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On the growth of derivatives of algebraic polynomials in regions with a piecewise quasicircle with zero angles

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Abstract. In this paper, we study the growth for the m-th derivatives of an arbitrary algebraic polynomial in bounded and unbounded regions with piecewise-quasicircle boundary having interior and exterior zero angles in the weighted Lebesgue spaces.

1. Introduction and definitions

Let $\overline{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ where $\mathbb{C}$ be a complex plane and $G \subset \mathbb{C}$ be a bounded Jordan region with boundary $L := \partial G$ (without loss of generality, let $0 \in G$); $\Omega := \overline{\mathbb{C}} \setminus \overline{G} = extL$. For $t \in \mathbb{C}$ and $\delta > 0$, let $\Delta(t, \delta) := \{w \in \mathbb{C} : |w - t| > \delta\}$; $\Delta := \Delta(0, 1)$ and $B(t, \delta) := \{w \in \mathbb{C} : |w - t| < \delta\}$; $B := B(0, 1)$. Let $\Phi : \Omega \to \Delta$ be the univalent conformal mapping normalized by $\Phi(\infty) = \infty$ and $\lim_{z\to\infty} \frac{\Phi(z)}{z} > 0$; $\Psi := \Phi^{-1}$.

On the other hand, let us set

$$L_t := \{z : \ |\Phi(z)| = t\}, \ L_1 \equiv L, \ G_t := intL_t, \ \Omega_t := extL_t;$$

for $t \geq 1$. Moreover, let $d(z, L) := dist(z, L) = \inf\{|\zeta - z| : \ \zeta \in L\}$ for $z \in \mathbb{C}$ and $L \subset \mathbb{C}$; and $\wp_n$ denotes the class of all algebraic polynomials $P_n(z)$ of degree at most $n \in \mathbb{N}$. Let $\left\{z_j\right\}_{j=1}^{l} \in L$ be the fixed system of distinct points. For some fixed R_0, $1 < R_0 < \infty$, and $z \in \overline{G}_{R_0}$, consider generalized Jacobi weight function by

$$h(z) := h_0(z) \prod_{j=1}^{l} \left|z - z_j\right|^{\gamma_j}, \tag{1}$$

where $\gamma_j > -1$, for all $j = 1, 2, ..., l$, $z \in G_{R_0}$ and $h_0(z) \geq c_0(L) > 0$ for some constant $c_0(L) > 0$.

For each $0 < p \leq \infty$ and rectifiable Jordan curve $L = \partial G$, we introduce the following norms:

$$\|P_n\|_p \ : = \|P_n\|_{\mathcal{L}_p(h,L)} := \left(\int_L h(z) \left|P_n(z)\right|^p |dz| \right)^{1/p} < \infty, 0 < p < \infty, \tag{2}$$

$$\|P_n\|_\infty \ : = \|P_n\|_{\mathcal{L}_\infty(1,L)} := \max_{z\in L} |P_n(z)|, \ p = \infty; \ \mathcal{L}_p(1, L) =: \mathcal{L}_p(L).$$

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It is well known that the Bernstein-Walsh inequality given below is often used in the theory of approximation of a function of a complex variable [45]:

$$\|P_n\|_{C(\overline{G}_R)} \leq |\Phi(z)|^n \|P_n\|_{C(\overline{G})}, \ \forall P_n \in \wp_n. \tag{3}$$

An analogue of this inequality in space $\mathcal{L}_p(h, L)$ is as follows [32]:

$$\|P_n\|_{\mathcal{L}_p(L_R)} \leq |\Phi(z)|^{n+\frac{1}{p}} \|P_n\|_{\mathcal{L}_p(L)}, \ \forall P_n \in \wp_n, \ p > 0.$$

This estimate has been generalized in [15, Lemma 2.4] for weight function $h(z) \neq 1$, defined as in (1), as below:

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \leq |\Phi(z)|^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\left\{0; \gamma_j : 1 \leq j \leq l\right\}. \tag{4}$$

If we consider the two-dimensional analogues of the quantities (2), i.e., integral over the region G, (we denote them by $\|P_n\|_{A_p(h,G)}$, $\|P_n\|_{A_p(1,G)}$ and $A_p(G)$ respectively), then corresponding estimations of the form (4) can also state for them. But to do this, the following definitions are needed.

Assume that $\varphi : G \to B$ is a conformal and univalent mapping which is normalized by $\varphi(0) = 0$, $\varphi'(0) > 0$; $\psi := \varphi^{-1}$. A bounded Jordan region G is called a κ *-quasidisk,* $0 \leq \kappa < 1$, if any conformal mapping ψ can be extended to a K -quasiconformal, $K = \frac{1+\kappa}{1-\kappa}$, homeomorphism of the plane $\overline{\mathbb{C}}$ on the $\overline{\mathbb{C}}$ (see [33, p. 100],[40]). In that case the curve $L := \partial G$ is called a κ *-quasicircle*. The region G (curve L) is called a *quasidisk (quasicircle)*, if it is *κ-quasidisk (κ -quasicircle)* for some $0 \leq \kappa < 1$. We denote this class by $Q(\kappa)$, $0 \leq \kappa < 1$, and say that $L = \partial G \in Q(\kappa)$, if $G \in Q(\kappa)$, $0 \leq \kappa < 1$. Also we say that $G \in \widetilde{Q}(\kappa)$, $0 \leq \kappa < 1$, if $G \in Q(\kappa)$ and ∂G is rectifiable. Furthermore, we denote that G $(L) \in Q$ (or $\widetilde{Q}$), if G $(L) \in Q(\kappa)$ (or $\widetilde{Q}(\kappa)$) for some $0 \leq \kappa < 1$. Note that quasicircles can be non-rectifiable (see, for example, [26], [33, p.104]). Recall that there is a geometric definition of quasiconformal curve in [33, p.102]. A curve L is said to be quasiconformal if for arbitrary points $z_1 \in L$ and $z_2 \in L$, the diameter of the shorter arc $l(z_1, z_2)$ of the curve L joining points z_1, z_2 satisfies the inequality

$$\frac{\text{diam } l(z_1, z_2)}{|z_1 - z_2|} \leq c < +\infty. \tag{5}$$

In [1] the analogue of the inequalities (3) and (4) holds as follows

$$\|P_n\|_{A_p(h,G_R)} \leq c_1 \ [1 + c_2(|\Phi(z)| - 1)]^{n+\frac{1}{p}} \|P_n\|_{A_p(h,G)}, \ |\Phi(z)| = R > 1, \ p > 0, \tag{6}$$

for arbitrary region $G \in Q$ and the weight function $h(z)$ given in (1) where $c_2 > 0$ and $c_1 := c_1(G, p, c_2) > 0$ constants, independent of n and z. Moreover, estimate (6) was generalized for arbitrary Jordan region G and $P_n \in \wp_n$ as below in [12, Theorem 1.1]:

$$\|P_n\|_{A_p(G_R)} \leq c_3 \, |\Phi(z)|^{n+\frac{2}{p}} \|P_n\|_{A_p(G_{R_1})}, \ |\Phi(z)| = R > R_1 = 1 + \frac{1}{n}, \ p > 0,$$

where $c_3 = \left(\frac{2}{e^p - 1}\right)^{\frac{1}{p}} \left[1 + O(\frac{1}{n})\right]$, $n \to \infty$, is asymptotically sharp constant.

A new version of the Bernstein-Walsh lemma for the regions with a rectifiable quasiconformal boundary is found as follows in [43]:

$$|P_n(z)| \leq c(L) \frac{\sqrt{n}}{d(z, L)} \|P_n\|_{A_2(G)} |\Phi(z)|^{n+1}, \ z \in \Omega,$$

where $c(L) > 0$ is a constant depending only of L.

