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On the behavior of derivatives of algebraic polynomials in regions with piecewise quasismooth boundary having cusps

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Abstract. In this paper, we study the growth for the m-th derivatives of an arbitrary algebraic polynomial in bounded and unbounded regions with piecewise quasismooth boundary having interior and exterior zero angles in weighted Lebesgue spaces.

1. Introduction

Let $\mathbb{C}$ be a complex plane and $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$; $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ (without loss of generality, let $0 \in G$); $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. For $t \in \mathbb{C}$ and $\delta > 0$, let $\Delta(t, \delta) := \{w \in \mathbb{C} : |w - t| > \delta\}$; $\Delta := \Delta(0, 1)$. Let $\Phi : \Omega \to \Delta$ be the univalent conformal mapping normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$.

For $t \geq 1$, let us set:

$$L_t := \{z : \ |\Phi(z)| = t\}, \ L_1 \equiv L, \ G_t := intL_t, \ \Omega_t := extL_t.$$

For $z \in \mathbb{C}$ and some set $S \subset \mathbb{C}$ let

$$d(z, S) := dist(z, S) = \inf \{|\zeta - z| : \ \zeta \in S\}.$$

Let $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$.

Let $\left\{z_j\right\}_{j=1}^{l} \in L$ be the fixed system of distinct points. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0}$, consider generalized Jacobi weight function $h(z)$:

$$h(z) := h_0(z) \prod_{j=1}^{l} \left|z - z_j\right|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., l$, $z \in G_{R_0}$ and $h_0(z) \geq c_0(L) > 0$ for some constant $c_0(L) > 0$.

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For each $0 < p \leq \infty$ and rectifiable Jordan curve $L = \partial G$, we denote

$$\|P_n\|_p \quad : \quad = \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left| P_n(z) \right|^p |dz| \right)^{1/p} < \infty, 0 < p < \infty, \tag{2}$$

$$\|P_n\|_\infty \quad : \quad = \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z \in L} |P_n(z)|, \ p = \infty; \ \mathcal{L}_p(1,L) =: \mathcal{L}_p(L).$$

It is well known that in the theory of approximation of a function of a complex variable, the following Bernstein-Walsh inequality is often used [45]:

$$\|P_n\|_{C(\overline{G}_R)} := \max_{z \in \overline{G}_R} |P_n(z)| \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \ \forall P_n \in \wp_n. \tag{3}$$

An analogue of this inequality in space $\mathcal{L}_p(h, L)$ is the following inequality[32]:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq R^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \ \forall P_n \in \wp_n, \ p > 0.$$

This estimate has been generalized in [9, Lemma 2.4] for weight function $h(z) \neq 1$, defined as in (1), as follows:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\left\{0; \gamma_j : 1 \leq j \leq l\right\}. \tag{4}$$

If we consider the two-dimensional analogues of the quantities (2), i.e., integral over the region G, (we denote them by $\|P_n\|_{A_p(h,G)}$, $\|P_n\|_{A_p(1,G)}$ and $A_p(G)$ respectively), then you can also specify the corresponding estimate of the form (4) for them. To do this, we give the following definition.

Following to [33, pp.100] (see, also, [40]), the curve L is called K -quasiconformal, ($K \geq 1$), (or $\kappa-$quasicircle, $0 \leq \kappa < 1$), if it is a image of the unit circle under the K -quasiconformal ($K \geq 1$) mapping of the plane $\overline{\mathbb{C}}$ onto $\overline{\mathbb{C}}$. The curve L is called quasiconformal (quasicircle), if is it K -quasiconformal ($\kappa-$quasicircle) for some $K \geq 1$ ($0 \leq \kappa < 1$). Note that quasicircles can be non-rectifiable (see, for example, [29], [33, p.104]).

In [2] the analogue of the inequalities (3) and (4) for arbitrary regions with $K-$quasiconformal boundary and weight function $h(z)$ defined as (1) is given by

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 \, {R^*}^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \ R > 1, \ p > 0, \tag{5}$$

where $R^* := 1 + c_2(R - 1)$, $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent of n and R. Moreover, this estimate was generalized for arbitrary Jordan region G and $P_n \in \wp_n$ as follows [6, Theorem 1.1]:

$$\|P_n\|_{A_p(G_R)} \leq c_3 R^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \ R > R_1 = 1 + \frac{1}{n}, \ p > 0,$$

where $c_3 = \left(\frac{2}{e^p-1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$, is asymptotically sharp constant.

In [43] for the regions with a rectifiable quasiconformal curve a new version of the Bernstein-Walsh lemma is found as follows:

$$|P_n(z)| \leq c(L) \frac{\sqrt{n}}{d(z,L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \ \ z \in \Omega,$$

where $c(L) > 0$ is a constant depending only of L.

Let $S \subset \mathbb{C}$ be rectifiable Jordan curve or arc and let $z = z(s)$, $s \in [0, |S|]$, $|S| := mes\, S$, be the natural parametrization of S. Let z_1, z_2 be an arbitrary points on S and $l(z_1, z_2)$ denotes the subarc of S of shorter diameter with endpoints z_1 and z_2 (including the endpoints). Following [41, p.163], we say that a bounded

Jordan curve S is $\lambda-$quasismooth (in the sense of Lavrentiev) curve, if for every pair $z_1,\ z_2 \in S$, there exists a constant $\lambda := \lambda(S) \geq 1$, such that

$$|l(z_1, z_2)| \leq \lambda\, |z_1 - z_2|\,,\ z_1,\ z_2 \in S,$$

where $|l(z_1, z_2)|$ is the linear measure (length) of $l(z_1, z_2)$.

In [11] (for $m = 0$ and L is $\lambda-$quasismooth curve) and [21] (for $m \geq 0$ and for more general class of curves, contained also $\lambda-$quasismooth curves) the authors investigated the problems on uniform and pointwise estimates for the $\left|P_n^{(m)}(z)\right|$, $m \geq 0$, in $\overline{G}$ and Ω, and the estimation was obtained by

$$\left|P_n^{(m)}(z)\right| \leq c_4\, \|P_n\|_p \begin{cases} \nu_n, & z \in \overline{G}, \\ \eta_n, & z \in \Omega, \end{cases} \tag{6}$$

where $c_4 = c_4(L, p, m, \gamma) > 0$ is a constant independent of n, h, P_n, $\nu_n = \nu_n(L, h, p) > 0$ and $\eta_n = \eta_n(L, h, p, z) \to \infty$, as $n \to \infty$, are a constants depending on the properties of the L and h.

Note that, $\lambda-$quasismooth curves doesn't have any cusps, i.e. interior and exterior zero angles with respect to $\overline{G}$.

Similar results of (6)-type in different spaces for $m = 0$ and for different weight function h, unbounded regions ($z \in \Omega$) were studied in [5], [7]-[16], [17], [19], [20], [31, p.418-428], [34], [38], [39], [43] and others.

Estimates of the (6)-type for bounded regions ($z \in \overline{G}$), for the norms $\|P_n\|_{\mathcal{L}_p(h,L)}$ or $\|P_n\|_{A_p(h,G)}$, $p > 0$, for some h ($h(z) \equiv 1$ or $h(z) \neq 1$) was studied in [28], [30], [44]), [2]-[4], [18], [25], [26] [27], [31, pp. 418-428], [34], [35, Sect. 5.3], [36], [37, pp.122-133], [38], [39], [42] (see also the references cited therein) and others. In this work we continue to study the similar problem for regions with cusps.

2. Definitions and main results

Throughout this paper, $c, c_0, c_1, c_2, ...$ are positive and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive constants (generally, different in different relations), which depends on L in general and, on parameters inessential for the argument, otherwise, the dependence will be explicitly stated. For any $k \geq 0$ and $m > k$, notation $i = \overline{k, m}$ means $i = k, k+1, ..., m$.

The region G is called a $\lambda-$quasismooth region, if $L = \partial G$ is a $\lambda-$quasismooth curve. Any subarc of a $\lambda-$quasismooth curve is called a $\lambda-$quasismooth arc. We denote this class of curves and arcs as $QS(\lambda)$ and say that a Jordan region $G \in QS(\lambda)$, if $\partial G \in QS(\lambda)$, $\lambda \geq 1$. Furthermore, we denote that L (or G) $\in QS$, if L (or G) $\in QS(\lambda)$ for some $\lambda \geq 1$.

We say that a bounded Jordan curve or arc L is *locally $\lambda-$quasismooth* at the point $z \in L$, if there exists a closed subarc $\ell \subset L$ containing z such that every open subarc of the ℓ containing z is the $\lambda-$quasismooth. According to the "three-point" criterion [33, p.100], [22] every quasismooth curve are quasiconformal.

Now, we will give a new class of regions with piecewise quasismooth boundary, which may have at the boundary points finite number of interior and exterior cusps with respect to the given region.

For any $j = 1, 2, ...$ and sufficiently small $\varepsilon_1 > 0$, we denote by f_j, $g_j : [0, \varepsilon_1] \to \mathbb{R}$ twice continuously differentiable functions such that $f_j(0) = g_j(0) = 0$ and $f_j^{(k)}(x) > 0$, $g_j^{(k)}(x) > 0$, for $x > 0$ and $k = 0, 1, 2$.

Definition 2.1. ([11] We say that a Jordan region $G \in PQS\left(\lambda\,; f_i,\ g_j\right)$, $\lambda \geq 1$, $f_i = f_i(x)$, $i = \overline{1, l_1}$, $g_j = g_j(x)$, $j = \overline{l_1 + 1, l}$, if $L := \partial G = \bigcup\limits_{j=0}^{l} L_j$ is the union of the finite number of λ-quasismooth arcs L_j, connecting at the points $\left\{z_j\right\}_{j=0}^{l} \in L$, and such that L is a locally λ-quasismooth arc at the $z_0 \in L \backslash \left\{z_j\right\}_{j=1}^{l}$ and, in the (x, y) local coordinate system with its origin at the z_j, $1 \leq j \leq l$, the following conditions are satisfied:

a) for every $z_j \in L$, $j = \overline{1, l_1}$, $l_1 \leq l$,

$$\left\{z = x + iy : |z| \leq \varepsilon_1,\ c_1 f_i(x) \leq y \leq c_2 f_i(x)\right\} \subset \overline{G},$$
$$\left\{z = x + iy : |z| \leq \varepsilon_1,\ \left|y\right| \geq \varepsilon_2 x\right\} \subset \overline{\Omega};$$

b) for every $z_j \in L,\ j=\overline{l_1+1,l}$,

$$\left\{z=x+iy:|z|<\varepsilon_3,\ \ c_3 g_j(x)\le y\le c_3 g_j(x),\ 0\le x\le\varepsilon_3\right\}\ \subset\ \overline{\Omega},$$
$$\left\{z=x+iy:|z|<\varepsilon_3,\ \ |y|\ge\varepsilon_3 x,\ 0\le x\le\varepsilon_3\right\}\ \subset\ \overline{G},$$

for some constants $-\infty<c_1<c_2<\infty$, $-\infty<c_3<c_4<\infty$, $\varepsilon_i>0$, $i=\overline{1,4}$.

It is clear from Definition 2.1 that each region $G\in PQS\left(\lambda;f_i,g_j\right)$ may have l_1 interior and $l-l_1$ exterior zero angles (with respect to $\overline{G}$). If a region G does not have interior zero angles ($l_1=0$) (exterior zero angles ($l_1=l$)), then it is written as $G\in PQS\left(\lambda;0,g_j\right)$ ($G\in PQS\left(\lambda;f_i,0\right)$). If a domain G does not have such angles ($l=0$), then G is bounded by a $\lambda-$ quasismooth circle and in this case we set $PQS\left(\lambda,0,0\right)\equiv QS(\lambda)$.

For $L=\partial G$ and $0<\delta_j<\delta_0:=\frac{1}{4}\min\left\{\left|z_i-z_j\right|:i,j=\overline{1,l},\ i\neq j\right\}$, we set: $U_\infty(L,\delta):=\bigcup\limits_{\zeta\in L}U(\zeta,\ \delta)-$infinite open cover of the curve L; $U_N(L,\delta):=\bigcup\limits_{j=1}^{N}U_j(L,\delta)\subset U_\infty(L,\delta)-$finite open cover of the curve L; $\Omega(z_j,\ \delta_j):=\Omega\cap\left\{z:\left|z-z_j\right|\le\delta_j\right\}$; $\delta:=\min\limits_{1\le j\le l}\delta_j$; $\Omega(\delta):=\bigcup\limits_{j=1}^{l}\Omega(z_j,\ \delta)$, $\widehat{\Omega}(\delta):=\Omega\setminus\Omega(\delta)$.

