}[(x_w-h_2(y_w))(h_3(y_w)-x_w) = \frac{(h_3(y_w)-h_2(y_w))^2}{4}$.

Thus,

$$
|(R_{x_w,m}F_G)(x_w,y_w)| \leq \left(1+\frac{(h_3(y_w)-h_2(y_w))\sqrt{m+4d^2}}{2\delta(m+d)}\right)\omega(F_G(\cdot,y_w);\delta), \quad y_w \in [0,w]
$$

and we obtain

$$
|(R_{x_w,m}F_G)(x_w,y_w)| \leq \left(1+\frac{(h_3(y_w)-h_2(y_w))\sqrt{m+4d^2}}{2(m+d)}\right)\omega\left(F_G(\cdot,y_w);\frac{\sqrt{m+4d^2}}{m+d}\right).
$$

□

Remark 2.4. *Similar conclusions are obtained for the remainder formula* $F_G = G^{(c,d)}_{y_w,n}F_G + R_{y_w,n}F_G$, $F_G(x_w,\cdot) \in C\left[g_1(x_w), g_3(x_w)\right]$, $x_w \in [0,w]$. *Then*

$$
|(R_{y_w,n}F_G)(x_w,y_w)| \leq \left(1+\frac{(g_3(x_w)-g_1(x_w))\sqrt{n+4d^2}}{2(n+d)}\right)\omega\left(F_G(x_w,\cdot);\frac{\sqrt{n+4d^2}}{n+d}\right).
$$

2.1. Product operator

Let P_{mn} and Q_{nm} denote the product operators defined as $P_{mn} = G^{(c,d)}_{x_w,m}G^{(c,d)}_{y_w,n}$ and $Q_{nm} = G^{(c,d)}_{y_w,n}G^{(c,d)}_{x_w,m}$.

$$
(P_{mn}F_G)(x_w,y_w) = \sum_{i=0}^{m}\sum_{j=0}^{n} s_{m,i}(x_w,y_w)s_{n,j}(x_i,y_w)\cdot F_G\left(x_i; g_1(x_i)+\frac{j+c}{n+d}(g_3(x_i)-g_1(x_i)\right),
$$

$$
(Q_{nm}F_G)(x_w,y_w) = \sum_{i=0}^{m}\sum_{j=0}^{n} s_{m,i}\left(x_w,y_j\right))s_{n,j}(x_w,y_w)\cdot F_G\left(h_2(y_j)+\frac{i+c}{m+d}(h_3(y_j)-h_2(y_j);y_j\right)
$$

with $x_i = h_2(y_w)+\dfrac{i+c}{m+d}(h_3(y_w)-h_2(y_w)$ and $y_j = g_1(x_w)+\dfrac{j+c}{n+d}(g_3(x_w)-g_1(x_w))$.

Theorem 2.5. *If* $F_G : GT_w \to GT_w$ *([1]) then*

(i) $(P_{mn}F_G)(C) = F_G(C)$, $P_{mn}F_G = F_G$ *on* Γ_3 ,

(ii) $(Q_{nm}F_G)(C) = F_G(C)\,, \quad Q_{mn}F_G = F_G \text{ on } \Gamma_3\,.$

Proof.

$$\begin{aligned}
&(P_{mn}F_G)(x_w, g_1(x_w)) = (G^{(c,d)}_{x_w,m}F_G)(x_w, g_1(x_w))\\
&(P_{mn}F_G)(g_2(y_w), y_w) = (G^{(c,d)}_{y_w,n}F_G)(h_2(y_w), y_w)\\
&(P_{mn}F_G)(x_w, g_3(x_w)) = F_G(x_w, g_3(x_w)), \quad x_w, y_w \in [0, w]
\end{aligned}$$

and

$$\begin{aligned}
&(Q_{nm}F_G)(x_w, g_1(x_w)) = (G^{(c,d)}_{x_w,m}F_G)(x_w, g_1(x_w))\\
&(Q_{nm}F_G)(h_2(y_w), y_w) = (G^{(c,d)}_{y_w,n}F_G)(h_2(y_w), y_w)\\
&(Q_{nm}F_G)(h_3(y_w), y_w) = F_G(h_3(y_w), y_w), \quad x_w, y_w \in [0, w]
\end{aligned}$$

are easy to verify by a straight forward calculation.