Suppose that S is a rectifiable Jordan curve (or arc) and $z = z(s)$ is the natural parametrization of S for $s \in [0, |S|]$, $|S| := mes\, S$. Let z_1, z_2 be an arbitrary points on S and $l(z_1, z_2)$ denotes the subarc of S of shorter diameter with endpoints z_1 and z_2 (including the endpoints) as mentioned above. Following [41, p.163], we say that a bounded Jordan curve S is $\lambda-$quasismooth (in the sense of Lavrentiev) curve, if for every pair $z_1, z_2 \in S$, there exists a constant $\lambda := \lambda(S) \geq 1$, such that

$$|l(z_1, z_2)| \leq \lambda |z_1 - z_2|, \ z_1, \ z_2 \in S, \tag{7}$$

where $|l(z_1, z_2)|$ is the linear measure (length) of $l(z_1, z_2)$.

The problem on uniform and pointwise estimates for the $\left|P_n^{(m)}(z)\right|$, $m \geq 0$, in $\overline{G}$ and Ω, was investigated in [17] where $m = 0$ and L is a $\lambda-$ quasismooth curve and the similar problem was considered in [10] for more general class of curves without any cusps, contained also $\lambda-$quasismooth curves in case of $m \geq 0$ and the following evaluations were obtained:

$$\left|P_n^{(m)}(z)\right| \leq c_4 \|P_n\|_p \begin{cases} \mu_n, & z \in \overline{G}, \\ \eta_n, & z \in \Omega, \end{cases} \tag{8}$$

where $c_4 = c_4(L, p, m, \gamma) > 0$ is a constant independent of n, h, P_n, $\mu_n = \mu_n(L, h, p) > 0$ and $\eta_n = \eta_n(L, h, p, z) \to \infty$, as $n \to \infty$, are a constants depending on the properties of the L, h. Moreover, we addressed the results to the statement (8) with G being a bounded region by a piecewise quasismooth curve having a finite number interior and exterior zero angles on the boundary in [30].

It is easy to see that quasismooth curves satisfy the inequality (5). Therefore, any quasismooth curve is quasicircle and they don't have any cusps. Moreover, according to [23, Lemma 3], there exists a rectifiable quasicircle which does not satisfy inequality (7). The estimate (8) for regions bounded by κ-quasicircle was studied in [14]. However, the evaluations of type (4) have not yet been studied for regions bounded by $k-$quasicircles with exterior and interior zero angles .

The aim of this study is to address this problem for regions bounded by piecewise rectifiable quasicircles having a finite number interior and exterior zero angles on the boundary.

After making some reminders, let's start giving the relevant definitions.

Throughout this study, $c, c_0, c_1, c_2, ...$ are positive constants and $\varepsilon_0, \varepsilon_1, \varepsilon_2, ...$ are sufficiently small positive ones (generally, different in different relations), which depend on G in general and on parameters inessential for the argument. Otherwise, such dependence will be explicitly stated.

Definition 1.1. A Jordan arc ℓ is called $\kappa-$quasiarc for some $0 \leq \kappa < 1$, if ℓ is a part of some κ-quasicircle for the same $0 \leq \kappa < 1$.

We say that a bounded Jordan curve (arc) L is *locally* $\kappa-$*quasicircle* ($\kappa-$quasiarc) at the point $z \in L$, if there exists a closed subarc $\ell \subset L$ containing z such that every open subarc of the ℓ containing z is the $\kappa-$*quasicircle* ($\kappa-$quasiarc).

For any $i = 1, 2, ...,$ $k = 0, 1, 2$ and $\varepsilon_1 > 0$, we denote by $f_i : [0, \varepsilon_1] \to \mathbb{R}$ and $g_i : [0, \varepsilon_1] \to \mathbb{R}$ twice differentiable functions such that

$$f_i(0) = g_i(0) = 0, \ f_i^{(k)}(x) > 0, \ g_i^{(k)}(x) > 0, \ 0 < x \leq \varepsilon_1. \tag{9}$$

Further, the notation $i = \overline{k, m}$ means $i = k, k+1, ..., m$ for any $k \geq 0$ and $m > k$. Now let us give a new class of regions bounded by piecewise quasicircle having interior and exterior cusps at the connecting points of boundary arcs.

Definition 1.2. We say that a Jordan region $G \in \widetilde{PQ}\left(\kappa; f_i, \ g_i\right)$, for some $0 \leq \kappa < 1$, $f_i = f_i(x)$, $i = \overline{1, l_1}$ and $g_i = g_i(x)$, $i = \overline{l_1 + 1, l}$, defined as in (9), if $L = \partial G = \bigcup\limits_{i=0}^{l} L_i$ is the union of the finite number of rectifiable κ_i-quasiarcs, $0 \leq \kappa_i < 1$, ($\kappa = \max\{\kappa_i, 0 \leq i \leq l\}$) L_i, connecting at the points $\{z_i\}_{i=0}^{l} \in L$ and such that L is a locally κ-quasiarc at the $z_0 \in L\setminus \{z_i\}_{i=1}^{l}$ and, in the (x,y) local coordinate system with its origin at the z_i, $1 \le i \le l$, the following conditions are satisfied:

$a)$ for every $z_i \in L,\ i=\overline{1,l_1},\ l_1 \le l$,

$$\left\{z=x+iy : |z|\le \varepsilon_1,\ c_{11}^{i} f_i(x) \le y \le c_{12}^{i} f_i(x),\ 0\le x\le \varepsilon_1\right\} \subset \overline{G},$$
$$\left\{z=x+iy : |z|\le \varepsilon_1,\ |y| \ge \varepsilon_2 x,\ 0\le x\le \varepsilon_1\right\} \subset \overline{\Omega};$$

$b)$ for every $z_i \in L,\ i=\overline{l_1+1,l}$,

$$\left\{z=x+iy : |z|< \varepsilon_3,\ c_{21}^{i} g_i(x) \le y \le c_{22}^{i} g_i(x),\ 0\le x\le \varepsilon_3\right\} \subset \overline{\Omega},$$
$$\left\{z=x+iy : |z|< \varepsilon_3,\ |y| \ge \varepsilon_4 x,\ 0\le x\le \varepsilon_3\right\} \subset \overline{G},$$

for some constants $-\infty < c_{11}^{i} < c_{12}^{i} < \infty$, $-\infty < c_{21}^{i} < c_{22}^{i} < \infty$ and $\varepsilon_s > 0$, $s=\overline{1,4}$.

From Definition 1.2 it is clear that each region $G \in \widetilde{PQ}\left(\kappa; f_i, g_i\right)$ may have l_1 interior and $l-l_1$ exterior zero angles (with respect to $\overline{G}$) at the points $\{z_i\}_{i=1}^{l} \in L$. If a region G does not have interior zero angles ($l_1=0$) (exterior zero angles ($l_1=l$)), then it is written as $G \in \widetilde{PQ}\left(\kappa; 0, g_i\right)$ ($G \in \widetilde{PQ}\left(\kappa; f_i, 0\right)$). If a region G does not have such angles ($l=0$), then G is bounded by a rectifiable $\kappa-$quasicircle and in this case we set $\widetilde{PQ}(\kappa,0,0) \equiv \widetilde{Q}(\kappa)$.

Throughout this work, we shall assume that the points $\{z_i\}_{i=1}^{l} \in L$ defined in (1) and Definition 1.2 are the same. Without loss of generality, we will also assume that the points $\{z_i\}_{i=0}^{l}$ are ordered in the positive direction on the curve L such that G has interior zero angles at the points $\{z_i\}_{i=1}^{l_1}$, if $l_1 \ge 1$ and exterior zero angles at the points $\{z_i\}_{i=l_1+1}^{l}$, if $l \ge l_1+1$.