Throughout this paper, we denote by

$$\gamma^*:=\max\{0;\gamma_k,\ k=\overline{1,l}\};\ \mu:=2(1-\frac{1}{\pi}\arcsin\frac{1}{\lambda}),1<\mu<2,\ \widetilde{\mu}:=\begin{cases}\mu, & \text{if } \alpha_i=0,\\ 2, & \text{if } \alpha_i\neq 0,\end{cases}\ \ i=\overline{1,l_1} \tag{7}$$

Now, we start to formulate the new results. Firstly we give recurrent estimate for $\left|P_n^{(m)}(z)\right|$, $m=1,2,...$

Theorem 2.2. *Let $p>1$; $G\in PQS\left(\lambda;f_i,g_i\right)$, for some $\lambda\ge 1$, $f_i(x)=C_i x^{1+\alpha_i}$, $\alpha_i\ge 0$, $i=\overline{1,l_1}$, and $g_i(x)=C_i x^{1+\beta_i}$, $\beta_i>0$, $i=\overline{l_1+1,l}$; $h(z)$ be defined as in (1). Then, for any $\gamma_i>-1$, $i=\overline{1,l}$, $P_n\in\wp_n$, $n\in\mathbb{N}$, and each $m=1,2,...$ the following inequality holds:*

$$\left|P_n^{(m)}(z)\right|\le c_1\ \left|\Phi^{n+1}(z)\right|\left\{\frac{\|P_n\|_p}{d(z,L)}A^1_{n,p}(z,m)+\sum_{j=1}^{m}C_m^j B^1_{n,j}(z)\left|P_n^{(m-j)}(z)\right|\right\}, \tag{8}$$

where $C_m^j:=\frac{m(m-1)...(m-j+1)}{j!}$ and $c_1=c_1(L,\gamma_i,\beta,m,p)>0$ is a constant independent of n and z;

$$A^1_{n,p}(z,m)\ \ :\ \ =\sum_{i=1}^{l_1}n^{\left(\frac{\gamma_i^*+1}{p}+m-1\right)\widetilde{\mu}}+\sum_{i=l_1+1}^{l}n^{\left(\frac{\gamma_i^*+1}{p}+m-1\right)\frac{\mu}{1+\beta_i}},$$
$$B^1_{n,j}(z,l)\ \ :\ \ =n^{j\widetilde{\mu}}+\sum_{i=l_1+1}^{l}n^{\frac{j\mu}{1+\beta_i}},\ j=\overline{1,m},$$

if $z\in\Omega(\delta)$,

$$A^1_{n,p}(z,m)\ \ :\ \ =\begin{cases}\sum\limits_{i=1}^{l_1}n^{\left(\frac{\gamma_1^*+1}{p}-1\right)\widetilde{\mu}}+\sum\limits_{i=l_1+1}^{l}n^{\left(\frac{\gamma_2^*+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & \gamma_1,\gamma_2>p-1,\\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases}\gamma_1=p-1,\gamma_2\le p-1\\ \textit{or}\ \gamma_1\le p-1,\gamma_2=p-1,\end{cases}\\ 1, & -1<\gamma_1,\gamma_2<p-1,\end{cases}$$
$$B^1_{n,j}(z)\ \ =\ \ 1,\ j=\overline{1,m},$$

if $z\in\widehat{\Omega}(\delta)$.

Now, for simplicity of our presentations, we assume that: $i = 1, 2$; $l_1 = 1$, $l = 2$; i.e. our region G may have one interior zero (or nonzero) angle having " f_1−touching" with $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, at the point z_1 and exterior zero angle having "g_2−touching" with $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$, at the point z_2, for some constants $-\infty < C_1 < +\infty$, $-\infty < C_2 < +\infty$, where $C_1 := C_1(c_1, c_2)$, $C_2 := C_2(c_3, c_4)$ and a constants c_i, $i = \overline{1,4}$, taken from Definition 2.1. In this case, combining the terms relating to the inner and outer corners, we obtain the following result.

Corollary 2.3. *Let $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, and $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ defined as in (1) for $l = 2$. Then, for any $\gamma_i > -1$, $i = 1, 2$, and $P_n \in \wp_n$, $n \in \mathbb{N}$, the following inequality holds:*

$$\left|P_n^{(m)}(z)\right| \leq c_2 \left|\Phi^{n+1}(z)\right| \left\{ \frac{\|P_n\|_p}{d(z,L)} A_{n,p}^2(z,m) + \sum_{j=1}^{m} C_m^j B_{n,j}^2(z) \left|P_n^{(m-j)}(z)\right| \right\}, \tag{9}$$

where $c_2 = c_2(L, \gamma_i, \beta, m, p) > 0$ is a constant independent of n and z;

$$A_{n,p}^2(z,m) \; : \; = \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+m-1\right)\widetilde{\mu}}, & \gamma_1 \geq \frac{\mu\left[(\gamma_2+1)+p(m-1)\right]}{\widetilde{\mu}(1+\beta_2)} - p(m-1) - 1, \quad \gamma_2 > 0. \\ n^{\left(\frac{\gamma_2+1}{p}+m-1\right)\frac{\mu}{1+\beta_2}}, & 0 < \gamma_1 < \frac{\mu\left[(\gamma_2+1)+p(m-1)\right]}{\widetilde{\mu}(1+\beta_2)} - p(m-1) - 1, \quad \gamma_2 > 0. \\ n^{\left(\frac{1}{p}+m-1\right)\widetilde{\mu}}, & -1 < \gamma_1 \leq 0, \quad -1 < \gamma_2 \leq 0, \end{cases}$$

$$B_{n,j}^2(z) \; : \; = n^{j\widetilde{\mu}} + n^{\frac{j\mu}{1+\beta_2}}, \; j = \overline{1,m},$$

if $z \in \Omega(\delta)$,

$$A_{n,p}^2(z,m) \; : \; = \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}} & \begin{cases} \gamma_1 \geq \frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)} + p - 1, \; \gamma_2 > p-1, \\ or \; \gamma_1 > p-1, \; -1 < \gamma_2 \leq p-1 \end{cases} \\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}} & -1 < \gamma_1 < \frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)} + p - 1, \gamma_2 > p-1. \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \gamma_1 = p-1, \gamma_2 \leq p-1 \\ or \; \gamma_1 \leq p-1, \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases}$$

$$B_{n,j}^2(z) \; = \; 1, \; j = \overline{1,m},$$

if $z \in \widehat{\Omega}(\delta)$.

The formula (8) allows one to sequentially obtain an evaluation for $\left|P_n^{(m)}(z)\right|$, for each $m \geq 1$. First, setting $m = 1$ and using $|P_n(z)|$ we obtain an evaluation for $\left|P_n'(z)\right|$. For $m \geq 2$, calculations are made sequentially by applying (8) (or (9)).

First, let us give the evaluation for $|P_n(z)|$.

Theorem 2.4. *Let $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, for some $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ and $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ be defined as in (1) for $l = 2$. Then, for any $\gamma_i > -1$, $i = 1, 2$, $P_n \in \wp_n$, $n \in \mathbb{N}$, and $z \in \Omega$, the following inequality holds:*

$$|P_n(z)| \leq c_3 A_{n,p}^3 \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} \|P_n\|_p, \tag{10}$$

where $c_3 = c_3(L, \gamma_i, \beta, p) > 0$ is a constant independent of n and z;

$$A_{n,p}^3 := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}, & \gamma_1 > \frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)} + p - 1, \; \gamma_2 > p-1, \\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{(1+\beta_2)}}, & p-1 < \gamma_1 \leq \frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)} + p - 1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \gamma_1 = p-1, -1 < \gamma_2 \leq p-1, \\ or \; -1 < \gamma_1 \leq p-1, \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases}$$

Note that, analogous result for $|P_n(z)|$, $p>0$, was obtained in [11]. In presenting this theorem for $p>1$, our goal is that here and in other inequalities for γ_1 and γ_2 we can choose the same intervals, which facilitate the calculations.

According to Corollary 2.3 and Theorem 2.4, we obtain an evaluation for $\left|P_n'(z)\right|$ at each point $z\in\Omega$.

Theorem 2.5. *Let $p>1$; $G\in PQS\left(\lambda;f_1,g_2\right)$, for some $\lambda\geq 1$, $f_1(x)=C_1x^{1+\alpha_1}$, $\alpha_1\geq 0$ and $g_2(x)=C_2x^{1+\beta_2}$, $\beta_2>0$; $h(z)$ be defined as in (1) for $l=2$. Then, for any $\gamma_i>-1$, $i=1,2$, $P_n\in\wp_n$, $n\in\mathbb{N}$, and $z\in\Omega$, the following inequality holds:*

$$\left|P_n'(z)\right|\leq c_4A_{n,p}^4(z)\frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)}\left\|P_n\right\|_p, \tag{11}$$

where $c_4=c_4(L,\gamma_i,\beta,p)>0$ is a constant independent of n and z;

$$A_{n,p}^4(z):=\begin{cases} n^{\frac{\gamma_1+1}{p}\widetilde{\mu}}, & \gamma_1>\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ n^{\widetilde{\mu}+(\frac{\gamma_2+1}{p}-1)\frac{\mu}{(1+\beta_2)}}, & p-1<\gamma_1\leq\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ n^{\widetilde{\mu}}\left(\ln n\right)^{1-\frac{1}{p}}, & 0<\gamma_1\leq p-1, & 0<\gamma_2\leq p-1,\\ n^{\widetilde{\mu}}, & -1<\gamma_1<p-1, & -1<\gamma_2<p-1,\end{cases}$$

if $z\in\Omega(\delta)$,

$$A_{n,p}^4(z):=\begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}, & \gamma_1\geq\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & p-1<\gamma_1<\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ \left(\ln n\right)^{1-\frac{1}{p}}, & -1<\gamma_1\leq p-1, & -1<\gamma_2\leq p-1,\\ 1, & -1<\gamma_1<p-1, & -1<\gamma_2<p-1,\end{cases}$$

if $z\in\widehat{\Omega}(\delta)$.

Considering Theorem 2.5 for $\left|P_n'(z)\right|$ and Theorem 2.4 for $|P_n(z)|$ in Corollary 2.3, we get the following result.

Theorem 2.6. *Let $p>1$; $G\in PQS\left(\lambda;f_1,g_2\right)$, for some $\lambda\geq 1$, $f_1(x)=C_1x^{1+\alpha_1}$, $\alpha_1\geq 0$ and $g_2(x)=C_2x^{1+\beta_2}$, $\beta_2>0$; $h(z)$ be defined as in (1) for $l=2$. Then, for any $\gamma_i>-1$, $i=1,2$, $P_n\in\wp_n$, $n\in\mathbb{N}$, and $z\in\Omega$, the following inequality holds:*

$$\left|P_n''(z)\right|\leq c_5A_{n,p}^5(z)\frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)}\left\|P_n\right\|_p, \tag{12}$$

where $c_5=c_5(L,\gamma_i,\beta,p)>0$ is a constant independent of n and z;

$$A_{n,p}^5(z):=\begin{cases} n^{\left(\frac{\gamma_1+1}{p}+1\right)\widetilde{\mu}}, & \gamma_1>\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ n^{2\widetilde{\mu}+(\frac{\gamma_2+1}{p}-1)\frac{\mu}{(1+\beta_2)}}, & p-1<\gamma_1\leq\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ n^{2\widetilde{\mu}}\left(\ln n\right)^{1-\frac{1}{p}}, & -1<\gamma_1\leq p-1, & -1<\gamma_2\leq p-1,\\ n^{2\widetilde{\mu}}, & -1<\gamma_1<p-1, & -1<\gamma_2<p-1,\end{cases}$$

if $z\in\Omega(\delta)$,

$$A_{n,p}^5(z):=\begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}, & \gamma_1\geq\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & p-1<\gamma_1<\frac{\mu\left[\gamma_2-(p-1)\right]}{\widetilde{\mu}(1+\beta_2)}+p-1, & \gamma_2>p-1,\\ \left(\ln n\right)^{1-\frac{1}{p}}, & -1<\gamma_1\leq p-1, & -1<\gamma_2\leq p-1,\\ 1, & -1<\gamma_1<p-1, & -1<\gamma_2<p-1,\end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Now, we can state evaluations for $\left|P_n^{(m)}(z)\right|$, $m \geq 0$.