In remainders formula $F_G = P_{mn}F_G + R^P_{m,n}F_G, \quad R^P_{m,n}$ is the corresponding remainder operator. □

Theorem 2.6. *If* $F_G \in C\,(GT_w)$ *then*

$$|(R^P_{m,n}F_G)(x_w, y_w)| \le \left(1 + \frac{(h_3(y_w) - h_2(y_w))\sqrt{m + 4d^2}}{2(m + d)} + \frac{(g_3(x_w) - g_1(x_w))\sqrt{n + 4d^2}}{2(n + 4d)}\right)$$
$$\times\, \omega\left(F_G; \frac{\sqrt{m + 4d^2}}{m + d}; \frac{\sqrt{n + 4d^2}}{n + d}\right), \quad (x_w, y_w) \in GT_w.$$

Proof.

$$\begin{aligned}
|(R^P_{m,n}F_G)(x_w, y_w)| &\le [\frac{1}{\delta_1}\sum_{i=0}^{m}\sum_{j=0}^{n} s_{m,i}\,(x_w, y_w)\, s_{n,j}\,(x_i, y_w)\cdot |x_w - x_i|\\
&+ \frac{1}{\delta_2}\sum_{i=0}^{m}\sum_{j=0}^{n} s_{m,i}(x_w, y_w) s_{n,j}\,(x_i, y_w)\cdot \left|y_w - \left(g_1(x_i) + \frac{j + c}{n + d}(g_3(x_i) - g_1(x_i))\right)\right|\\
&+ \sum_{i=0}^{m}\sum_{j=0}^{n} s_{m,i}\,(x_w, y_w)\, s_{n,j}\,(x_i, y_w)]\cdot \omega(F_G; \delta_1; \delta_2)\\
&\le \left[1 + \frac{1}{\delta_1}\sqrt{\frac{m(x_w - h_2(y_w))(h_3(y_w) - x_w) + (c - dx_w)(2mx_w + dx_w + c)}{(m + d)^2}}\right.\\
&\left. + \frac{1}{\delta_2}\sqrt{\frac{n(y_w - g_1(x))(g_3(x) - y_w) + (c - dy_w)(2ny_w + dy_w + c)}{(n + d)^2}}\right]\omega(F_G; \delta_1; \delta_2).
\end{aligned}$$

But

$$m(x_w - h_2(y_w))(h_3(y_w) - x_w) + (c - dx_w)(2mx_w + dx_w + c) \le \frac{w^2(m + 4d^2)}{4}(h_3(y_w) - h_2(y_w)),$$

$$n(y_w - g_1(x_w))(g_3(x_w) - y_w) + (c - dy_w)(2ny_w + dy_w + c) \le \frac{w^2(n + 4d^2)}{4}(g_3(x_w) - g_1(x_w)),$$

$$|(R^P_{m,n}F_G)(x_w, y_w)| \leq \left[1 + \frac{1}{\delta_1}\frac{(h_3(y_w) - h_2(y_w))\sqrt{m + 4d^2}}{2(m + d)} + \frac{1}{\delta_2}\frac{(g_3(x_w) - g_1(x_w))\sqrt{n + d^2}}{2(n + d)}\right]$$
$$\times\, \omega(F_G; \delta_1; \delta_2), \delta_1 = \frac{\sqrt{m + 4d^2}}{m + d}, \delta_2 = \frac{\sqrt{n + 4d^2}}{n + d}$$
$$|(R^P_{m,n}F_G)(x_w, y_w)| \leq \left(1 + \frac{(h_3(y_w) - h_2(y_w))\sqrt{m + 4d^2}}{2(m + d)} + \frac{(g_3(x_w) - g_1(x_w))\sqrt{n + d^2}}{2(n + d)}\right)$$
$$\times\, \omega\left(F_G; \frac{\sqrt{m + 4d^2}}{m + d}; \frac{\sqrt{n + 4d^2}}{n + d}\right).$$