Note that the similar results of the type (8) in various spaces for $m=0$, the different weight functions and, unbounded regions ($z \in \Omega$) were studied in [6], [13]-[20], [4], [9], [31, p.418-428], [34], [39], [38], [43] and others. Moreover, the estimates of the type (8) for bounded regions ($z \in \overline{G}$), for the norms $\|P_n\|_{\mathcal{L}_p(h,L)}$ or $\|P_n\|_{A_p(h,G)}$, $p>0$, for some weight functions $h(z)$ ($h(z)\equiv 1$ or $h(z) \ne 1$) were investigated in [1]-[7], [8], [22]-[28], [31, pp. 418-428], [34], [35, Sect. 5.3], [36], [37, pp.122-133], [39], [38], [42], [44] (see also the references cited therein) and others.

2. Main results

Let $U_\infty(L,\delta) := \bigcup\limits_{\zeta\in L} U(\zeta,\ \delta)$ show the infinite open cover of the curve L and $U_N(L,\delta) := \bigcup\limits_{j=1}^{N} U_j(L,\delta) \subset U_\infty(L,\delta)$ denote the finite open cover of the curve L where $\delta := \min\limits_{1\le j\le l}\delta_j$ for $L=\partial G$ and $0<\delta_j<\delta_0 := \frac{1}{4}\min\left\{\left|z_i-z_j\right| : i,j=\overline{1,l},\ i\ne j\right\}$. Besides that $\Omega_t(z_j,\ \delta_j) := \Omega_t \cap \left\{z : \left|z-z_j\right| \le \delta_j\right\}$; $\Omega_t(\delta) := \bigcup\limits_{j=1}^{l} \Omega_t(z_j,\ \delta)$, $\widehat{\Omega}_t(\delta) := \Omega_t \setminus \Omega_t(\delta)$ for $t \ge 1$. Additionally, let $\Delta_j := \Phi(\Omega_t(z_j,\ \delta))$, $\Delta_t(\delta) := \bigcup\limits_{j=1}^{l} \Phi(\Omega_t(z_j,\ \delta))$, $\widehat{\Delta}_t(\delta) := \Delta_t \setminus \Delta_t(\delta)$. Clearly, $\Omega_t = \bigcup\limits_{j=1}^{l} \Omega_{t,j}$. $F^i := \Phi(L^i) = \overline{\Delta}_{t,i}' \cap \{\tau : |\tau|=1\}$, $F_t^i := \Phi(L_t^i) = \overline{\Delta}_{t,i}' \cap \{\tau : |\tau|=t\}$ where $\Omega_{t,j} := \Psi(\Delta_{t,j}')$ and $L_t^j := L_t \cap \overline{\Omega}_{t,j}$, $i=\overline{1,l}$.

During this study, we will use the abbreviations defined below

$$
\begin{aligned}
p_1^i &:= \frac{1+(\gamma_i+1)(1+\widetilde{\kappa})}{2+\widetilde{\kappa}};\ p_2^i := \frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\kappa)+(1+\beta_i)}; \\
p_3 &:= \frac{(\gamma_2+1)(1+\kappa)-(1+\widetilde{\kappa})(1+\beta_2)}{(1+\widetilde{\kappa})(1+\beta_2)-(1+\kappa)};\ p_4^i := 1+(\gamma_i+1)(1+\widetilde{\kappa}); \\
p_5^i &:= 1+(\gamma_i+1)\frac{1+\kappa}{1+\beta_i};\ p_6^i(m) := \frac{(\gamma_i+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-(m-1)(1+\kappa)}; \\
p_6(2) &:= \frac{(\gamma_2+1)(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-(1+\kappa)};\ p_6^*(m;i) := \frac{(\gamma_i^*+1)(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-m(1+\kappa)}; \\
p_7 &:= \frac{(\gamma_2+1)(1+\kappa)-(1+\widetilde{\kappa})(1+\beta_2)}{\left[(1+\widetilde{\kappa})(1+\beta_2)-(1+\kappa)\right](m-1)};\ p_8^i := \frac{(1+\kappa)+(1+\beta_i)}{(1+\beta_i)-(m-1)(1+\kappa)};\ i=1,2; \\
\gamma_k^* &:= \max\{0;\gamma_k,\ k=\overline{1,l}\};\ \widetilde{\gamma_2} := \frac{\gamma_2(1+\kappa)}{(1+\kappa)+(1+\beta_2)}\cdot\frac{2+\widetilde{\kappa}}{1+\widetilde{\kappa}};\ \widetilde{\gamma_3} := \frac{\gamma_2(1+\kappa)}{(1+\widetilde{\kappa})(1+\beta_2)}; \\
\widetilde{\gamma_4} &:= 2\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)}-1\right];\ \widetilde{\gamma_5} := \frac{(\gamma_2+1)(1+\kappa)}{(1+\widetilde{\kappa})(1+\beta_2)}-1;\ \widetilde{\gamma_6} := m\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)}-1\right]; \\
\widetilde{\gamma_1}(m) &:= \frac{\left[(\gamma_2+1)+p(m-1)\right](1+\kappa)}{(1+\widetilde{\kappa})(1+\beta_2)}-p(m-1)-1;\ \widetilde{\gamma_7} := (p+1)\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{(1+\kappa)}-1\right]
\end{aligned}
\tag{10}
$$

where $\gamma_i > -1$, $\alpha_i \geq 0$, $\beta_i > 0$, and $\widetilde{\kappa} = \begin{cases} 1, & \alpha_i > 0, \\ \kappa, & \alpha_i = 0, \end{cases}$ with $0 < \kappa < 1$ for any $i = \overline{1,l}$ and $m \geq 1$. Now, we start to formulate the new results. We note that all parameters p and γ with different labels are taken from (10). Firstly we give recurrent estimate for $\left|P_n^{(m)}(z)\right|$ with $m = 1, 2, ...$

Theorem 2.1. *Let $p > 1$; $G \in \widetilde{PQ}\left(\kappa; f_i, g_i\right)$, for some $0 \leq \kappa < 1$, $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1,l_1}$, and $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{l_1+1,l}$; $h(z)$ be defined as in (1). Then the inequality*

$$
\left|P_n^{(m)}(z)\right| \leq c_1 \left|\Phi^{n+1}(z)\right| \left\{ \frac{\|P_n\|_p}{d(z,L)} A^1_{n,p}(z,m) + \sum_{v=1}^{m} C_m^v B_{n,v}(z) \left|P_n^{(m-v)}(z)\right| \right\} \tag{11}
$$

holds for any $\gamma_i > -1$, $i = \overline{1,l}$, $P_n \in \wp_n$, $n \in \mathbb{N}$, and each $m = 1, 2, ...$ where $c_1 = c_1(L, \gamma_i, \beta, m, p) > 0$ is a constant independent of n and z. Here, $C_m^v := \frac{m(m-1)...(m-v+1)}{v!}$ and

$$A^1_{n,p}(z,m) :=$$

$$
= \begin{cases}
\sum\limits_{i=1}^{l_1} n^{\left(\frac{\gamma_i+1}{p}+m-1\right)(1+\widetilde{\kappa})}, & p > 1, \quad m \geq 2 \\
\sum\limits_{i=1}^{l_1} n^{\frac{\gamma_i+1}{p}(1+\widetilde{\kappa})}, & p < p_4^i, \quad m = 1 \\
(n \ln n)^{1-\frac{1}{p}}, & p = p_4^i, \quad m = 1 \\
n^{1-\frac{1}{p}}, & p > p_4^i, \quad m = 1
\end{cases}
+ \begin{cases}
\sum\limits_{i=l_1+1}^{l} n^{\left(\frac{\gamma_i+1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_i}}, & p > 1, \quad m \geq 2,\ \beta_i < (m-1)(1+\kappa)-1, \\
\sum\limits_{i=l_1+1}^{l} n^{\left(\frac{\gamma_i+1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_i}}, & p < p_6^i(m), \quad m \geq 2,\ \beta_i \geq (m-1)(1+\kappa)-1, \\
\sum\limits_{i=l_1+1}^{l} n^{\frac{\gamma_i+1}{p}\frac{1+\kappa}{1+\beta_i}}, & p < p_5^i, \quad m = 1, \beta_i > 0, \\
(n \ln n)^{1-\frac{1}{p}}, & p = p_5^i, \quad m = 1, \beta_i > 0, \\
n^{1-\frac{1}{p}}, & p > p_5^i, \quad m = 1, \beta_i > 0,
\end{cases}
$$