Theorem 2.7. *Let* $p > 0$; $G \in PQS\left(\lambda; f_i, g_i\right)$, *for some* $\lambda \geq 1$, $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, l_1}$, *and* $g_i(x) = c_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{l_1+1, l}$; $h(z)$ *be defined as in (1). Then, for any* $\gamma_i > -1$, $i = \overline{1, l}$, *and* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_6 = c_6(G, p, \lambda, \gamma_i, \beta_i) > 0$ *such that*

$$\|P_n\|_\infty \leq c_6 \left(\sum_{i=1}^{l_1} E_i^1(m) + \sum_{i=l_1+1}^{l} E_i^2(m) \right) \|P_n\|_p , \tag{13}$$

where

$$E_i^1(m) \ : \ = n^{\left(\frac{\gamma_i^*+1}{p}+m\right)\widetilde{\mu}}, \ \text{for } i = \overline{1, l_1};$$

$$E_i^2(m) \ : \ = \begin{cases} n^{\left(\frac{\gamma_i^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p > 1, & \beta_i < m\mu - 1, \\ n^{\left(\frac{\gamma_i^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p < \frac{(\gamma_i^*+1)\mu+1+\beta_i}{1+\beta_i - m\mu}, & \beta_i \geq m\mu - 1, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{(\gamma_i^*+1)\mu+1+\beta_i}{1+\beta_i - m\mu}, & \beta_i \geq m\mu - 1, \\ n^{1-\frac{1}{p}}, & p > \frac{(\gamma_i^*+1)\mu+1+\beta_i}{1+\beta_i - m\mu}, & \beta_i \geq m\mu - 1. \end{cases} \quad \text{for } i = \overline{l_1+1, l}.$$

Analogously to the Corollary 2.3, we get next corollary for the cases of $i = 1, 2$; $l_1 = 1$, $l = 2$.

Corollary 2.8. *Let* $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, *for some* $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *defined as in (1) for* $l = 2$. *Then, for any* $\gamma_i > -1$, $i = 1, 2$, *and* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_7 = c_7(L, p, \lambda, \gamma_i, \beta) > 0$ *such that*

$$\|P_n\|_\infty \leq c_7 D_{n,p}(0) \|P_n\|_p , \tag{14}$$

where $D_{n,p}(0)$ *is defined as*

$$D_{n,p}(0) := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}\right)\widetilde{\mu}}, & p < 1 + (\gamma_1+1)\widetilde{\mu}, & \gamma_1 \geq \frac{(\gamma_2+1)\mu}{\widetilde{\mu}(1+\beta_2)} - 1, & \gamma_2 \geq \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1, \\ n^{\left(\frac{\gamma_2+1}{p}\right)\frac{\mu}{1+\beta_2}}, & p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2}, & 0 < \gamma_1 < \frac{(\gamma_2+1)\mu}{\widetilde{\mu}(1+\beta_2)} - 1, & \gamma_2 \geq \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}\right)\widetilde{\mu}}, & p < 1 + \left(\gamma_1^*+1\right)\widetilde{\mu}, & \gamma_1 > -1, & -1 < \gamma_2 < \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1, \\ (n \ln n)^{1-\frac{1}{p}}, & p = \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2}, & 0 < \gamma_1 < \frac{(\gamma_2+1)\mu}{\widetilde{\mu}(1+\beta_2)} - 1, & \gamma_2 \geq \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1, \\ n^{1-\frac{1}{p}}, & p > \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2}, & 0 < \gamma_1 < \frac{(\gamma_2+1)\mu}{\widetilde{\mu}(1+\beta_2)} - 1, & \gamma_2 \geq \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1, \\ n^{1-\frac{1}{p}}, & p > 1 + (\gamma_1+1)\widetilde{\mu}, & \gamma_1 \geq \frac{(\gamma_2+1)\mu}{\widetilde{\mu}(1+\beta_2)} - 1, & \gamma_2 \geq \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1, \\ n^{1-\frac{1}{p}}, & p > 1 + \left(\gamma_1^*+1\right)\widetilde{\mu}, & \gamma_1 > -1, & -1 < \gamma_2 < \frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1. \end{cases}$$

Corollary 2.9. *Let* $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, *for some* $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *defined as in (1) for* $l = 2$. *Then, for any* $\gamma_i > -1$, $i = 1, 2$, *and* $P_n \in \wp_n$, $n \in \mathbb{N}$, *there exists* $c_8 = c_8(L, p, \lambda, \gamma_i, \beta) > 0$ *such that*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_8 D_{n,p}(m) \|P_n\|_p , \ m \geq 1, \tag{15}$$

where $D_{n,p}(m)$ *is defined as*

$$D_{n,p}(m) := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+m\right)\widetilde{\mu}}, & p > 1, & \gamma_1 \geq \widetilde{\gamma_1} & \gamma_2 \geq \widetilde{\gamma_2}, & \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & p > 1, & \gamma_1 > -1, & -1 < \gamma_2 < \widetilde{\gamma_2}, & \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_2+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p > 1, & 0 < \gamma_1 < \widetilde{\gamma_1} & \gamma_2 \geq \widetilde{\gamma_2}, & \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_1+1}{p}+m\right)\widetilde{\mu}}, & p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & \gamma_1 \geq \widetilde{\gamma_1}, & \gamma_2 \geq \widetilde{\gamma_2}, & \beta_2 \geq m\mu - 1, \\ n^{\left(\frac{\gamma_1+1}{p}+m\right)\widetilde{\mu}}, & p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & \gamma_1 > 0, & 0 < \gamma_2 < \widetilde{\gamma_2}, & \beta_2 \geq m\mu - 1, \\ n^{\left(\frac{\gamma_2+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p < \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & 0 < \gamma_1 < \widetilde{\gamma_1}, & \gamma_2 \geq \widetilde{\gamma_2}, & \beta_2 \geq m\mu - 1, \\ n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}}, & p \geq \frac{(\gamma_2+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & \gamma_1 > -1, & \gamma_2 > -1, & \beta_2 \geq m\mu - 1, \end{cases}$$

where $\widetilde{\gamma_1} := \frac{(\gamma_2+pm+1)\mu}{\widetilde{\mu}(1+\beta_2)} - (pm+1)$; $\widetilde{\gamma_2} := (pm+1)\left[\frac{\widetilde{\mu}(1+\beta_2)}{\mu} - 1\right]$.

Remark 2.10. [38, Remark 2.16] The inequalities (14) and (15) are exact in the sense of order.

Combining the statement (15) with (11) and (12), we will obtain estimation on the growth for $|P_n^{(m)}(z)|$, $m \geq 0$, in the whole complex plane.

Theorem 2.11. *Let* $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, *for some* $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, for any* $\gamma_i > -1$, $i = 1, 2$, $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $z \in \mathbb{C}$, *we have*

$$|P_n(z)| \leq c_9 \|P_n\|_p \begin{cases} D_{n,p}(0), & z \in \overline{G}, \\ \frac{|\Phi^{n+1}(z)|}{d(z,L)} A^3_{n,p}(z), & z \in \Omega, \end{cases}$$

where $c_9 = c_9(L, \gamma_i, p) > 0$ *is a constant independent of n and z;* $D_{n,p}(1)$ *and* $A^4_{n,p}(z)$ *are defined as in Corollary 2.9 for all* $z \in \overline{G}$ *and Theorem 2.4 for all* $z \in \Omega$, *respectively.*

Theorem 2.12. *Let* $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, *for some* $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, for any* $\gamma_i > -1$, $i = 1, 2$, $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $z \in \mathbb{C}$, *we have*

$$\left|P'_n(z)\right| \leq c_{10} \|P_n\|_p \begin{cases} D_{n,p}(1), & z \in \overline{G}, \\ \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} A^4_{n,p}(z), & z \in \Omega, \end{cases}$$

where $c_{10} = c_{10}(L, \gamma_i, p) > 0$ *is a constant independent of n and z;* $D_{n,p}(1)$ *and* $A^4_{n,p}(z)$ *are defined as in Corollary 2.9 for all* $z \in \overline{G}$ *and Theorem 2.5 for all* $z \in \Omega$, *respectively.*

Theorem 2.13. *Let* $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, *for some* $\lambda \geq 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, for any* $\gamma_i > -1$, $i = 1, 2$, $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $z \in \mathbb{C}$, *we have*

$$\left|P''_n(z)\right| \leq c_{11} \|P_n\|_p \begin{cases} D_{n,p}(2), & z \in \overline{G}, \\ \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A^5_{n,p}(z), & z \in \Omega, \end{cases}$$

where $c_{11} = c_{11}(L, \gamma_i, p) > 0$ *is a constant independent of n and z;* $D_{n,p}(2)$ *and* $A^5_{n,p}(z)$ *are defined as in Corollary 2.9 for all* $z \in \overline{G}$ *and Theorem 2.6 for all* $z \in \Omega$, *respectively.*

Thus, using Theorem 2.2 and the estimation $\left|P_n^{(m)}(z)\right|$ sequentially for each $m \geq 3$, and combining the obtained estimates with Corollary 2.9, we acquire evaluations for the $\left|P_n^{(m)}(z)\right|$, for each points $z \in \mathbb{C}$.

3. Some auxiliary results

For $a > 0$ and $b > 0$ we use expressions "$a \preceq b$" and "$a \asymp b$" if $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2, respectively.

Lemma 3.1. ([1]) *Let G be a quasidisk, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1, 2, 3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. Therefore, $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ a constant, depending on G.

Corollary 3.2. *Under the conditions of Lemma 3.1, we have:*

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{\varepsilon},$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 3.3. ([46], [47]) *Let $L \in QS(\lambda)$ for some $\lambda \geq 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{\mu},$$

for all $w_1, w_2 \in \overline{\Delta}$, where $\mu := 2(1 - \frac{1}{\pi} \arcsin \frac{1}{\lambda})$.

This fact follows from an appropriate result for the mapping $f \in \sum(\kappa)$[40, p.287] and estimation for Ψ'[24, Th.2.8]:

$$d(\Psi(\tau), L) \asymp |\Psi'(\tau)| (|\tau| - 1). \tag{16}$$

Lemma 3.4. ([3]) *Let $L = G$ be a rectifiable Jordan curve and $P_n(z)$, $\deg P_n \leq n$, $n = 1, 2, ...$, be arbitrary polynomial and weight function $h(z)$ satisfy the condition (1). Then for any $R > 1$, $p > 0$ and $n = 1, 2, ...$*

$$\|P_n\|_{\mathcal{L}_p(h, L_R)} \preceq R^{n + \frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h, L)}, \ \gamma^* = \max\left\{0; \gamma_j : 1 \leq j \leq l\right\}.$$

4. Proofs of theorems

Proof of Theorem 2.2. Let $G \in PQS\left(\lambda; f_i, g_i\right)$, for some $\lambda \geq 1$, $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, l_1}$, and $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{l_1 + 1, l}$. For $z \in \Omega$, let us define $H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}$. By Leibnitz rule

$$H_n^{(m)}(z) = \sum_{j=0}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z) = \frac{P_n^{(m)}(z)}{\Phi^{n+1}(z)} + \sum_{j=1}^{m} C_m^j \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} P_n^{(m-j)}(z),$$

where $C_m^j := \frac{m(m-1)...(m-j+1)}{j!}$ and consequence after the module, we get

$$\left|P_n^{(m)}(z)\right| \leq \left|\Phi^{n+1}(z)\right| \left\{\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| + \sum_{j=1}^{m} C_m^j \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)}\right| \left|P_n^{(m-j)}(z)\right|\right\}. \tag{17}$$

Therefore, for the $\left|P_n^{(m)}(z)\right|$ at the points $z \in \Omega$ it is sufficient to evaluate for the cases

A) $\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right|$, $m = 1, 2, ...;$

B) $\left|\left(\Phi^{-n-1}(z)\right)^{(j)}\right|$, $j = \overline{1, m}$.

Now let us start with evaluations A) and B).