□

Remark 2.7. *Similar conclusions are obtained for the remainder of the formula*

$$F_G = Q_{n,m}F_G + R^Q_{n,m}F_G.$$

Now we get

$$|(R^P_{n,m}F_G)(x_w, y_w)| \leq \left(1 + \frac{(h_3(y_w) - h_2(y_w))\sqrt{m + 4d^2}}{2(m + d)} + \frac{(g_3(x_w) - g_1(x_w))\sqrt{n + d^2}}{2(n + d)}\right)$$
$$\times\, \omega\left(F_G; \frac{\sqrt{m + 4d^2}}{m + d}; \frac{\sqrt{n + 4d^2}}{n + d}\right).$$

2.2. Boolean sum operators

Boolean sum operators $G^{(c,d)}_{x_w,m}$ and $G^{(c,d)}_{y_w,n}$ are defined by,

$$G^{(c,d)}_{mn} = G^{(c,d)}_{x_w,m} \oplus G^{(c,d)}_{y_w,n} = G^{(c,d)}_{x_w,m} + G^{(c,d)}_{y_w,n} - G^{(c,d)}_{x_w,m}G^{(c,d)}_{y_w,n}$$

respectively

$$T^{(c,d)}_{nm} = G^{(c,d)}_{y_w,n} \oplus G^{(c,d)}_{x_w,m} = G^{(c,d)}_{y_w,n} + G^{(c,d)}_{x_w,m} - G^{(c,d)}_{y_w,n}G^{(c,d)}_{x_w,m}.$$

Theorem 2.8. *If $F_G : GT_w \to GT_w$, then*

$$G_{m,n}F_G|_{\partial GT_w} = F_G|_{\partial GT_w} \text{ and } T_{n,m}F_G|_{\partial GT_w} = F_G|_{\partial GT_w}.$$

Proof. As $(P_{mn}F_G)(x_w, g_1(x_w)) = (G^{(c,d)}_{x_w,m}F_G)(x_w, g_1(x_w))$,
$(P_{mn}F_G)(h_2(y_w), y_w) = (G^{(c,d)}_{y_w,n}F_G)(h_2(y_w), y_w)$ and $(P_{mn}F_G)(x_w, g_3(x_w)) = F_G(x_w, g_3(x_w))$,
the conclusion follows. □

Thus, remainder of the boolean sum estimation formula are defined as follows

$$F_G = G_{mn}F_G + R^G_{m,n}F_G.$$

Theorem 2.9. *If $F_G \in C(GT_w)$, then*

$$|(R^G_{m,n}F_G)(x_w, y_w)| \leq \left[1 + \frac{(h_3(y_w) - h_2(y_w))\sqrt{m + 4d^2}}{2}\right]\omega\left(F_G(\cdot, y_w); \frac{\sqrt{m + 4d^2}}{m + d}\right)$$
$$+ \left[1 + \frac{(g_3(x_w) - g_1(x_w))\sqrt{n + 4d^2}}{2}\right]\omega\left(F_S(x_w, \cdot); \frac{\sqrt{n + 4d^2}}{n + d}\right)$$
$$+ \left(1 + \frac{(h_3(y_w) - h_2(y_w))(g_3(x_w) - g_1(x_w))}{2}\right)\omega\left(F_G; \frac{\sqrt{m + 4d^2}}{m + d}; \frac{\sqrt{n + 4d^2}}{n + d}\right),$$
$$(x_w, y_w) \in GT_w.$$

Proof. $F_G - G_{mn}F_G = F_G - G^{(c,d)}_{x_w,m}F_G + F_G - G^{(c,d)}_{y_w,n}F_G - (F_G - P_{mn}F_G)$ implies that

$$|(R^G_{m,n}F_G)(x_w, y_w)| \leq |(R_{x_w,m}F_G)(x_w, y_w)| + |(R_{y_w,n}F_G)(x_w, y_w)| + |(R^P_{m,n}F_G)(x_w, y_w)|$$

and the conclusion follows. An equivalent form of inequity can be derived in regards to the $R^T_{m,n}F_G$ error.