if $\gamma_i > 0$ and

$$A^1_{n,p}(z,m) :=$$

$$= \begin{cases} n^{\left(\frac{1}{p}+m-1\right)(1+\widetilde{\kappa})}, & p>1, \quad m\geq 2,\\ n^{\frac{1}{p}(1+\widetilde{\kappa})}, & p<2+\widetilde{\kappa}, \quad m=1,\\ (n\ln n)^{1-\frac{1}{p}}, & p=2+\widetilde{\kappa}, \quad m=1,\\ n^{1-\frac{1}{p}}, & p>2+\widetilde{\kappa}, \quad m=1, \end{cases} + \begin{cases} \sum\limits_{i=l_1+1}^{l} n^{\left(\frac{1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_i}}, & p>1, \quad m\geq 2, \ \beta_i\leq (m-1)(1+\kappa)-1,\\ \sum\limits_{i=l_1+1}^{l} n^{\left(\frac{1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_i}}, & 1<p<p_8^i, \quad m\geq 2, \ \beta_i> (m-1)(1+\kappa)-1,\\ \sum\limits_{i=l_1+1}^{l} n^{\frac{1}{p}\frac{1+\kappa}{1+\beta_i}}, & p<1+\frac{1+\kappa}{1+\beta_i}, \quad m=1, \beta_i>0,\\ (n\ln n)^{1-\frac{1}{p}}, & p=1+\frac{1+\kappa}{1+\beta_i}, \quad m=1, \beta_i>0,\\ n^{1-\frac{1}{p}}, & p>1+\frac{1+\kappa}{1+\beta_i}, \quad m=1, \beta_i>0, \end{cases}$$

if $-1<\gamma_i\leq 0$,

$$B^1_{n,v}(z) := n^{v(1+\widetilde{\kappa})}, \ v=\overline{1,m},$$

if $z\in\Omega_R(\delta)$;

$$A^1_{n,p}(z,m) := \begin{cases} \sum\limits_{i=1}^{l_1} n^{\left(\frac{\gamma_i+1}{p}-1\right)(1+\widetilde{\kappa})}, & p<p_1^i, \quad \gamma_i>0,\\ (n\ln n)^{1-\frac{1}{p}}, & p=p_1^i, \quad \gamma_i>0,\\ n^{1-\frac{1}{p}}, & p>p_1^i, \quad \gamma_i>0,\\ n^{\kappa(1-\frac{1}{p})} & p>1, \quad -1<\gamma_i\leq 0, \end{cases} + \begin{cases} \sum\limits_{i=l_1+1}^{l} n^{\frac{\gamma_i+1-p}{p(1+\beta_i)}(1+\kappa)}, & p<p_2^i, \quad \gamma_i>0,\\ (n\ln n)^{1-\frac{1}{p}}, & p=p_2^i, \quad \gamma_i>0,\\ n^{1-\frac{1}{p}}, & p>p_2^i, \quad \gamma_i>0,\\ n^{\kappa(1-\frac{1}{p})} & p>1, \quad -1<\gamma_i\leq 0, \end{cases}$$

$$B_{n,v}(z) := n^{\kappa}, \ v=\overline{1,m},$$

if $z\in\widehat{\Omega}_R(\delta)$.

Now, we assume that $i=1,2$; $l_1=1$, $l=2$ for simplicity of our presentations, i.e. the region G may have one interior zero (or nonzero) angle having " f_1-touching" with $f_1(x)=C_1x^{1+\alpha_1}$, $\alpha_1\geq 0$, at the point z_1 and exterior zero angle having "g_2-touching" with $g_2(x)=C_2x^{1+\beta_2}$, $\beta_2>0$, at the point z_2, for some constants $-\infty<C_1<+\infty$, $-\infty<C_2<+\infty$, where $C_1:=C_1(c_1,c_2)$, $C_2:=C_2(c_3,c_4)$ and constants c_i, $i=\overline{1,4}$, taken from Definition 1.2. In this case, combining the terms relating to the inner and outer corners, we obtain the following result.

Corollary 2.2. *Let* $p>1$; $G\in\widetilde{PQ}\left(\kappa;f_1,\ g_2\right)$, *for some* $0\leq\kappa<1$, $f_1(x)=C_1x^{1+\alpha_1}$, $\alpha_1\geq 0$, *and* $g_2(x)=C_2x^{1+\beta_2}$, $\beta_2>0$; $h(z)$ *defined as in (1) for* $l=2$. *Then, the following inequality holds*

$$\left|P_n^{(m)}(z)\right|\leq c_2\left|\Phi^{n+1}(z)\right|\left\{\frac{\|P_n\|_p}{d(z,L)}A^2_{n,p}(z,m)+\sum_{v=1}^{m}C_m^vB^2_{n,v}(z)\left|P_n^{(m-v)}(z)\right|\right\} \tag{12}$$

for any $\gamma_i>-1$, $i=1,2$, *and* $P_n\in\wp_n$, $n\in\mathbb{N}$, *where* $c_2=c_2(L,\gamma_i,\beta,m,p)>0$ *is a constant independent of* n. *Here,*

$$A^2_{n,p}(z,m) := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+m-1\right)(1+\widetilde{\kappa})}, & 1<p<p_7 \quad \gamma_1\geq\widetilde{\gamma}_1(m), \quad \gamma_2>\widetilde{\gamma}_6, \quad \beta_2>0,\\ n^{\left(\frac{\gamma_1+1}{p}+m-1\right)(1+\widetilde{\kappa})}, & p>1, \quad \gamma_1>0, \quad \gamma_2\leq\widetilde{\gamma}_6, \quad \beta_2<(m-1)(1+\kappa)-1,\\ n^{\left(\frac{\gamma_2+1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_2}}, & 1<p<p_7, \quad 0<\gamma_1<\widetilde{\gamma}_1(m), \quad \gamma_2>\widetilde{\gamma}_6, \quad \beta_2>0, \end{cases}$$

if $\gamma_1,\gamma_2>0$,

$$A^2_{n,p}(z,m) = \begin{cases} n^{\left(\frac{1}{p}+m-1\right)(1+\widetilde{\kappa})}, & p>1, \quad m\geq 2, \quad \beta_2\leq (m-1)(1+\kappa)-1,\\ n^{\left(\frac{1}{p}+m-1\right)(1+\widetilde{\kappa})}, & 1<p<p_8^2, \quad m\geq 2, \quad \beta_2>(m-1)(1+\kappa)-1,\\ n^{\frac{1}{p}(1+\widetilde{\kappa})}, & p<2+\widetilde{\kappa}, \quad m=1, \quad \beta_2>0,\\ (n\ln n)^{1-\frac{1}{p}}, & p=2+\widetilde{\kappa}, \quad m=1, \quad \beta_2>0,\\ n^{1-\frac{1}{p}}, & p>2+\widetilde{\kappa} \quad m=1, \quad \beta_2>0, \end{cases}$$

if $-1 < \gamma_1, \gamma_2 \le 0$ *and*

$$B^2_{n,v}(z) := n^{v(1+\widetilde{\kappa})}, v = \overline{1,m},$$

for the $z \in \Omega_R(\delta)$ *and*

$$A^2_{n,p}(z,m) = \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & 1 < p < p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_3, \quad \beta_2 > 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \quad \beta_2 > 0, \\ n^{1-\frac{1}{p}}, & p > p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \quad \beta_2 > 0, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}}, & 1 < p < p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_3, \quad \beta_2 > 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \quad \beta_2 > 0, \\ n^{1-\frac{1}{p}}, & p > p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \quad \beta_2 > 0, \end{cases}$$

if $\gamma_1, \gamma_2 > 0$,

$$A^2_{n,p}(z,m) = \left\{ n^{\kappa(1-\frac{1}{p})} \quad p > 1, \quad \beta_2 > 0, \right.$$

if $-1 < \gamma_1, \gamma_2 \le 0$ *and* $B_{n,v}(z) := n^{\kappa}, v = \overline{1,m},$ *for the* $z \in \widehat{\Omega}_R(\delta)$.