A) Since the function $H_n(z)$ is analytic in Ω, continuous on $\overline{\Omega}$ and $H_n(\infty) = 0$, then Cauchy integral representation for the $m - th$ derivatives gives:

$$H_n^{(m)}(z) = -\frac{m!}{2\pi i} \int\limits_L H_n(\zeta) \frac{d\zeta}{(\zeta - z)^{m+1}}, \; z \in \Omega, \; m \geq 1.$$

Then,

$$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| \leq \frac{m!}{2\pi} \int\limits_L \left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right| \frac{|d\zeta|}{|\zeta - z|^{m+1}} \leq \frac{m!}{2\pi d(z, L_R)} \int\limits_L |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}. \tag{18}$$

Let us denote the integral in (18) with

$$A_n(z) := \int\limits_L |P_n(\zeta)| \frac{|d\zeta|}{|\zeta - z|^m}, \tag{19}$$

and evaluate this integral. Multiplying the numerator and denominator of the integrand by $h^{1/p}(\zeta)$, according to the Hölder inequality, we get

$$A_n(z) \leq \|P_n\|_p \left(\int\limits_L \frac{|d\zeta|}{h^{\frac{q}{p}}(\zeta) |\zeta - z|^{qm}} \right)^{\frac{1}{q}}. \tag{20}$$

Denote by $J_n(z)$ the last integral, we have

$$\begin{aligned} [J_n(z)]^q \; &:= \int\limits_L \frac{|d\zeta|}{h^{q-1}(\zeta) |\zeta - z|^{qm}} \\ &\preceq \int\limits_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1} |\zeta - z|^{qm}} + \int\limits_{L^2} \frac{|d\zeta|}{|\zeta - z_2|^{(q-1)\gamma_2} |\zeta - z|^{qm}} \\ &=: J_{n,1}^1(z) + J_{n,2}^2(z). \end{aligned} \tag{21}$$

To simplify the calculations, let us prove for two points $z_1, z_2 \in L$ and put $z_1 = -1$, $z_2 = 1$; $(-1, 1) \subset G$. On the other hand, let local coordinate axis in Definition 2.1 be parallel to natural axis OX and OY in the coordinate system XOY; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : \mathrm{Im} z \geq 0\}$, $L^- := \{z \in L : \mathrm{Im} z < 0\}$. Moreover, let $z^{\pm} \in \Psi(w^{\pm})$ where $w^{\pm} := \left\{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, and $L_i^{\pm}(z_i, z^{\pm})$ denote the arcs, connected the points z_i which $z^{\pm}$, respectively; $\left|L_i^{\pm}\right| := mes\, L_i^{\pm}(z_i, z^{\pm})$, $i = 1, 2$. Assume that $z_0 \in L^+$ is taken as an arbitrary point (or $z_0 \in L^-$ subject to the chosen direction). Then, from (20) and (21), we have

$$A_n(z) \leq \|P_n\|_p \left\{ \left[J_{n,1}^1(z)\right]^{\frac{1}{q}} + \left[J_{n,2}^2(z)\right]^{\frac{1}{q}} \right\}. \tag{22}$$

Now, let us introduce some notation with $R = 1 + \frac{1}{n}$; $d_{i,R} := d(z_i, L_R)$;

$$\begin{aligned} E_1^{1,\pm} \quad &: \quad = \left\{\zeta \in L^1 : \ |\zeta - z_1| < c_1 d_{1,R}\right\}, \ E_2^{1,\pm} := \left\{\zeta \in L^1 : \ c_1 d_{1,R} \le |\zeta - z_1| \le \left| L_1^{\pm}\right|\right\}, \\ E_1^{2,\pm} \quad &: \quad = \left\{\zeta \in L^2 : \ |\zeta - z_2| < c_2 d_{2,R}\right\}, E_2^{2,\pm} := \left\{\zeta \in L^2 : \ c_2 d_{2,R} \le |\zeta - z_2| \le \left| L_2^{\pm}\right|\right\}; \\ \mathfrak{I}_{n,k}^{i,\pm}(z) \quad &: \quad = \int\limits_{E_k^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)} |\zeta - z|^{qm}} \ ; \ i,k = 1,2. \end{aligned} \tag{23}$$

Then, under the these notations, from (22) we get

$$A_n(z) \quad \le \quad \|P_n\|_p \sum_{i=1}^{2} \left[\mathfrak{I}_{n,1}^{i,\pm}(z) + \mathfrak{I}_{n,2}^{i,\pm}(z)\right]^{\frac{1}{q}}, \ i = 1,2. \tag{24}$$

Therefore we can start to estimate the integrals $\mathfrak{I}_{n,k}^{i,\pm}$ for each $i,k = 1,2$, $z \in \Omega(\delta)$ and $z \in \widehat{\Omega}(\delta)$.

1. Let us use the following notations where $z \in \Omega(\delta)$.

$$\begin{aligned} \left(E_k^{i,\pm}\right)_1 \quad &: \quad = \left\{\zeta \in E_k^{i,\pm} : \ |\zeta - z_i| < \ |\zeta - z|\right\}, \ \left(E_k^{i,\pm}\right)_2 := \ E_k^{i,\pm} \ \setminus \ \left(E_k^{i,\pm}\right)_1; \\ \mathfrak{I}_{n,k,1}^{i,\pm}(z) \quad &: \quad = \int\limits_{E_{k,1}^{i,\pm}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i(q-1)+qm}}; \ \mathfrak{I}_{n,k,2}^{i,\pm}(z) := \int\limits_{E_{k,2}^{i,\pm}} \frac{|d\zeta|}{|\zeta - z|^{\gamma_i(q-1)+qm}}. \end{aligned}$$

1.1. Let $\gamma_1, \gamma_2 \ge 0$. Then we get

$$\begin{aligned} \mathfrak{I}_{n,1,1}^{1,\pm}(z) \quad &\le \quad \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)+qm}} \le d_{1,R}^{-\left[\gamma_1(q-1)+qm\right]+1}; \\ \mathfrak{I}_{n,1,2}^{1,\pm}(z) \quad &\le \quad d_{1,R}^{-\left[(\gamma_1+1)+qm\right]} \cdot mes\left(E_{1,2}^{1,\pm}\right) \le d_{1,R}^{-\left[\gamma_1(q-1)+qm\right]+1}; \\ \mathfrak{I}_{n,1}^{1,\pm}(z) &= \mathfrak{I}_{n,1,1}^{1,\pm}(z) + \mathfrak{I}_{n,1,2}^{1,\pm}(z) \le d_{1,R}^{-\left[\gamma_1(q-1)+qm\right]+1}. \end{aligned} \tag{25}$$

$$\begin{aligned} \mathfrak{I}_{n,2,1}^{1,\pm}(z) \quad &\le \quad \int\limits_{c_1 d_{1,R}}^{|L_1^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)+qm}} \le d_{1,R}^{-\left[\gamma_1(q-1)+qm\right]+1}; \ \ \mathfrak{I}_{n,2,2}^{1,\pm}(z) \le d_{1,R}^{-\left[\gamma_1(q-1)+qm\right]+1}; \\ \mathfrak{I}_{n,2}^{1,\pm}(z) \quad &= \quad \mathfrak{I}_{n,2,1}^{1,\pm}(z) + \mathfrak{I}_{n,2,2}^{1,\pm}(z) \le d_{1,R}^{-\left[\gamma_1(q-1)+qm\right]+1}. \end{aligned}$$

Similarly, for the $J_{n,2}^2(z)$ in neighborhood of the point z_2, we have

$$\begin{aligned} \mathfrak{I}_{n,1,1}^{2,\pm}(z) \quad &\le \quad \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{\gamma_2(q-1)+qm}} \le d_{2,R}^{-\left[\gamma_2(q-1)+qm\right]+1}; \\ \mathfrak{I}_{n,1,2}^{2,\pm}(z) \quad &\le \quad \int\limits_{\left(E_1^{2,\pm}\right)_2} \frac{|d\zeta|}{|\zeta - z|^{\gamma_2(q-1)+qm}} \le d_{2,R}^{-\left[\gamma_2(q-1)+qm\right]} \cdot mes\left(E_{1,2}^{2,\pm}\right) \le d_{2,R}^{-\left[\gamma_2(q-1)+qm\right]+1}; \\ \mathfrak{I}_{n,1}^{2,\pm}(z) &= \mathfrak{I}_{n,1,1}^{2,\pm}(z) + \mathfrak{I}_{n,1,2}^{2,\pm}(z) \le d_{2,R}^{-\left[\gamma_2(q-1)+qm\right]+1}. \end{aligned} \tag{26}$$

$$\begin{aligned}\mathfrak{I}^{2,\pm}_{n,2,1}(z) &\leq \int\limits_{E^{2,\pm}_{2,1}} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2(q-1)+qm}} \leq \int\limits_{c_2 d_{2,R}}^{|L^{\pm}_2|} \frac{ds}{s^{\gamma_2(q-1)+qm}} \leq d_{2,R}^{-[\gamma_2(q-1)+qm]+1};\\
\mathfrak{I}^{2,\pm}_{n,2,2}(z) &\leq \int\limits_{E^{2,\pm}_{2,2}} \frac{|d\zeta|}{|\zeta - z|^{\gamma_2(q-1)+qm}} \leq \int\limits_{c_2 d_{2,R}}^{|L^{\pm}_2|} \frac{ds}{s^{\gamma_2(q-1)+qm}} \leq d_{2,R}^{-[\gamma_2(q-1)+qm]+1};\\
\mathfrak{I}^{2,\pm}_{n,2}(z) &= \mathfrak{I}^{2,\pm}_{n,2,1}(z) + \mathfrak{I}^{2,\pm}_{n,2,2}(z) \leq d_{2,R}^{-[\gamma_1(q-1)+qm]+1}.\end{aligned}$$

1.2. Let $\gamma_1, \gamma_2 < 0$. Then, analogously to the (25) and (26), we obtain

$$\mathfrak{I}^{1,\pm}_{n,1}(z) = \int\limits_{E^{1,\pm}_1} \frac{|\zeta - z_1|^{-\gamma_1(q-1)}\,|d\zeta|}{|\zeta - z|^{qm}} \leq d_{1,R}^{-\gamma_1(q-1)-qm} mesE^{1,\pm}_1 \leq d_{1,R}^{-[\gamma_1(q-1)+qm]+1}; \tag{27}$$

$$\mathfrak{I}^{1,\pm}_{n,2}(z) \leq \int\limits_{E^{1,\pm}_2} \frac{|\zeta - z_1|^{-\gamma_1(q-1)}\,|d\zeta|}{|\zeta - z|^{qm}} \leq \int\limits_{c_1 d_{1,R}}^{|L^{\pm}_1|} \frac{ds}{s^{qm}} \leq d_{1,R}^{-qm+1};$$

and

$$\begin{aligned}\mathfrak{I}^{2,\pm}_{n,1}(z) &\leq \int\limits_{E^{2,\pm}_1} \frac{|\zeta - z_2|^{-\gamma_2(q-1)}}{|\zeta - z|^{qm}}\,|d\zeta| \leq d_{2,R}^{(-\gamma_2)(q-1)-qm} mesE^{2,\pm}_1 \leq d_{2,R}^{-[\gamma_2(q-1)+qm]+1};\\
\mathfrak{I}^{2,\pm}_{n,2}(z) &\leq \int\limits_{E^{2,\pm}_2} \frac{|\zeta - z_2|^{-\gamma_2(q-1)}}{|\zeta - z|^{qm}}\,|d\zeta| \leq \int\limits_{c_2 d_{2,R}}^{|L^{\pm}_2|} \frac{ds}{s^{qm}} \leq d_{2,R}^{-qm+1}.\end{aligned}$$

In this case, combining (24) - (27) we get

$$A_n(z) \leq \|P_n\|_p \left[d_{1,R}^{-\frac{(\gamma^*_1+1)}{p}-m+1} + d_{2,R}^{-\frac{(\gamma^*_2+1)}{p}-m+1} \right]. \tag{28}$$

B) By Cauchy integral representation for the $j-th$ derivatives of $\left(\frac{1}{\Phi^{n+1}(z)}\right)$, we have

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(j)} = -\frac{j!}{2\pi i} \int\limits_L \frac{1}{\Phi^{n+1}(\zeta)} \frac{d\zeta}{(\zeta - z)^{j+1}}\ ,\ z \in \Omega.$$

Thus

$$B_{n,j}(z) := \left| \left(\Phi^{-n-1}(z)\right)^{(j)} \right| \leq \frac{j!}{2\pi} \int\limits_L \left| \frac{1}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|^{j+1}} = \frac{j!}{2\pi} \int\limits_L \frac{|d\zeta|}{|\zeta - z|^{j+1}}.$$