Boolean sum operators, often referred to as logical OR operators, are used to combine Boolean values (true or false) in logic and computer programming. They allow you to express conditions where at least one of the operands needs to be true for the entire expression to be true. When considering the use of Boolean sum operators, it is important to note the disparity between their application on a triangular domain versus their use in closed intervals $[a, b]$. The Boolean sum operators themselves do not differ in their fundamental logic when applied to a triangular domain versus a closed interval. The difference lies in how the conditions are defined and how they relate to the specific context of the domain. Triangular domains involve spatial relationships and geometric regions, while closed intervals involve numerical values and their ranges. □

3. Graphical Analysis

We take the functions [21] for graphical analysis which are usually used in the literature:

$$GENTLE : H_1(u, v) = \frac{1}{3} exp\left[\frac{-81}{16}((u - 0.5)^2 + (v - 0.5))^2\right],$$

$$SADDLE : H_2(u, v) = \frac{125 + \cos(5.4v)}{6 + 6 * (3u - 1)^2},$$

We have taken the triangular region $\{(u, v) : u + v \leq 1, u, v \geq 0\}$ as domain of the functions H_1 and H_2. We have used Python 3 to plot some graphics for operator convergence in Figure 2 for H_2 and in Figure 3 for H_1 [21], considering $w = 1,\ n = 5,\ n = 6,\ c = 1,\ \ d = 1$ and $g_1, h_2, g_3, h_3 : [0, 1] \to [0, 1]$ defined by

$$g_1(u) = -\frac{\sqrt{15}}{2} - \sqrt{4 - (u - 0.5)^2},$$
$$h_2(v) = -\frac{\sqrt{15}}{2} - \sqrt{4 - (v - 0.5)^2},$$
$$g_3(u) = \sqrt{1 - u^2},$$
$$h_3(u) = \sqrt{1 - v^2}.$$