The formula (11) allows one to sequentially obtain an estimate for $\left|P_n^{(m)}(z)\right|$, for each $m \ge 1$. First, we get an estimate for $\left|P_n'(z)\right|$ by setting $m = 1$ and using the estimate $|P_n(z)|$. In case of $m \ge 2$, the estimations are made sequentially by applying the estimates (11) (or (12)).

First, let's give the evaluation for $|P_n(z)|$.

Theorem 2.3. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa; f_1, g_2\right)$, *for some* $0 \le \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \ge 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, the following inequality holds*

$$|P_n(z)| \le c_3 \frac{\left|\Phi^{n+1}(z)\right| \|P_n\|_p}{d(z,L)} A^3_{n,p}(z) \tag{13}$$

for any $\gamma_i > -1$, $i = 1,2$, $P_n \in \wp_n$, $n \in \mathbb{N}$, *and* $z \in \Omega_R$, *where* $c_3 = c_3(L, \gamma_i, \beta, p) > 0$ *is a constant independent of n and z. Here,*

$$A^3_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & 1 < p < p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_3, \quad \gamma_2 > 0, \quad \beta_2 > 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \quad \gamma_2 > 0, \quad \beta_2 > 0, \\ n^{1-\frac{1}{p}}, & p > p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \quad \gamma_2 > 0, \quad \beta_2 > 0, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}}, & 1 < p < p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_3, \quad \gamma_2 > 0, \quad \beta_2 > 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \quad \gamma_2 > 0, \quad \beta_2 > 0, \\ n^{1-\frac{1}{p}}, & p > p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \quad \gamma_2 > 0, \quad \beta_2 > 0, \\ n^{\kappa(1-\frac{1}{p})} & p > 1, \quad -1 < \gamma_1 \le 0, \quad -1 < \gamma_2 \le 0, \quad \beta_2 > 0. \end{cases}$$

Note that, analogous result for $|P_n(z)|$, $p > 0$, obtained in [14]. But, this theorem for $p > 2$ gives a better estimate.

According to Corollary 2.2 and Theorem 2.3, we obtain the following result for $\left|P_n'(z)\right|$ at each point $z \in \Omega_R$.

Theorem 2.4. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa; f_1, g_2\right)$, *for some* $0 \le \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \ge 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, the inequality*

$$\left|P_n'(z)\right| \le c_4 \frac{\left|\Phi^{2(n+1)}(z)\right| \|P_n\|_p}{d(z,L)} A^4_{n,p}(z) \tag{14}$$

holds for any $\gamma_i > -1,\ i = 1,2,\ \beta_2 > 0,\ P_n \in \wp_n,\ n \in \mathbb{N}$, *and* $z \in \Omega_R$, *where* $c_4 = c_4(L, \gamma_i, \beta, p) > 0$ *is a constant independent of n and z. Here,*

$$A^4_{n,p}(z) := \begin{cases} n^{\frac{\gamma_1+1}{p}(1+\widetilde{\kappa})}, & 1 < p < p_1^1, \quad \gamma_1 \geq \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+(1+\widetilde{\kappa})}, & 1 < p < p_1^1, \quad \widetilde{\gamma}_5 \leq \gamma_1 < \widetilde{\gamma}_3, \\ n^{\left(1-\frac{1}{p}\right)+(1+\widetilde{\kappa})}(\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{\left(1-\frac{1}{p}\right)+(1+\widetilde{\kappa})}, & p > p_1^1, \quad \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{\left(1-\frac{1}{p}\right)+(1+\widetilde{\kappa})}, & p > p_4^1, \quad \widetilde{\gamma}_5 \leq \gamma_1 < \widetilde{\gamma}_2, \\ n^{\left(1-\frac{1}{p}\right)+(1+\widetilde{\kappa})}(\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad \widetilde{\gamma}_5 \leq \gamma_1 < \widetilde{\gamma}_2, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+(1+\widetilde{\kappa})}, & 1 < p < p_2^2, \quad \gamma_1 < \widetilde{\gamma}_5, \\ n^{\left(1-\frac{1}{p}\right)+(1+\widetilde{\kappa})}, & p > p_2^2, \quad \gamma_1 < \widetilde{\gamma}_2, \end{cases}$$

if $\gamma_1, \gamma_2 > 0$,

$$A^4_{n,p}(z) := n^{\kappa(1-\frac{1}{p})+(1+\widetilde{\kappa})},\ p > 1,$$

if $-1 < \gamma_1, \gamma_2 \leq 0$ *for the* $z \in \Omega_R(\delta)$;

$$A^4_{n,p}(z) := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})+\kappa}, & 1 < p < p_1^1, \quad \gamma_1 \geq \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+\kappa}, & 1 < p < p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_3, \\ n^{1-\frac{1}{p}+\kappa}(\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+\kappa}(\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+\kappa}, & p > p_1^1, \quad \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+\kappa}, & p > p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \end{cases}$$

if $\gamma_1, \gamma_2 > 0$,

$$A^4_{n,p}(z) := n^{\kappa(1-\frac{1}{p})+\kappa},\ p > 1,$$

if $-1 < \gamma_1, \gamma_2 \leq 0$ *for the* $z \in \widehat{\Omega}_R(\delta)$.

By considering Theorem 2.4 for $\left|P_n'(z)\right|$ and Theorem 2.3 for $|P_n(z)|$ in Corollary 2.2, we have the following result.

Theorem 2.5. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa; f_1,\ g_2\right)$, *for some* $0 \leq \kappa < 1,\ f_1(x) = C_1 x^{1+\alpha_1}, \alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2},\ \beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, the inequality*

$$\left|P_n''(z)\right| \leq c_5 \frac{\left|\Phi^{3(n+1)}(z)\right| \|P_n\|_p}{d(z,L)} A^6_{n,p}(z) \tag{15}$$

holds for any $\gamma_i > -1,\ i = 1,2,\ P_n \in \wp_n,\ n \in \mathbb{N}$, *and* $z \in \Omega_R$, *where* $c_5 = c_5(L, \gamma_i, \beta, p) > 0$ *is a constant independent of n and z. Here,*