Using notation $L^{\pm} = E^{1,\pm}_1 \cup E^{1,\pm}_2 \cup E^{2,\pm}_1 \cup E^{2,\pm}_2$ from (23), we have

$$B_{n,j}(z) \leq \sum_{i,k=1}^{2} \int\limits_{E^{i,+}_k \cup E^{i,-}_k} \frac{|d\zeta|}{|\zeta - z|^{j+1}} =: M(E^{i,+}_k) + M(E^{i,-}_k). \tag{29}$$

Since the integrals $M(E_k^{i,+})$ and $M(E_k^{i,-})$, $i,k=1,2$, are estimated to be similar, we will estimate only $M(E_k^{i,+})$.

$$M(E_1^{1,+}) = \int_{|z_1-z|}^{c_1 d_{1,R}} \frac{ds}{s^{j+1}} \preceq d_{1,R}^{-j};\ M(E_2^{1,+}) = \int_{d_{1,R}}^{|L_1^{\pm}|} \frac{ds}{s^{j+1}} \preceq d_{1,R}^{-j};$$

$$M(E_1^{2,+}) = \int_0^{|z_2-z_2^+|} \frac{ds}{s^{j+1}} \preceq \int_0^{c_2 d_{2,R}} \frac{ds}{s^{j+1}} \preceq d_{2,R}^{-j};\ M(E_2^{2,+}) = \int_{|z_2-z_2^+|}^{|L_2^{\pm}|} \frac{ds}{s^{j+1}} \preceq \int_{c_2 d_{2,R}}^{|L_2^{\pm}|} \frac{ds}{s^{j+1}} \preceq d_{2,R}^{-j}.$$

Then, from (29), we have

$$B_{n,j}(z) \preceq d_{1,R}^{-j} + d_{2,R}^{-j}. \tag{30}$$

Comparing (17), (18), (28)-(30), we get

$$\left|P_n^{(m)}(z)\right| \preceq \left|\Phi^{n+1}(z)\right| \left[\frac{1}{d(z,L)} \|P_n\|_p \left(d_{1,R}^{-\frac{(\gamma_1^*+1)}{p}-m+1} + d_{2,R}^{-\frac{(\gamma_2^*+1)}{p}-m+1}\right) + \sum_{j=1}^{m} C_m^j \left(d_{1,R}^{-j} + d_{2,R}^{-j}\right) \left|P_n^{(m-j)}(z)\right|\right]. \tag{31}$$

According to Lemma 3.3, [24, p.61] and [23, p.10], we obtain

$$d_{1,R} \succeq \begin{cases} n^{-\mu}, & \text{if } \alpha_1 = 0; \\ n^{-2}, & \text{if } \alpha_1 \neq 0, \end{cases} \tag{32}$$

Let us define: $z_R \in L_R : d_{2,R} = |z_2 - z_R|$; $\zeta^{\pm} \in L^{\pm} : d(z_R, L^2 \cap L^{\pm}) := d(z_R, L^+)$; $z_2^{\pm} := \zeta \in L^2 : |\zeta - z_2| = c_2 d_{2,R}$ for the estimate $d_{2,R}$, . Then, we have

$$d_R^{\pm} := d(z_R, L^2 \cap L^{\pm}) \asymp \left|z_R - z_2^{\pm}\right| \asymp d_{2,R}^{1+\beta_2} \tag{33}$$

from Lemma 3.1. Hence $d_{2,R} = \left(d_R^{\pm}\right)^{\frac{1}{1+\beta_2}}$. Further, according to Lemma 3.3 and [23, Corollary 2], we get $d_R^{\pm} \succeq n^{-\mu}$. Therefore, the result

$$d_{2,R} \succeq n^{\frac{-\mu}{1+\beta_2}}. \tag{34}$$

is obtained. We get

$$\left|P_n^{(m)}(z)\right| \preceq \left|\Phi^{n+1}(z)\right| \left[\frac{\|P_n\|_p}{d(z,L)} \left(n^{\left(\frac{\gamma_1^*+1}{p}+m-1\right)\widetilde{\mu}} + n^{\left(\frac{\gamma_2^*+1}{p}+m-1\right)\frac{\mu}{1+\beta_2}}\right) + \sum_{j=1}^{m} C_m^j \left(n^{j\widetilde{\mu}} + n^{\frac{j\mu}{1+\beta_2}}\right) \left|P_n^{(m-j)}(z)\right|\right] \tag{35}$$

from (17)-(34), and we complete the proof for the points $z \in \Omega(\delta)$.

2. Now suppose that $z \in \widehat{\Omega}(\delta)$.

2.1. Let $\gamma_1, \gamma_2 \geq 0$. From (23), we successively find that

$$\mathfrak{I}_{n,1}^{1,\pm}(z) \preceq \int_0^{c_1 d_{1,R}} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1) > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1; \end{cases}$$

$$\mathfrak{I}_{n,2}^{1,\pm}(z) \preceq \int_{c_1 d_{1,R}}^{|L_1^{\pm}|} \frac{ds}{s^{\gamma_1(q-1)}} \preceq \begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1) > 1, \\ \ln \frac{1}{d_{1,R}}, & \gamma_1(q-1) = 1, \\ 1, & \gamma_1(q-1) < 1; \end{cases}$$

$$\mathfrak{I}_{n,1}^{1,\pm}(z)+\mathfrak{I}_{n,2}^{1,\pm}(z)\preceq\begin{cases} d_{1,R}^{1-\gamma_1(q-1)}, & \gamma_1(q-1)>1,\\ \ln\frac{1}{d_{1,R}}, & \gamma_1(q-1)=1,\\ 1, & \gamma_1(q-1)<1;\end{cases} \tag{36}$$

$$\mathfrak{I}_{n,1}^{2,\pm}(z) \preceq \int\limits_0^{c_2 d_{2,R}}\frac{ds}{s^{\gamma_2(q-1)}}\preceq\begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1)>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1;\end{cases}$$

$$\mathfrak{I}_{n,2}^{2,\pm}(z) \preceq \int\limits_{c_2 d_{2,R}}^{|L_2^{\pm}|}\frac{ds}{s^{\gamma_2(q-1)}}\preceq\begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1)>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1;\end{cases}$$

$$\mathfrak{I}_{n,1}^{2}(z)+\mathfrak{I}_{n,2}^{2,\pm}(z)\preceq\begin{cases} d_{2,R}^{1-\gamma_2(q-1)}, & \gamma_2(q-1)>1,\\ \ln\frac{1}{d_{2,R}}, & \gamma_2(q-1)=1,\\ 1, & \gamma_2(q-1)<1.\end{cases} \tag{37}$$

2.2. Let $\gamma_1,\gamma_2<0$. Then, analogously to the estimates (27), we get that

$$\mathfrak{I}_{n,1}^{1,\pm}(z)\preceq d_{1,R}^{(-\gamma_1)(q-1)} mesE_1^1 \preceq 1,\ \mathfrak{I}_{n,2}^{1,\pm}(z)\preceq \left|L_1^{\pm}\right|^{(-\gamma_1)(q-1)+1}\preceq 1;$$

$$\mathfrak{I}_{n,1}^{1}(z)+\mathfrak{I}_{n,2}^{1}(z)\preceq 1; \tag{38}$$

$$\mathfrak{I}_{n,1}^{2,\pm}(z)\preceq d_{2,R}^{(-\gamma_2)(q-1)} mesE_1^{2,\pm} \preceq 1,\ \mathfrak{I}_{n,2}^{2,\pm}(z)\preceq \left|L_2^{\pm}\right|^{(-\gamma_2)(q-1)+1}\preceq 1,$$

$$\mathfrak{I}_{n,1}^{2,\pm}(z)+\mathfrak{I}_{n,2}^{2,\pm}(z)\preceq 1. \tag{39}$$

Therefore combining estimates (24) - (52) we obtain that

$$A_n(z)\preceq \|P_n\|_p\begin{cases} d_{1,R}^{1-\frac{\gamma_1+1}{p}}+d_{2,R}^{1-\frac{\gamma_2+1}{p}}, & \gamma_1,\gamma_2>p-1.\\ \left(\ln\frac{1}{d_{1,R}}\right)^{1-\frac{1}{p}}+\left(\ln\frac{1}{d_{2,R}}\right)^{1-\frac{1}{p}}, & \gamma_1,\gamma_2=p-1,\\ 1, & \gamma_1,\gamma_2<p-1.\end{cases} \tag{40}$$

$$B_{n,j}(z)\preceq\int\limits_L\frac{|d\zeta|}{|\zeta-z|^{j+1}}\preceq\int\limits_L|d\zeta|\preceq 1. \tag{41}$$

Taking into account of (40), (32) and (34), we complete the proof for the points $z\in\widehat{\Omega}(\delta)$, and so, the proof of Theorem 2.2 is completed.

Proof of Theorem 2.4. Suppose $G\in PQS\left(\lambda;f_1,g_2\right)$, for some $\lambda\geq 1$, $f_1(x)=C_1x^{1+\alpha_1}$, $\alpha_1\geq 0$ and $g_2(x)=C_2x^{1+\beta_2}$, $\beta_2>0$; $h(z)$ be defined as in (1). By Cauchy integral formula for $H_n(z)=\frac{P_n(z)}{\Phi^{n+1}(z)}$, $z\in\Omega$, we have

$$\left|\frac{P_n(z)}{\Phi^{n+1}(z)}\right|\leq\frac{1}{2\pi}\int\limits_L\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|} \tag{42}$$

$$\preceq\frac{1}{d(z,L)}\int\limits_L|P_n(\zeta)|\,|d\zeta| =: \frac{1}{d(z,L)}A_n.$$

Therefore, it turns out that

$$\left|\frac{P_n(z)}{\Phi^{n+1}(z)}\right|\preceq\frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)}A_n,\ \text{where } A_n=\sum_{i=1}^{2}\int\limits_{L^i}|P_n(\zeta)|\,|d\zeta|. \tag{43}$$

Multiplying the numerator and denominator of the integrand by $h^{1/p}(\zeta)$ and applying the Hölder inequality, we obtain that

$$A_n \le \sum_{i=1}^{2}\left(\int_{L^i} h(\zeta)\,|P_n(\zeta)|^p\,|d\zeta|\right)^{1/p} \times \left(\int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^{l}\left|\zeta - z_j\right|^{\gamma_j}}\right)^{1/q} \tag{44}$$
$$=: \sum_{i=1}^{2}\left(\widetilde{J}^i_{n,1}\cdot \widetilde{J}^i_{n,2}\right)^{1/p}, \frac{1}{p}+\frac{1}{q}=1.$$

According to Lemma 3.3, we get

$$\widetilde{J}^i_{n,1} \preceq \|P_n\|_p\,,\ i=1,2. \tag{45}$$

for $\widetilde{J}^i_{n,1}$. Then, from (44) and (45) we have

$$A_n \preceq \|P_n\|_p \sum_{i=1}^{2}\left(\widetilde{J}^i_{n,2}\right)^{1/q}.$$

For the integral $\widetilde{J}^i_{n,2}$, we obtain

$$\widetilde{J}^i_{n,2} := \int_{L^i} \frac{|d\zeta|}{\prod_{j=1}^{l}\left|\zeta - z_j\right|^{\gamma_i}} \asymp \int_{L^i} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}},\ i=1,2. \tag{46}$$

Thus, from (46), we write

$$A_n \preceq \|P_n\|_p \sum_{i=1}^{2}\left(\widetilde{J}^i_{n,2}\right)^{1/q},\ \text{where}\ \widetilde{J}^1_{n,2} = \int_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}}\ ;\ \widetilde{J}^2_{n,2} = \int_{L^2} \frac{|d\zeta|}{|\zeta - z_2|^{\gamma_2}}. \tag{47}$$

Taking into consideration above notations from (47), we have

$$A_n \preceq \|P_n\|_p \sum_{i=1}^{2}\left(\widetilde{J}^i_{n,2}\right)^{1/q} \tag{48}$$
$$=: \|P_n\|_p \sum_{i=1}^{2}\left[\widetilde{\mathfrak{I}}^i_{n,1}(E^{i,\pm}_1) + \widetilde{\mathfrak{I}}^i_{n,2}(E^{i,\pm}_2)\right]^{1/q} =: \|P_n\|_p \sum_{i=1}^{2}\left[\widetilde{\mathfrak{I}}^{i,\pm}_{n,1} + \widetilde{\mathfrak{I}}^{i,\pm}_{n,2}\right]^{1/q},\ i=1,2,$$

where

$$\widetilde{\mathfrak{I}}^{i,\pm}_{n,k} := \widetilde{\mathfrak{I}}^i_{n,k}(E^{i,\pm}_k) := \int_{E^{i,\pm}_k} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}}\ ;\ i,\ k=1,2.$$

Therefore, it is sufficient to estimate the integrals $\mathfrak{I}^{i,\pm}_{n,k}$ for each $i=1,2$ and $k=1,2$.