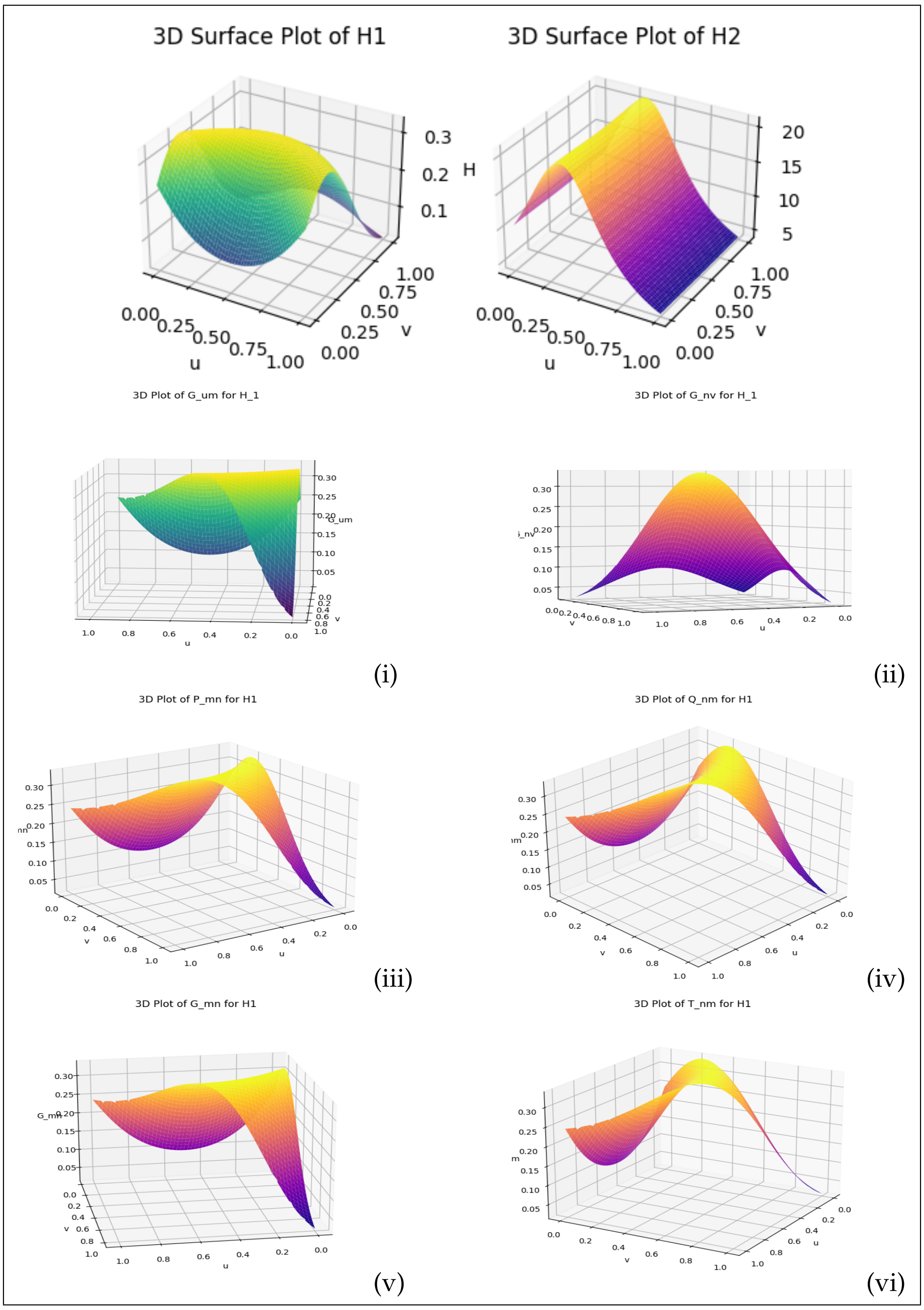

Figure 2: Graphs of H_1 and H_2(i), $G_{um}H_1$ (ii), $G_{nv}H_1$ (iii), $P_{mn}H_1$(iv), $Q_{nm}H_1$(v), $G_{mn}H_1$(vi), $T_{nm}H_1$(vii)

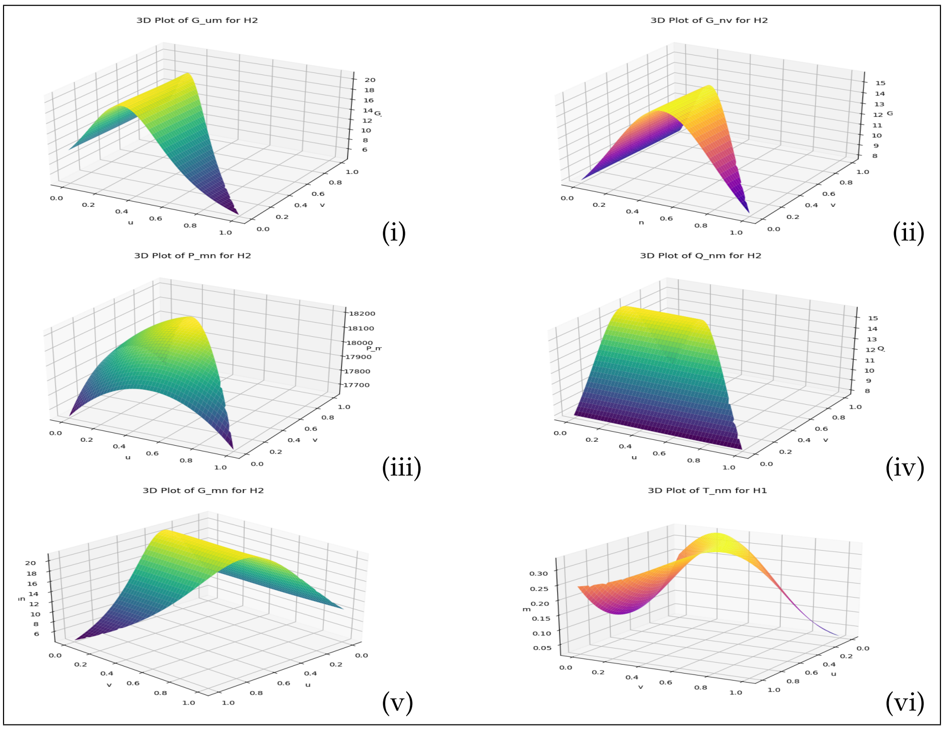

Figure 3: Graphs of $G_{um}H_2$(i), $G_{nv}H_2$(ii), $P_{mn}H_2$(iii), $Q_{nm}H_2$(iv), $G_{mn}H_2$(v), $T_{nm}H_2$(vi)