$$A^6_{n,p}(z) :=$$

$$= \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+1\right)(1+\widetilde{\kappa})}, & 1<p<p_3, & \gamma_1 \geq \widetilde{\gamma}_1(2), \ \beta_2 \leq \kappa, \\ n^{\left(\frac{\gamma_1+1}{p}+1\right)(1+\widetilde{\kappa})}, & p \geq p_3, & \gamma_1 > 0, \ \beta_2 \leq \kappa, \\ n^{\left(\frac{\gamma_2+1}{p}+1\right)\frac{1+\kappa}{1+\beta_2}}, & p_3 \leq p < p_6(2), & \gamma_1 < \widetilde{\gamma}_1(2), \ \beta_2 > \kappa, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+2\right)(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \gamma_1 \geq \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+3(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \widetilde{\gamma}_5 \leq \gamma_1 < \widetilde{\gamma}_3, \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}(\ln n)^{1-\frac{1}{p}}, & p=p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}, & p>p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}, & p>q_1, & \widetilde{\gamma}_5 \leq \gamma_1 < \widetilde{\gamma}_2, \\ n^{\left(1-\frac{1}{p}\right)+2(1+\widetilde{\kappa})}(\ln n)^{1-\frac{1}{p}}, & p=p_2^2, & \widetilde{\gamma}_5 \leq \gamma_1 < \widetilde{\gamma}_2, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+3(1+\widetilde{\kappa})}, & 1<p<p_2^2, & \gamma_1 < \widetilde{\gamma}_5, \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}, & p>p_2^2, & \gamma_1 < \widetilde{\gamma}_2, \end{cases}$$

if $\gamma_1,\gamma_2 > 0,$

$$A^6_{n,p}(z) = \begin{cases} n^{\kappa(1-\frac{1}{p})+3(1+\widetilde{\kappa})}, & p>1, & \beta_2 \leq \kappa, \\ n^{\kappa(1-\frac{1}{p})+3(1+\widetilde{\kappa})}, & 1<p<\frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-(1+\kappa)}, & \beta_2 > \kappa, \end{cases}$$

if $-1 < \gamma_1,\gamma_2 \leq 0,$ *for the* $z \in \Omega_R(\delta)$*;*

$$A^6_{n,p}(z) = \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \gamma_1 \geq \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}}, & 1<p<p_2^2, & 0<\gamma_1<\widetilde{\gamma}_3, \\ (n\ln n)^{1-\frac{1}{p}}, & p=p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, \\ (n\ln n)^{1-\frac{1}{p}}, & p=p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}}, & p>p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}}, & p>p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})+2\kappa}, & 1<p<p_1^1, & \gamma_1 \geq \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+2\kappa}, & 1<p<p_2^2, & 0<\gamma_1<\widetilde{\gamma}_3, \\ n^{1-\frac{1}{p}+2\kappa}(\ln n)^{1-\frac{1}{p}}, & p=p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+2\kappa}(\ln n)^{1-\frac{1}{p}}, & p=p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+2\kappa}, & p>p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+2\kappa}, & p>p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, \end{cases}$$

if $\gamma_1, \gamma_2 > 0$,

$$A^6_{n,p}(z) := n^{\kappa(1-\frac{1}{p})+2\kappa}$$

if $-1 < \gamma_1,\gamma_2 \leq 0,$ *for the* $z \in \widehat{\Omega}_R(\delta).$

Now, we can state estimates for $\left|P_n^{(m)}(z)\right|$, $m \geq 0$.

Theorem 2.6. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa\,; f_i,\ g_i\right)$, *for some* $0 \leq \kappa < 1$, $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, l_1}$, *and* $g_i(x) = c_i x^{1+\beta_i}, \beta_i > 0,\ i = \overline{l_1+1, l}$; $h(z)$ *be defined as in (1). Then, the inequality*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_6 \left(\sum_{i=1}^{l_1} n^{\left(\frac{\gamma_i^*+1}{p}+m\right)(1+\widetilde{\kappa})} + \sum_{i=l_1+1}^{l} n^{\left(\frac{\gamma_i^*+pm}{1+\beta_i}+1\right)\frac{(1+\kappa)}{p}} \right) \|P_n\|_p\,, \tag{16}$$

holds for any $\gamma_i > -1,\ i = \overline{1, l}$, *and* $P_n \in \wp_n,\ n \in \mathbb{N}$ *where* $c_6 = c_6(L, \gamma_i, \beta, p) > 0$ *is a constant independent of* n *and* z.

Analogously to Corollary 2.2, we obtain the next result for $i = 1, 2;\ l_1 = 1$, and $l = 2$.

Corollary 2.7. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa\,; f_1,\ g_2\right)$, *for some* $0 \leq \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$, *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *defined as in (1) for* $l = 2$. *Then, the inequality*

$$\left\|P_n^{(m)}\right\|_\infty \leq c_7 M_n(m)\, \|P_n\|_p\,,\ m \geq 0 \tag{17}$$

holds for any $\gamma_i > -1,\ i = 1, 2$, *and* $P_n \in \wp_n,\ n \in \mathbb{N}$, *where* $c_7 = c_7(L, \gamma_i, \beta, p) > 0$ *is a constant independent of n and z. Here,*

$$M_n(m) := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+m\right)(1+\widetilde{\kappa})}, & \gamma_1 \geq \left(\frac{\gamma_2+pm}{1+\beta_2}+1\right)\frac{1+\kappa}{1+\widetilde{\kappa}} - pm - 1, \quad \gamma_2 \geq (pm+1)\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{1+\kappa} - 1\right] - \beta_2, \\ n^{\left(\frac{\gamma_2+pm}{1+\beta_2}+1\right)\frac{1+\kappa}{p}}, & 0 < \gamma_1 < \left(\frac{\gamma_2+pm}{1+\beta_2}+1\right)\frac{1+\kappa}{1+\widetilde{\kappa}} - pm - 1, \quad \gamma_2 \geq (pm+1)\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{1+\kappa} - 1\right] - \beta_2, \\ n^{\left(\frac{\gamma_1+1}{p}+m\right)(1+\widetilde{\kappa})}, & \gamma_1 > 0, \quad \gamma_2 < (pm+1)\left[\frac{(1+\widetilde{\kappa})(1+\beta_2)}{1+\kappa} - 1\right] - \beta_2, \\ n^{\left(\frac{1}{p}+m\right)(1+\widetilde{\kappa})}, & -1 < \gamma_1 \leq 0, \quad -1 < \gamma_2 \leq 0. \end{cases}$$

Remark 2.8. ([39, Remark 2.16]) The inequality (17) is sharp.

Combining (17) with (13), (14) and (15), we obtain the estimations on the growth of $|P_n(z)|$, $|P_n'(z)|$ and $\left|P_n''(z)\right|$, respectively, in the whole complex plane.

Theorem 2.9. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa; f_i,\ g_i\right)$, *for some* $0 \leq \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, we have the inequality*

$$|P_n(z)| \leq c_8 \|P_n\|_p \begin{cases} M_n(0), & z \in \overline{G}_R, \\ \frac{|\Phi^{n+1}(z)|}{d(z,L)} A^3_{n,p}(z), & z \in \Omega_R, \end{cases}$$

for any $\gamma_i > -1,\ i = 1, 2$, $P_n \in \wp_n,\ n \in \mathbb{N}$, *and* $z \in \Omega$, *where* $c_8 = c_8(L, \gamma_i, p) > 0$ *is a constant independent of n and z. Here,* $M_n(0)$ *and* $A^3_{n,p}(z)$ *are defined as in Corollary 2.7 for* $m = 0$, $z \in \overline{G}_R$ *and Theorem 2.3 for all* $z \in \Omega_R$, *respectively.*

Theorem 2.10. *Let* $p > 1$; $G \in \widetilde{PQ}\left(\kappa; f_i,\ g_i\right)$, *for some* $0 \leq \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, we obtain*

$$\left|P_n'(z)\right| \leq c_9 \|P_n\|_p \begin{cases} M_n(1), & z \in \overline{G}_R, \\ \frac{|\Phi^{2(n+1)}(z)|}{d(z,L)} A^4_{n,p}(z), & z \in \Omega_R, \end{cases}$$

for any $\gamma_i > -1,\ i = 1, 2$, $P_n \in \wp_n,\ n \in \mathbb{N}$, *and* $z \in \Omega$, *where* $c_9 = c_9(L, \gamma_i, p) > 0$ *is a constant independent of n and z. Here,* $M_n(1)$ *and* $A^4_{n,p}(z)$ *are defined as in Corollary 2.7 for* $m = 0$, $z \in \overline{G}_R$ *and Theorem 2.4 for all* $z \in \Omega_R$, *respectively.*