Taking into account (47) and (48), let us start with the evaluations of the integral $\widetilde{J}^1_{n,2}$:

$$\widetilde{J}^1_{n,2} = \int_{L^1} \frac{|d\zeta|}{|\zeta - z_1|^{\gamma_1}} = \sum_{k=1}^{2}\int_{E^{1,\pm}_k} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_1}} =: \widetilde{\mathfrak{I}}^{1,\pm}_{n,1} + \widetilde{\mathfrak{I}}^{1,\pm}_{n,2}. \tag{49}$$

Given the possible values of γ_i ($-1 < \gamma_i < 0,\ \gamma_i \geq 0,\ i = 1, 2$), we will consider the estimates for the $\widetilde{J}^1_{n,2}$ separately.

Let $\gamma_1 \geq 0$ and $\gamma_2 \geq 0$. In this case for the integral $\widetilde{J}^1_{n,2}$, we get that

$$\widetilde{\mathfrak{J}}^{1,\pm}_{n,1} \leq \int\limits_{E^{1,\pm}_1} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1}} \leq \int\limits_0^{c_1 d_{1,R}} \frac{ds}{s^{(q-1)\gamma_1}} \leq \begin{cases} d_{1,R}^{1-(q-1)\gamma_1}, & (q-1)\gamma_1 > 1, \\ 1, & (q-1)\gamma_1 \leq 1, \end{cases} \quad c_1 > 1; \tag{50}$$

$$\widetilde{\mathfrak{J}}^{1,\pm}_{n,2} \leq \int\limits_{E^{1,\pm}_2} \frac{|d\zeta|}{|\zeta - z_1|^{(q-1)\gamma_1}} \leq \int\limits_{c_1 d_{1,R}}^{|l^{\pm}_1|} \frac{ds}{s^{(q-1)\gamma_1}} \leq \begin{cases} d_{1,R}^{1-(q-1)\gamma_1}, & (q-1)\gamma_1 > 1, \\ \ln \frac{1}{d_{1,R}}, & (q-1)\gamma_1 = 1, \\ 1, & (q-1)\gamma_1 < 1. \end{cases}$$

Similar estimate for the integral $\widetilde{J}^2_{n,2}$ is given as follows:

$$\widetilde{\mathfrak{J}}^{2,\pm}_{n,1} \leq \int\limits_{E^{2,\pm}_1} \frac{|d\zeta|}{|\zeta - z_2|^{(q-1)\gamma_2}} \leq \int\limits_0^{c_2 d_{2,R}} \frac{ds}{s^{(q-1)\gamma_2}} \leq \begin{cases} d_{2,R}^{1-(q-1)\gamma_2}, & (q-1)\gamma_2 > 1, \\ 1, & (q-1)\gamma_2 \leq 1, \end{cases} \quad c_2 > 1; \tag{51}$$

$$\widetilde{\mathfrak{J}}^{2,\pm}_{n,2} \leq \int\limits_{E^{2,\pm}_2} \frac{|d\zeta|}{|\zeta - z_2|^{(q-1)\gamma_2}} \leq \int\limits_{c_2 d_{2,R}}^{|l^{\pm}_2|} \frac{ds}{s^{(q-1)\gamma_2}} \leq \begin{cases} d_{2,R}^{1-(q-1)\gamma_2}, & (q-1)\gamma_2 > 1, \\ \ln \frac{1}{d_{2,R}}, & (q-1)\gamma_2 = 1, \\ 1, & (q-1)\gamma_2 < 1. \end{cases}$$

If $\gamma_1 < 0$ and $\gamma_2 < 0$, analogously to the (50) and (51), we obtain

$$\begin{aligned} \widetilde{\mathfrak{J}}^{1,\pm}_{n,1} &\leq \int\limits_{E^{1,\pm}_1} |\zeta - z_1|^{-(q-1)\gamma_1} |d\zeta| \leq d_{1,R}^{-(q-1)\gamma_1} mes E^1_1 \leq 1, \\ \widetilde{\mathfrak{J}}^{1}_{n,2} &\leq \int\limits_{E^{1,\pm}_2} |\zeta - z_1|^{-(q-1)\gamma_1} |d\zeta| \leq \left| l^{\pm}_1 \right|^{-(q-1)\gamma_1+1} \leq 1, \end{aligned}$$

and

$$\begin{aligned} \widetilde{\mathfrak{J}}^{2,\pm}_{n,1} &\leq \int\limits_{E^{2,\pm}_1} |\zeta - z_2|^{(-(q-1)\gamma_2)} |d\zeta| \leq d_{2,R}^{(-(q-1)\gamma_2)} mes E^{2,\pm}_1 \leq 1, \\ \widetilde{\mathfrak{J}}^{2,\pm}_{n,2} &\leq \int\limits_{E^{2,\pm}_2} |\zeta - z_2|^{(-(q-1)\gamma_2)} |d\zeta| \leq \left| l^{\pm}_2 \right|^{(-(q-1)\gamma_2)+1} \leq 1. \end{aligned} \tag{52}$$

Therefore, from (24) - (52), it turns out that

$$A_n(z) \leq \|P_n\|_p \begin{cases} d_{1,R}^{\frac{1-(q-1)\gamma_1}{q}} + d_{2,R}^{\frac{1-(q-1)\gamma_2}{q}}, & (q-1)\gamma_1 > 1, (q-1)\gamma_2 > 1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{\frac{1}{q}} + \left(\ln \frac{1}{d_{2,R}}\right)^{\frac{1}{q}}, & (q-1)\gamma_1 = 1, (q-1)\gamma_2 = 1, \\ 1, & (q-1)\gamma_1 < 1, (q-1)\gamma_2 < 1. \end{cases} \tag{53}$$

Comparing (42), (43) and (53), we see that

$$|P_n(z)| \leq c \frac{B^0_{n,1}}{d(z, L)} \|P_n\|_p |\Phi(z)|^{n+1}, \tag{54}$$

where $c = c(L, p, \gamma_i) > 0,\ i = 1, 2$, is the constant independent from n and z, and

$$B^0_{n,1} := \begin{cases} d_{1,R}^{\frac{\gamma_1+1}{p}-1} + d_{2,R}^{\frac{\gamma_2+1}{p}-1}, & \gamma_1 > p-1, \gamma_2 > p-1, \\ \left(\ln \frac{1}{d_{1,R}}\right)^{1-\frac{1}{p}} + \left(\ln \frac{1}{d_{2,R}}\right)^{1-\frac{1}{p}}, & \gamma_1 = p-1, \gamma_2 = p-1 \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases} \tag{55}$$

According to (32) and (34), we acquire

$$|P_n(z)| \le \frac{B_{n,1}}{d(z,L)} \|P_n\|_p |\Phi(z)|^{n+1},$$

where

$$B_{n,1} := \begin{cases} n^{(\frac{(\gamma_1+1)}{p}-1)\widetilde{\mu}} + n^{(\frac{\gamma_2+1}{p}-1)\frac{\mu}{(1+\beta_2)}}, & \gamma_1 > p-1, \gamma_2 > p-1,, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \gamma_1 = p-1, -1 < \gamma_2 \le p-1, \\ \text{or } -1 < \gamma_1 \le p-1, \gamma_2 = p-1 \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases} \tag{56}$$

So, we complete the proof of Theorem 2.4.

Proof of Theorem 2.5. From Corollary 3.2 and Theorem 2.4, we get

$$\left|P_n'(z)\right| \le \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} \left[A_n(z,1) + |P_n(z)| \begin{cases} n^{\widetilde{\mu}}, & \text{if } z \in \Omega(\delta), \\ 1, & \text{if } z \in \widehat{\Omega}(\delta), \end{cases} \right] \tag{57}$$

where for any $\gamma > -1, m = 1$

$$A_n(z,1) \le \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}} & \begin{cases} \gamma_1 \ge \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1,\ \gamma_2 > p-1, \\ \text{or } \gamma_1 > p-1,\ -1 < \gamma_2 \le p-1, \end{cases} \\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}} & -1 < \gamma_1 < \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & \begin{cases} \gamma_1 = p-1, \gamma_2 \le p-1 \\ \text{or } \gamma_1 \le p-1, \gamma_2 = p-1, \end{cases} \\ 1, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases} \quad \text{if } z \in \Omega(\delta);$$

$$A_n(z,1) \le \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}} & \gamma_1 \ge \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1,\ \gamma_2 > p-1, \\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}} & p-1 < \gamma_1 < \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & -1 < \gamma_1, \gamma_2 \le p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1. \end{cases} \quad \text{if } z \in \widehat{\Omega}(\delta);$$

Taking into account estimates (9) for $A_n(z,1)$ and (10) for $|P_n(z)|$ and substituting them into (57), after simple calculations, we prove the required result.

$$\left|P_n'(z)\right| \le \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p \begin{cases} n^{\widetilde{\mu}}, & p > 1 & -1 < \gamma_1 \le 0, & -1 < \gamma_2 \le 0, \\ n^{\frac{\gamma_1+1}{p}\widetilde{\mu}}, & p > 1 & \gamma_1 > \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, & \gamma_2 > p-1, \\ n^{\widetilde{\mu}} (\ln n)^{1-\frac{1}{p}}, & p > 1 & 0 < \gamma_1 \le p-1, & 0 < \gamma_2 \le p-1, \\ n^{\widetilde{\mu}}, & p > 1 & 0 < \gamma_1 < p-1, & 0 < \gamma_2 < p-1, \\ n^{\widetilde{\mu}+(\frac{\gamma_2+1}{p}-1)\frac{\mu}{(1+\beta_2)}}, & p > 1 & p-1 < \gamma_1 \le \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, & \gamma_2 > p-1, \end{cases}$$

if $z \in \Omega(\delta)$;

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}, & \gamma_1 \geq \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1,\ \gamma_2 > p-1, \\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & p-1 < \gamma_1 < \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & -1 < \gamma_1, \gamma_2 \leq p-1, \\ 1, & -1 < \gamma_1, \gamma_2 < p-1, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$.

Proof of Theorem 2.6. Taking into account estimates (9) for $A_n(z,2)$, (10) for $|P_n(z)|$, (11) for $\left|P_n'(z)\right|$, estimates for $B_{n,j}^2$, $j=1,2$, and substituting in the formula below

$$\left|P_n''(z)\right| \leq \left|\Phi^{n+1}(z)\right| \left[\frac{\|P_n\|_p}{d(z,L)} A_n^1(z,2) + C_2^1 B_{n,1}^1 \left|P_n'(z)\right| + C_2^2 B_{n,2}^1 |P_n(z)|\right],$$

after the simple calculations, we obtain that

$$\left|P_n''(z)\right| \leq \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p \begin{cases} n^{2\widetilde{\mu}}, & -1 < \gamma_1 < p-1, & -1 < \gamma_2 < p-1, \\ n^{\left(\frac{\gamma_1+1}{p}+1\right)\widetilde{\mu}}, & \gamma_1 > \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, & \gamma_2 > p-1, \\ n^{2\widetilde{\mu}} (\ln n)^{1-\frac{1}{p}}, & 1 < \gamma_1 \leq p-1, & 1 < \gamma_2 \leq p-1, \\ n^{2\widetilde{\mu}+\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{(1+\beta_2)}}, & p-1 < \gamma_1 \leq \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, & \gamma_2 > p-1, \end{cases}$$

if $z \in \Omega(\delta)$;

$$\left|P_n''(z)\right| \leq \frac{\left|\Phi^{3(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)\widetilde{\mu}}, & \gamma_1 \geq \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, & \gamma_2 > p-1, \\ n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{\mu}{1+\beta_2}}, & p-1 < \gamma_1 < \frac{\mu[\gamma_2-(p-1)]}{\widetilde{\mu}(1+\beta_2)} + p-1, & \gamma_2 > p-1, \\ (\ln n)^{1-\frac{1}{p}}, & -1 < \gamma_1 \leq p-1, & -1 < \gamma_2 \leq p-1, \\ 1, & -1 < \gamma_1 < p-1, & -1 < \gamma_2 < p-1, \end{cases}$$

if $z \in \widehat{\Omega}(\delta)$. Therefore, the proof of Theorem 2.6 is completed.