The Maple software was utilized to create a table of approximations with the highest allowable amount of error.

Table 1: The approximation error.

$Max\quad error$	H_1	H_2
$G_{u,m}^{(1,1)}$	0.0872	0.1114
$G_{n,v}^{(1,1)}$	0.0617	0.0925
$P_{mn}^{(1,1)}$	0.1479	0.1576
$Q_{nm}^{(1,1)}$	0.1346	0.1443
$G_{mn}^{(1,1)}$	0.0383	0.0478
$T_{nm}^{(1,1)}$	0.0331	0.0356

4. Conclusion

The approximation operators on triangles have been the subject of extensive research due to their applications in the fields like computer-aided geometric design (CAGD) and finite element analysis. In this work, we have constructed some Stancu-type operators, their products and boolean sum operators on a triangular domain with curved sides and described their interpolation features. Further, we have discussed remainders of approximation formula. Graphical representations have been given to demonstrate the theoretical findings. We observed with the aid of figures and a table that displays errors that the approximation properties are quite satisfactory. As the selection parameters m and n are expanded, the operators $G_{x_w,m}^{(c,d)}$ and $G_{y_w,n}^{(c,d)}$ are able to provide a close approximation of the function.

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References

[1] Barnhill R.E., Birkhoff G., Gordon W.J., Smooth interpolation in triangles. J. Approx. Theory **8**, 114–128 (1973).
[2] Barnhill R.E., Gregory J.A., Compatible smooth interpolation in triangles. J. Approx. Theory **15**, 214–225 (1975).
[3] Barnhill R.E., Gregory J.A., Sard kernels theorems on triangular domains with applications to finite element error bounds. Numer. Math. **25**, 215–229 (1976).
[4] Blaga P., Cătinaş T., Coman G., Bernstein-type operators on triangle with all curved sides. Appl. Math. Comput. **218**, 3072–3082 (2011).
[5] Cătinaş T., Coman G., Interpolation operators on a triangle with one curved side. BIT Numerical Mathematics **50**(2), 243–267 (2010).
[6] Cheregi L., Interpolation of Stancu-type operators on a triangle with one curved side. ROMAI J. **18**(1), 17–30(2022).
[7] Caglar H., Caglar N., Elfaituri K., B-spline interpolation compared with finite difference, finite element and finite volume methods which applied to two-point boundary value problems. J. Comput. Appl. Math. **139**(1), 9–19(2002).
[8] Fang Q., Tsuchiya T., Yamamoto T., Finite difference, finite element and finite volume methods applied to two-point boundary value problems. Journal of Computational and Applied Mathematics**139** (1), 9–19(2002).
[9] Finta Z., On approximation properties of Stancu's operators. Stud. Univ. Babeş-Bolyai, Math **47**(4) (2002).
[10] Iliyas M., Khan A., Arif M., Mursaleen M., Lone M.R., Lupaş type Bernstein operators on triangle with one curve side. J. Analysis, DOI: 10.1007/s41478-023-00561-y.
[11] Kalyani P., Rao P.S.R.C., Solution of Boundary Value Problems by Approaching Spline Techniques. Int. J. Eng. Math.(2013), DOI: http://dx.doi.org/10.1155/2013/482050.
[12] Koubaiti O., Fakkoussi S.E. , El-Mekkaoui J., Moustachir H., Elkhalfi A., Pruncu C.I., The treatment of constraints due to standard boundary conditions in the context of the mixed Web-spline finite element method. Engineering Computations(2021) **38**(7), 2937-2968(2021).
[13] Mansoori M.S., Qasim M., Khan A., Mursaleen M., Abbas Z., Iterates of q-Bernstein operators on triangular domain with all curved sides. Demonstratio Mathematica **55**, 891–899 (2022).