Theorem 2.11. *Let* $p > 1$; $G \in PQS\left(\lambda; f_1, g_2\right)$, *for some* $0 \leq \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ *and* $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$; $h(z)$ *be defined as in (1) for* $l = 2$. *Then, we get*

$$\left|P_n''(z)\right| \leq c_{10} \|P_n\|_p \begin{cases} M_n(2), & z \in \overline{G}_R, \\ \frac{|\Phi^{3(n+1)}(z)|}{d(z,L)} A^6_{n,p}(z), & z \in \Omega_R, \end{cases}$$

for any $\gamma_i > -1,\ i = 1, 2$, $P_n \in \wp_n,\ n \in \mathbb{N}$, *and* $z \in \Omega_R$, *where* $c_{10} = c_{10}(L, \gamma_i, p) > 0$ *is a constant independent of n and z. Here,* $M_n(2)$ *and* $A^6_{n,p}(z)$ *are defined as in Corollary 2.7 for all* $z \in \overline{G}_R$ *and Theorem 2.5 for all* $z \in \Omega_R$, *respectively.*

Therefore, by using Theorem 2.1 and the estimation $\left|P_n^{(m)}(z)\right|$ sequentially for each $m \geq 3$, and by combining the obtained results in Corollary 2.7 we obtain the estimates for $\left|P_n^{(m)}(z)\right|$, at each $z \in \mathbb{C}$.

3. Some auxiliary results

For $a>0$ and $b>0$ we use expressions "$a \preceq b$" and "$a \asymp b$" if $a \leq cb$ and $c_1 a \leq b \leq c_2 a$ for some constants c, c_1, c_2, respectively.

Lemma 3.1. ([5]) *Let G be a quasidisk, $z_1 \in L$, $z_2, z_3 \in \Omega \cap \{z : |z - z_1| \leq d(z_1, L_{r_0})\}$; $w_j = \Phi(z_j)$, $j = 1,2,3$. Then*

a) The statements $|z_1 - z_2| \preceq |z_1 - z_3|$ and $|w_1 - w_2| \preceq |w_1 - w_3|$ are equivalent. Therefore, $|z_1 - z_2| \asymp |z_1 - z_3|$ and $|w_1 - w_2| \asymp |w_1 - w_3|$ also are equivalent.

b) If $|z_1 - z_2| \preceq |z_1 - z_3|$, then

$$\left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_1} \preceq \left|\frac{z_1 - z_3}{z_1 - z_2}\right| \preceq \left|\frac{w_1 - w_3}{w_1 - w_2}\right|^{c_2},$$

where $0 < r_0 < 1$ a constant, depending on G.

Corollary 3.2. *Under the conditions of Lemma 3.1, we have*

$$|w_1 - w_2|^{c_1} \preceq |z_1 - z_2| \preceq |w_1 - w_2|^{\varepsilon}$$

where $\varepsilon = \varepsilon(G) < 1$.

Lemma 3.3. *Let $G \in Q(\kappa)$ for some $0 \leq \kappa < 1$. Then*

$$|\Psi(w_1) - \Psi(w_2)| \succeq |w_1 - w_2|^{1+\kappa}$$

for all $w_1, w_2 \in \overline{\Delta}$.

This fact follows from an appropriate result for the mapping $f \in \sum(\kappa)$[40, p.287] and estimation for the Ψ'[24, Th.2.8]:

$$\frac{d(\Psi(\tau), L)}{|\tau| - 1} \asymp |\Psi'(\tau)|. \tag{18}$$

Let $\left\{z_j\right\}_{j=1}^{l}$ be a fixed system of the points on L and the weight function $h(z)$ defined as (1).

Lemma 3.4. ([2]) *Let $L = G$ be a rectifiable Jordan curve and $P_n(z)$, be an arbitrary polynomial with* $\deg P_n \leq n$, $n = 1,2,...$, *and $h(z)$ satisfies the condition (1). Then for any $R > 1$, $p > 0$ and $n = 1,2,...$*

$$\|P_n\|_{\mathcal{L}_p(h,L_R)} \preceq R^{n+\frac{1+\gamma^*}{p}} \|P_n\|_{\mathcal{L}_p(h,L)}, \ \gamma^* = \max\left\{0; \gamma_j : 1 \leq j \leq l\right\}.$$

4. Proofs of theorems

Proof of Theorem 2.1. Assume that $G \in \widetilde{PQ}\left(\kappa; f_i, g_i\right)$, for some $0 < \kappa < 1$, $f_i(x) = C_i x^{1+\alpha_i}$, $\alpha_i \geq 0$, $i = \overline{1, l_1}$, and $g_i(x) = C_i x^{1+\beta_i}$, $\beta_i > 0$, $i = \overline{l_1 + 1, l}$. Moreover, let $R_1 := 1 + \frac{R-1}{2}$ where $R = 1 + \frac{1}{n}$ and $H_n(z) := \frac{P_n(z)}{\Phi^{n+1}(z)}$ for $z \in \Omega$. The $m-$ th derivative of $H_n(z)$ is in the form

$$H_n^{(m)}(z) = \sum_{v=0}^{m} C_m^v \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(v)} P_n^{(m-v)}(z) = \frac{P_n^{(m)}(z)}{\Phi^{n+1}(z)} + \sum_{v=1}^{m} C_m^v \left(\frac{1}{\Phi^{n+1}(z)}\right)^{(v)} P_n^{(m-v)}(z),$$

where $C_m^v := \frac{m(m-1)...(m-v+1)}{v!}$. After the transition to modulus, we get that

$$\left|P_n^{(m)}(z)\right| \leq \left|\Phi^{n+1}(z)\right| \left\{\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right| + \sum_{v=1}^{m} C_m^v \left|\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(v)}\right| \left|P_n^{(m-v)}(z)\right|\right\}. \tag{19}$$

Therefore, it is sufficient to evaluate the cases A)$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right|$, $m=1,2,...$; B) $\left|\left(\Phi^{-n-1}(z)\right)^{(v)}\right|$, $v=\overline{1,m}$ for calculation of $\left|P_n^{(m)}(z)\right|$ at the points $z\in\Omega$. Now let us consider the evaluations of the cases A) and B).

A) Since the function $H_n(z)$ is analytic in Ω, continuous on $\overline{\Omega}$ and $H_n(\infty)=0$, then the Cauchy integral representation for the $m-$ th derivative gives that

$$H_n^{(m)}(z)=-\frac{1}{2\pi i}\int\limits_{L_{R_1}}H_n(\zeta)\frac{d\zeta}{(\zeta-z)^{m+1}},\ z\in\Omega_R,\ m\geq 1.$$

Then we get

$$\left|\left(\frac{P_n(z)}{\Phi^{n+1}(z)}\right)^{(m)}\right|\leq\frac{1}{2\pi}\int\limits_{L_{R_1}}\left|\frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|^{m+1}}\leq\frac{1}{2\pi d(z,L_{R_1})}\int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|^{m}}. \tag{20}$$

Now let us estimate the integral denoted by

$$A_n(z):=\int\limits_{L_{R_1}}|P_n(\zeta)|\frac{|d\zeta|}{|\zeta-z|^{m}}. \tag{21}$$

Multiplying the numerator and denominator of the integrand by $h^{1/p}(\zeta)$, according to the Hölder inequality, we obtain that

$$A_n(z)\leq\|P_n\|_p\left(\int\limits_{L_{R_1}}\frac{|d\zeta|}{h^{\frac{q}{p}}(\zeta)\,|\zeta-z|^{qm}}\right)^{\frac{1}{q}}. \tag{22}$$