Proof of Theorem 2.7. Let $p > 1$ and $G \in PQS\left(\lambda; f_i, g_i\right)$, for some $\lambda \geq 1$, where $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, l_1}$, and $g_i(x) = c_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{l_1+1, l}$,. Cauchy integral representation for $m-th$ derivatives of $P_n(z)$ in the region G_R gives that

$$P_n^{(m)}(z) = \frac{m!}{2\pi i} \int\limits_{L_R} \frac{P_n(\zeta)}{(\zeta - z)^{m+1}} d\zeta, \ \ z \in G_R.$$

Moving on to both parts of the modules, multiplying the numerator and determinator of the integrand by $h^{1/p}(\zeta)$, according to the Hölder inequality, it turns out that

$$\left|P_n^{(m)}(z)\right| \leq \frac{m!}{2\pi} \left(\int\limits_{L_R} h(\zeta) |P_n(\zeta)|^p |d\zeta|\right)^{1/p} \times \left(\int\limits_{L_R} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} \left|\zeta - z_j\right|^{(q-1)\gamma_j} |\zeta - z|^{q(m+1)}}\right)^{1/q} =: \frac{m!}{2\pi} Y_{n,1} \times Y_{n,2},$$

where

$$Y_{n,1} := \|P_n\|_{\mathcal{L}_p(h,L_R)}, \ \ (Y_{n,2})^q := \int\limits_{L_R} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} \left|\zeta - z_j\right|^{(q-1)\gamma_j} |\zeta - z|^{q(m+1)}}.$$

Let $z \in L$. Then, from Lemma 3.4 we write

$$\left|P_n^{(m)}(z)\right| \leq Y_{n,1} \cdot Y_{n,2} \leq \|P_n\|_p \cdot Y_{n,2}\,. \tag{58}$$

To estimate the integral $Y_{n,2}$, we introduce: $w_j := \Phi(z_j)$, $\varphi_j := \arg w_j$. Without loss of generality, we will assume that $\varphi_l < 2\pi$. For $\eta := \min\left\{\eta_j, j = \overline{1,l}\right\}$, where $\eta_j = \min\limits_{t \in \partial\Phi(\Omega(z_j,\ \delta_j))} \left|t - w_j\right| > 0$, let us set

$$\begin{aligned}
\Delta(\eta_j) \quad &: \quad = \left\{t : \left|t - w_j\right| \leq \eta_j\right\} \subset \Phi(\Omega(z_j,\ \delta_j)),\\
\Delta(\eta) \quad &: \quad = \bigcup_{j=1}^{l} \Delta_j(\eta),\ \widehat{\Delta}_j = \Delta \backslash \Delta(\eta_j); \widehat{\Delta}(\eta) := \Delta \backslash \Delta(\eta);\ \Delta_1^{'} := \Delta_1^{'}(1),\\
\Delta_1^{'}(\rho) \quad &: \quad = \left\{t = Re^{i\theta} : R \geq \rho > 1,\ \frac{\varphi_0 + \varphi_1}{2} \leq \theta < \frac{\varphi_1 + \varphi_2}{2}\right\},\\
\Delta_j^{'} \quad &: \quad = \Delta_j^{'}(1), \Delta_j^{'}(\rho) := \left\{t = Re^{i\theta} : R \geq \rho > 1,\ \frac{\varphi_{j-1} + \varphi_j}{2} \leq \theta < \frac{\varphi_j + \varphi_0}{2}\right\},\ j = 2, 3..., \ l,\\
\text{where } \varphi_0 \quad &= \quad 2\pi - \varphi_l;\ \Omega^j := \Psi(\Delta_j^{'}), L_{R_1}^j := L_{R_1} \cap \Omega^j;\ \Omega = \bigcup_{j=1}^{l} \Omega^j.
\end{aligned}$$

Then, we get

$$(Y_{n,2})^q = \sum_{i=1}^{l} \int\limits_{L_R^i} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} \left|\zeta - z_j\right|^{(q-1)\gamma_j} |\zeta - z|^{q(m+1)}} \asymp \sum_{i=1}^{l} \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{(q-1)\gamma_i} |\zeta - z|^{q(m+1)}} =: \sum_{i=1}^{l} Y_{n,2}^i, \tag{59}$$

where

$$Y_{n,2}^i := \int\limits_{L_R^i} \frac{|d\zeta|}{|\zeta - z_i|^{(q-1)\gamma_i} |\zeta - z|^{q(m+1)}},\ i = \overline{1,l}, \tag{60}$$

since the points $\{z_i\}_{i=1}^l \in L$ are distinct. For simplicity of our next calculations, we assume that $i = 1,2$; $l_1 = 1$, $l = 2$; $z_1 = -1$, $z_2 = 1$; $(-1,1) \subset G$; $R = 1 + \frac{\varepsilon_0}{n}$, and let local coordinate axis in Definition 2.1 is parallel to OX and OY in the OXY coordinate system; $L = L^+ \cup L^-$, where $L^+ := \{z \in L : Imz \geq 0\}$, $L^- := \{z \in L : Imz < 0\}$. Let $z^\pm \in \Psi(w^\pm)$ where $w^\pm := \left\{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, and L^i denote the arcs which connecting the points z^+, z_i, $z^- \in L$; $L^{i,\pm} := L^i \cap L^\pm$, $i = 1,2$. Let z_0 be taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). For simplicity, without loss of generality, we assume that $z_0 = z^+$ ($z_0 = z^-$). Analogously, we introduce: $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : Imz \geq 0\}$, $L_R^- := \{z \in L_R : Imz < 0\}$. Let $w_R^\pm := \left\{w = Re^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, $z_R^\pm \in \Psi(w_R^\pm)$.We set $z_{i,R} \in L_R$, such that $d_{i,R} = \left|z_i - z_{i,R}\right|$ and $\zeta^\pm \in L^\pm$, such that $d(z_{2,R}, L^2 \cap L^\pm) := d(z_{2,R}, L^\pm)$; $z_i^\pm := \left\{\zeta \in L^i :\ |\zeta - z_i| = c_i d(z_i, L_R)\right\}$, $z_{i,R}^\pm := \left\{\zeta \in L_R^i :\ \left|\zeta - z_{i,R}\right| = c_i d(z_{i,R}, L_R)\right\}$, $w_{i,R}^\pm = \Phi(z_{i,R}^\pm)$. Let L_R^i, $i = 1,2$, denote arcs, connecting the points z_R^+, $z_{i,R}$, $z_R^- \in L_R$, $L_R^{i,\pm} := L_R^i \cap L_R^\pm$ and $l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$ denote arcs, connecting the points $z_{i,R}^\pm$ with $z_R^\pm$, respectively and $\left|\, l_{i,R}^\pm\right| := mes\ l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$, $i = 1,2$. On the other hand, we denote

$$\begin{aligned}
E_{1,R}^{i,\pm} \quad &: \quad = \left\{\zeta \in L_R^{i,\pm} :\ |\zeta - z_i| < c_i d_{i,R}\right\},\\
E_{2,R}^{i,\pm} \quad &: \quad = \left\{\zeta \in L_R^{i,\pm} : c_i d_{i,R} \leq |\zeta - z_i| \leq \left|\, l_{i,R}^\pm\right|\right\},\ F_{j,R}^{i,\pm} := \Phi(E_{j,R}^{i,\pm});\\
E_1^{i,\pm} \quad &: \quad = \left\{\zeta \in L^{i,\pm} :\ |\zeta - z_i| < c_i d_{i,R}\right\},\\
E_2^{i,\pm} \quad &: \quad = \left\{\zeta \in L^{i,\pm} : c_i d_{i,R} \leq |\zeta - z_i| \leq \left|\, l_{i,R}^\pm\right|\right\},\ F_j^{i,\pm} := \Phi(E_j^{i,\pm}),\ i,j = 1,2.
\end{aligned}$$

Taking into consideration these designations, from (60), we have

$$Y^i_{n,2} \asymp \sum_{i,j=1}^{2} \int\limits_{E^{i,+}_{j,R} \cup E^{i,-}_{j,R}} \frac{|d\zeta|}{|\zeta - z_i|^{(q-1)\gamma_i} |\zeta - z|^{q(m+1)}} =: \sum_{i,j=1}^{2} \left[Y(E^{i,+}_{j,R}) + Y(E^{i,-}_{j,R}) \right].$$

So, we need to evaluate the integrals $Y(E^{i,+}_{j,R})$ and $Y(E^{i,-}_{j,R})$ for each $i, j = 1, 2$.

Let z' be such that

$$\|P_n\|_\infty =: |P_n(z')|, \; z' \in L = L^1 \cup L^2. \tag{61}$$

There are two possible cases: the point z' may lie on L^1 or L^2.

1) Suppose first that $z' \in L^1$. Consider the individual cases.

1.1) If $z' \in E^{1,\pm}_1 \cup E^{1,\pm}_2$, then

$$Y(E^{1,+}_{1,R}) + Y(E^{1,-}_{1,R}) \preceq \int\limits_{E^{1,+}_{1,R} \cup E^{1,-}_{1,R}} \frac{|d\zeta|}{[\min\{|\zeta - z_1|; |\zeta - z'|\}]^{(q-1)\gamma_1 + q(m+1)}} \preceq \int\limits_{d_{1,R}}^{cd_{1,R}} \frac{ds}{s^{(q-1)\gamma_1 + q(m+1)}} \preceq \frac{1}{d_{1,R}^{(q-1)\gamma_1 + q(m+1) - 1}}, \tag{62}$$

for $\gamma_1 > 0$, and

$$Y(E^{1,+}_{1,R}) + Y(E^{1,-}_{1,R}) \preceq (cd_{1,R})^{-(q-1)\gamma_1} \int\limits_{d_{1,R}}^{cd_{1,R}} \frac{ds}{s^{q(m+1)}} \preceq \frac{1}{d_{1,R}^{(q-1)\gamma_1 + q(m+1) - 1}}, \tag{63}$$

for $-1 < \gamma_1 \leq 0$.

1.2) If $z' \in E^{1,\pm}_1$, then

$$Y(E^{1,+}_{2,R}) + Y(E^{1,-}_{2,R}) \preceq \int\limits_{E^{1,+}_{2,R} \cup E^{1,-}_{2,R}} \frac{|d\zeta|}{[\min\{|\zeta - z_1|; |\zeta - z'|\}]^{(q-1)\gamma_1 + q(m+1)}} \preceq \int\limits_{cd_{1,R}}^{\left| l^{\pm}_{1,R} \right|} \frac{ds}{s^{(q-1)\gamma_1 + q(m+1)}} \preceq \frac{1}{d_{1,R}^{(q-1)\gamma_1 + q(m+1) - 1}}, \tag{64}$$

for $\gamma_1 > 0$ and

$$Y(E^{1,+}_{2,R}) + Y(E^{1,-}_{2,R}) \preceq \int\limits_{E^{1,+}_{2,R} \cup E^{1,-}_{2,R}} \frac{|d\zeta|}{|\zeta - z'|^{q(m+1)}} \preceq \int\limits_{cd_{1,R}}^{\left| l^{\pm}_{1,R} \right|} \frac{ds}{s^{q(m+1)}} \preceq \frac{1}{d_{1,R}^{q(m+1) - 1}}. \tag{65}$$

for $-1 < \gamma_1 \leq 0$.