[14] Marshall J.A., McLeod R., Curved elements in the finite element method. Conference on Numer. Sol. Diff. Eq., Lectures Notes in Math., Springer Verlag **363**, 89–104 (1974).
[15] Mohammadi R., Numerical solution of general singular perturbation boundary value problems based on adaptive cubic spline. TWMS J. Pure Appl. Math. **3**, No. 1, 176–193(2012).
[16] Mohammadi R. , Alavi A.S., Quadratic spline solution of calculus of variation problems. TWMS J. App. Eng. Math. **5**(2), 276–285(2015).
[17] Rashidinia J., Ghasemi M., B-spline collocation for solution of two-point boundary value problems. J. Comput. Appl. Math **235**, 2325–2342(2011).
[18] Rashidinia J., Mohammadi R., Jalilian R., Cubic spline method for two-point boundary value problems. IUST International Journal of Engineering Science **19**(5-2), 39–43(2008).
[19] Moroşanu C., Cubic spline method and fractional steps scheme to approximate the phase-field system with non-homogeneous Cauchy-Neumann boundary conditions. ROMAI J. **8**(1), 73–91 (2012).
[20] Raiz M., Rao N.,Mishra V. N., Szász-Type Operators Involving q-Appell Polynomials, Book Chapter entitled Approximation Theory, Sequence Spaces and Applications. Springer,(2022), doi:10.1007/978-981-19-6116-810.
[21] Renka R.L., Cline A.K., A triangle-based C^1 interpolation method. Rocky Mt. J. Math **14**, 223-–237 (1984).
[22] Rus I., Iterates of Stancu operators (via fixed point principles) revisited. Fixed Point Theory **11**(2), 369–374 (2010).
[23] Stancu, D.D., Asupra unei generalizări a polinoamelor lui Bernstein. Studia Universitatis Babes-Bolyai **14**(2), 31–45 (1969).
[24] Stancu, D.D., Approximation properties of a class of multiparameter positive linear operators. Approximation and Optimization Transilvania Press, Cluj-Napoca **1**, 107–120 (1996).
[25] Stancu, D.D., Approximation of function by a new class of polynomial operators. Rev. Roum. Math. Pures et Appl. **13**(8), 1173–1194 (1968).
[26] Stancu, D.D., On the remainder of approximation of functions by means of a parameter-dependent linear polynomial operator. Studia Univ. Babes-Bolyai, Ser. Math.-Mech **16**(2), 59–66 (1971).

[27] Stancu D.D., Coman G., Agratini O., Trâmbiţaş: R. Analiză numerică şi teoria aproximării , Presa Universitară Clujeană, Cluj-Napoca **1**(2001).

[28] Tayebi S., Momani S., Arqub O.A., The cubic B-spline interpolation method for numerical point solutions of conformable boundary value problems. Alex. Eng. J. **61** (2), 1519–1528(2022).

[29] Mishra V. N., Raiz M., Rao N., Dunkl analogue of Szász Schurer Beta bivariate operators. Mathematical Foundations of Computing, **6**(4), 651–669(2023) doi: 10.3934/mfc.2022037.

[30] Raiz M., Kumar A., Mishra V. N., Rao N., Dunkl analogue of Szász Beta bivariate operators and their approximating behaviour. Mathematical Foundations of Computing, **5**(4), 315–330(2023) doi:10.3934/mfc.2022007.

[31] Ware A.N., Ashine A.B., Cubic spline and finite difference method for solving boundary value problems of ordinary differential equation. Appl. Math. Comput. **175**(1), 72–79 (2006).

[32] Zin S.M., Majid A.A., Ismail A.I.Md., Abbas M., Application of hybrid cubic b-Spline collocation approach for solving a generalized nonlinear klien-gordon equation. Math. Probl. Eng. (2014), DOI: https://doi.org/10.1155/2014/108560.