1.3) If $z' \in E^{1,\pm}_2$, then

$$Y(E^{1,+}_{2,R}) + Y(E^{1,-}_{2,R}) \preceq \frac{1}{d_{1,R}^{(q-1)\gamma_1}} \int\limits_{E^{1,+}_{2,R} \cup E^{1,-}_{2,R}} \frac{|d\zeta|}{|\zeta - z'|^{q(m+1)}} \preceq \frac{1}{d_{1,R}^{(q-1)\gamma_1}} \int\limits_{cd_{1,R}}^{\left| l^{\pm}_{1,R} \right|} \frac{ds}{s^{q(m+1)}} \preceq \frac{1}{d_{1,R}^{(q-1)\gamma_1 + q(m+1) - 1}}, \tag{66}$$

for $\gamma_1 > 0$, and

$$Y(E^{1,+}_{2,R}) + Y(E^{1,-}_{2,R}) \preceq \int\limits_{E^{1,+}_{2,R} \cup E^{1,-}_{2,R}} \frac{|d\zeta|}{|\zeta - z'|^{q(m+1)}} \preceq \int\limits_{cd_{1,R}}^{\left| l^{\pm}_{1,R} \right|} \frac{ds}{s^{q(m+1)}} \preceq \frac{1}{d_{1,R}^{q(m+1) - 1}}, \tag{67}$$

for $-1 < \gamma_1 \leq 0$. Combining the cases (62)-(67), we obtain

$$\sum_{i=1}^{2}\left[Y(E_{i,R}^{1,+}) + Y(E_{i,R}^{1,-})\right] \leq \frac{1}{d_{1,R}^{(q-1)\gamma_1^* + q(m+1)-1}}. \tag{68}$$

According to Lemma 3.3 and [23, p.10], from (68), we have

$$d_{1,R} \succeq \begin{cases} n^{-\mu}, & \text{if } \alpha_1 = 0; \\ n^{-2}, & \text{if } \alpha_1 \neq 0, \end{cases}$$

and, consequently, from (68)-(32), we get

$$\sum_{i=1}^{2}\left[Y(E_{i,R}^{1,+}) + Y(E_{i,R}^{1,-})\right] \preceq \begin{cases} n^{2\left[(q-1)\gamma_1^* + q(m+1)-1\right]}, & \text{if } \alpha_1 \neq 0; \\ n^{\mu\left[(q-1)\gamma_1^* + q(m+1)-1\right]}, & \text{if } \alpha_1 = 0. \end{cases} \tag{69}$$

2) Now, suppose that $z' \in L^2$. In this case, replacing the variable $\tau = \Phi(\zeta)$, according to (16), we have

$$Y_{n,2}^{i} \asymp \sum_{i,j=1}^{2} \int_{F_{j,R}^{i,\pm}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau) - \Psi(w')|^{q(m+1)}} =: \sum_{i,j=1}^{2}\left[Y(F_{j,R}^{i,+}) + Y(F_{j,R}^{i,-})\right]. \tag{70}$$

2.1) If $z' \in E_1^{2,\pm}$, then

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \asymp \int_{F_{1,R}^{2,+}\cup F_{1,R}^{2,-}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau) - \Psi(w')|^{q(m+1)}\,(|\tau| - 1)} \tag{71}$$

$$\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2}\left|\Psi(\tau) - \Psi(w')^{q(m+1)-1}\right|} + n \int_{F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2}\,|\Psi(\tau) - \Psi(w')|^{q(m+1)-1}},$$

for all $\gamma_2 > -1$. The last two integrals are evaluated identically. Therefore, we evaluate one of them, say the first. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau) - \Psi(w')|$, we obtain

$$|\Psi(\tau) - \Psi(w')| \succeq \max\left\{|\Psi(\tau) - \Psi(w_2)|;\ \left|\Psi(\tau) - z_2^+\right|\right\} \succeq |\Psi(\tau) - \Psi(w_2)| \succeq \left|\Psi(\tau) - z_2^+\right|^{\frac{1}{1+\beta_2}}.$$

Then, from (71), we get

$$\begin{aligned} Y(F_{1,R}^{2,+}) &\preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}}} \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu}} \\ &\preceq \begin{cases} n^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases} \end{aligned}$$

if $\gamma_2 > 0$, and

$$Y(F_{1,R}^{2,+}) \preceq n \int_{F_{1,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{-(q-1)\gamma_2}\,|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{q(m+1)-1}{1+\beta_2}}} \preceq n \int_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{q(m+1)-1}{1+\beta_2}\mu}} \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}\mu}, & \frac{q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases}$$

if $-1 < \gamma_2 \leq 0$,. Therefore, we have

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \leq \begin{cases} n^{\frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases} \tag{72}$$

2.2) If $z' \in E_2^{2,\pm}$, then

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \leq n \int\limits_{F_{1,R}^{2,+} \cup F_{1,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} |\Psi(\tau) - \Psi(w')|^{q(m+1)-1}}, \tag{73}$$

for all $\gamma_2 > -1$. When $\tau \in F_{1,R}^{2,+}$ for the $|\Psi(\tau) - \Psi(w')|$, we obtain $|\Psi(\tau) - \Psi(w')| \geq \left|\Psi(\tau) - z_2^+\right|$ and, analogous to previous case, we write

$$Y(F_{1,R}^{2,+}) \leq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} \left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}} \leq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}}} \tag{74}$$

$$\leq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu}} \leq \begin{cases} n^{\frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2 + q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases}$$

if $\gamma_2 > 0$, and

$$Y(F_{1,R}^{2,+}) \leq n \int\limits_{F_{1,R}^{2,+}} \frac{|\Psi(\tau) - \Psi(w_2)|^{-(q-1)\gamma_2} |d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}} \leq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}} \tag{75}$$

$$\leq n \int\limits_{F_{1,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{q(m+1)-1}{1+\beta_2}\mu}} \leq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}\mu}, & \frac{q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases}$$

if $-1 < \gamma_2 \leq 0$.

So, from (73)-(75), we have

$$Y(F_{1,R}^{2,+}) + Y(F_{1,R}^{2,-}) \leq \begin{cases} n^{\frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2^* + q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases} \tag{76}$$

2.3) If $z' \in E_1^{2,\pm}$, then

$$Y(F_{2,R}^{2,+}) + Y(F_{2,R}^{2,-}) \leq n \int\limits_{F_{2,R}^{2,+} \cup F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} |\Psi(\tau) - \Psi(w')|^{q(m+1)-1}} \tag{77}$$

$$\leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} |\Psi(\tau) - \Psi(w')|^{q(m+1)-1}} + n \int\limits_{F_{2,R}^{2,+} \cup F_{2,R}^{2,-}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{(q-1)\gamma_2} |\Psi(\tau) - \Psi(w')|^{q(m+1)-1}},$$

for $\gamma_2 > -1$. The last two integrals are evaluated identically. Let us evaluate the first integral.

For $\tau \in F_{2,R}^{2,\pm}$ and $z' \in E_1^{2,\pm}$, we have

$$|\Psi(\tau) - \Psi(w')| \geq \left|\Psi(\tau) - z_2^+\right|; \ |\Psi(\tau) - \Psi(w_2)| \geq \left|\Psi(\tau) - z_2^+\right| \geq \left|z_{2,R} - z_2^+\right|^{\frac{1}{1+\beta_2}}.$$

Then

$$\begin{aligned} \Upsilon(F_{2,R}^{2,+}) &\leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{(q-1)\gamma_2} \left|\Psi(\tau) - z_2^+\right|^{q(m+1)-1}} \leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu}} \\ &\leq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases} \end{aligned}$$

and so, for $\gamma_2 > 0$ we obtain that

$$\Upsilon(F_{2,R}^{2,+}) + \Upsilon(F_{2,R}^{2,-}) \leq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases}$$

For $-1 < \gamma_2 \leq 0$, we get that

$$\Upsilon(F_{2,R}^{2,+}) + \Upsilon(F_{2,R}^{2,-}) \leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w')|^{q(m+1)-1}} \leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\Psi(\tau) - z_2^+\right|^{\frac{q(m+1)-1}{1+\beta_2}}} \tag{78}$$

$$\leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{q(m+1)-1}{1+\beta_2}\mu}} \leq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}\mu}, & \frac{q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases}$$

Then, for $\gamma_2 > -1$, we have

$$\Upsilon(F_{2,R}^{2,+}) + \Upsilon(F_{2,R}^{2,-}) \leq \begin{cases} n^{\frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n \ln n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases} \tag{79}$$

2.4) If $z' \in E_2^{2,+}$, then for $\gamma_2 > 0$, $\tau \in F_{2,R}^{2,+}$ and $z' \in E_1^{2,\pm}$, we acquire that

$$\begin{aligned} |\Psi(\tau) - \Psi(w')| &\geq \left|z_{2,R}^+ - z_2^+\right| \geq \left|z_2 - z_{2,R}^+\right| \geq \left|z_2^+ - z_{2,R}\right|^{\frac{1}{1+\beta_2}}; \\ |\Psi(\tau) - \Psi(w_2)| &\geq \left|z_{2,R}^+ - z_2\right| \geq \left|z_{2,R} - z_2\right| \geq \left|z_2^+ - z_{2,R}\right|^{\frac{1}{1+\beta_2}}. \end{aligned}$$

Therefore, we obtain that

$$\Upsilon(F_{2,R}^{2,\pm}) \leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|z_2^+ - z_{2,R}\right|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}}} \leq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|w_2^+ - w_{2,R}\right|^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu}} \tag{80}$$

$$\preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n^{\frac{(q-1)\gamma_2}{1+\beta_2}\mu} n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n^{\frac{(q-1)\gamma_2}{1+\beta_2}\mu} n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases}$$

and

$$Y(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n^{\frac{(q-1)\gamma_2}{1+\beta_2}\mu} n\ln n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n^{\frac{(q-1)\gamma_2}{1+\beta_2}\mu} n, & \frac{(q-1)\gamma_2+q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases}$$

The case of $z' \in E_2^{2,-}$ is absolutely identical to the case $z' \in E_2^{2,+}$.

If $-1 < \gamma_2 \le 0$, then

$$Y(F_{2,R}^{2,+}) \preceq n \int\limits_{F_{2,R}^{2,+}} \frac{|d\tau|}{\left|w_2^+ - w_{2,R}\right|^{\frac{q(m+1)-1}{1+\beta_2}\mu}} \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}\mu}, & \frac{q(m+1)-1}{1+\beta_2}\mu > 1, \\ n\ln n, & \frac{q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases} \tag{81}$$

and

$$Y(F_{2,R}^{2,-}) \preceq \begin{cases} n^{\frac{q(m+1)-1}{1+\beta_2}\mu}, & \frac{q(m+1)-1}{1+\beta_2}\mu > 1, \\ n\ln n, & \frac{q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{q(m+1)-1}{1+\beta_2}\mu < 1. \end{cases} \tag{82}$$

Combining the relations (69)-(82), for $l_1 = 1,\ l_2 = 1$, for each $\gamma_1 > -1$ and $\gamma_2 > -1$ and any $p > 1$, we write

$$\begin{aligned} Y_{n,2}^1 + Y_{n,2}^2 \quad &\preceq \quad \begin{cases} n^{2\left[(q-1)\gamma_1^*+q(m+1)-1\right]}, & \text{if } \alpha_1 \neq 0; \\ n^{\mu\left[(q-1)\gamma_1^*+q(m+1)-1\right]}, & \text{if } \alpha_1 = 0. \end{cases} + \begin{cases} n^{\frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n\ln n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases} \\ &\preceq \quad n^{\widetilde{\mu}\left[(q-1)\gamma_1^*+q(m+1)-1\right]} + \begin{cases} n^{\frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu}, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu > 1, \\ n\ln n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu = 1, \\ n, & \frac{(q-1)\gamma_2^*+q(m+1)-1}{1+\beta_2}\mu < 1, \end{cases} \end{aligned} \tag{83}$$

where $\widetilde{\mu} := \begin{cases} \mu, & \text{if } \alpha_1 = 0, \\ 2, & \text{if } \alpha_1 \neq 0, \end{cases}$. Then, from (58)-(61) and (83), for all $z \in L$, we obtain that

$$|P_n(z)| \preceq \|P_n\|_p \cdot \left[n^{\left(\frac{\gamma_1^*+1}{p}+m\right)\widetilde{\mu}} + \begin{cases} n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p > 1, & \beta_2 < m\mu - 1, \\ n^{\left(\frac{\gamma_2^*+1}{p}+m\right)\frac{\mu}{1+\beta_2}}, & p < \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & \beta_2 \ge m\mu - 1, \\ (n\ln n)^{1-\frac{1}{p}}, & p = \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & \beta_2 \ge m\mu - 1, \\ n^{1-\frac{1}{p}}, & p > \frac{(\gamma_2^*+1)\mu+1+\beta_2}{1+\beta_2-m\mu}, & \beta_2 \ge m\mu - 1. \end{cases} \right]$$

To complete the proof of Theorem 2.7, it remains for us to combine the estimates over all points $z \in L$.

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