For the last integral denote by $J_n(z)$, we get

$$\begin{aligned}[J_n(z)]^q\ :&=\int\limits_{L_{R_1}}\frac{|d\zeta|}{h^{q-1}(\zeta)\,|\zeta-z|^{qm}}=\sum_{i=1}^{l}\int\limits_{L_{R_1}^i}\frac{|d\zeta|}{\prod\limits_{j=1}^{l}\left|\zeta-z_j\right|^{(q-1)\gamma_j}|\zeta-z|^{qm}}\\ &\leq\ \sum_{i=1}^{l}\int\limits_{L_{R_1}^i}\frac{|d\zeta|}{|\zeta-z_i|^{(q-1)\gamma_i}\,|\zeta-z|^{qm}}=:\sum_{i=1}^{l}J_n^i(z)\end{aligned} \tag{23}$$

since the points $\left\{z_j\right\}_{j=1}^{l}$ are distinct on L. First of all, we need to introduce some notations to estimate the integrals $J_n^i(z)$, $i=\overline{1,l}$,. Let $w_j:=\Phi(z_j)$, $\varphi_j:=\arg w_j$. Without loss of generality, we will assume that $\varphi_l<2\pi$. For $\eta:=\min\left\{\eta_j,\ j=\overline{1,l}\right\}$, where $\eta_j=\min\limits_{t\in\partial\Phi(\Omega(z_j,\ \delta_j))}\left|t-w_j\right|>0$, let us set

$$\begin{aligned}\Delta(\eta_j)\ &:\ =\left\{t:\left|t-w_j\right|\leq\eta_j\right\}\subset\Phi(\Omega(z_j,\ \delta_j)),\\ \Delta(\eta)\ &:\ =\bigcup_{j=1}^{l}\Delta_j(\eta),\ \widehat{\Delta}_j=\Delta\backslash\Delta(\eta_j);\widehat{\Delta}(\eta):=\Delta\backslash\Delta(\eta);\ \Delta_1^{'}:=\Delta_1^{'}(1),\\ \Delta_1^{'}(\rho)\ &:\ =\left\{t=Re^{i\theta}:R\geq\rho>1,\ \frac{\varphi_0+\varphi_1}{2}\leq\theta<\frac{\varphi_1+\varphi_2}{2}\right\},\\ \Delta_j^{'}\ &:\ =\Delta_j^{'}(1),\Delta_j^{'}(\rho):=\left\{t=Re^{i\theta}:R\geq\rho>1,\ \frac{\varphi_{j-1}+\varphi_j}{2}\leq\theta<\frac{\varphi_j+\varphi_0}{2}\right\},\ j=2,3...,\ l,\end{aligned}$$

where

$$\varphi_0 \quad : \quad = 2\pi - \varphi_l;\ \Omega_j := \Psi(\Delta_j^{'}), L_{R_1}^j := L_{R_1} \cap \Omega_j;\ \Omega = \bigcup_{j=1}^{l} \Omega_j.$$

It remains to estimate the integrals $J_n^i(z)$ for each $i = \overline{1,l}$. For simplicity of our next calculations, we assume that

$$l_1 = 1,\ l_2 = 2,\ i = 1,2;\ z_1 = -1,\ z_2 = 1;\ (-1,1) \subset G;\ R = 1 + \frac{\varepsilon_0}{n}, \tag{24}$$

and let us consider $L = L^+ \cup L^-$, where $L^+ := \{z \in L : Imz \geq 0\}$, $L^- := \{z \in L : Imz < 0\}$; by taking local coordinate axis in Definition 1.2. Moreover, let $w^\pm := \left\{w = e^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, $z^\pm \in \Psi(w^\pm)$ and let L^i be arcs which is connecting the points $z^+, z_i, z^- \in L$ and $L^{i,\pm} := L^i \cap L^\pm$ for $i = 1,2$. For simplicity, without loss of generality, we assume that $z_0 = z^+$ ($z_0 = z^-$) where z_0 is taken as an arbitrary point on L^+ (or on L^- subject to the chosen direction). Analogously to the previous notations, we introduce $L_R = L_R^+ \cup L_R^-$, where $L_R^+ := \{z \in L_R : Imz \geq 0\}$, $L_R^- := \{z \in L_R : Imz < 0\}$ and let $w_R^\pm := \left\{w = Re^{i\theta} : \theta = \frac{\varphi_1 \pm \varphi_2}{2}\right\}$, $z_R^\pm \in \Psi(w_R^\pm)$. On the other hand, we set $z_{i,R} \in L_R$ such that $d_{i,R} = \left|z_i - z_{i,R}\right|$ and $d(z_{2,R}, L^2 \cap L^\pm) := d(z_{2,R}, L^\pm)$; $z_i^\pm := \left\{\zeta \in L^i : |\zeta - z_i| = c_i d(z_i, L_R)\right\}$, $z_{i,R}^\pm := \left\{\zeta \in L_R^i : \left|\zeta - z_{i,R}\right| = c_i d(z_{i,R}, L_R)\right\}$, $w_{i,R}^\pm = \Phi(z_{i,R}^\pm)$. Let $L_R^{i,\pm} := L_R^i \cap L_R^\pm$ where L_R^i, $i = 1,2$, denote arcs, connecting the points $z_R^+, z_{i,R}, z_R^- \in L_R$, and $\left|l_{i,R}^\pm\right| := mes\ l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$, where $l_{i,R}^\pm(z_{i,R}^\pm, z_R^\pm)$ denote arcs connecting the points $z_{i,R}^\pm$ with $z_R^\pm$, and let $d_{i,R,R_1} := d(L_R^{i,\pm}, L_{R_1}^{i,\pm})$ for $i = 1,2$. Besides, we use the notations given by

$$\begin{aligned} E_{1,R_1}^{i,\pm} \quad &: \quad = \left\{\zeta \in L_{R_1}^{i,\pm} : |\zeta - z_i| < c_i d_{i,R_1}\right\}, \\ E_{2,R_1}^{i,\pm} \quad &: \quad = \left\{\zeta \in L_{R_1}^{i,\pm} : c_i d_{i,R_1} \leq |\zeta - z_i| \leq \left|l_{i,R_1}^\pm\right|\right\},\ F_{j,R_1}^{i,\pm} := \Phi(E_{j,R_1}^{i,\pm}); \\ E_1^{i,\pm} \quad &: \quad = \left\{\zeta \in L^{i,\pm} : |\zeta - z_i| < c_i d_{i,R_1}\right\}, \\ E_2^{i,\pm} \quad &: \quad = \left\{\zeta \in L^{i,\pm} : c_i d_{i,R_1} \leq |\zeta - z_i| \leq \left|l_{i,R_1}^\pm\right|\right\},\ F_j^{i,\pm} := \Phi(E_j^{i,\pm}),\ i, j = 1,2. \end{aligned} \tag{25}$$

Taking into consideration these designations and replacing the variable $\tau = \Phi(\zeta)$, from (18) and (23), we have

$$J_n^i(z) \asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R_1}^{i,+} \cup F_{j,R_1}^{i,-}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{(q-1)\gamma_i}\,|\Psi(\tau) - \Psi(w)|^{qm}} =: \sum_{i,j=1}^{2} \left[J(F_{j,R_1}^{i,+}) + J(F_{j,R_1}^{i,-})\right], \tag{26}$$

and we have

$$A_n(z) \quad \leq \quad \|P_n\|_p \sum_{i=1}^{2} \left[J_n^i(z)\right]^{\frac{1}{q}} =: \|P_n\|_p \sum_{i=1}^{2} \left[I_1^i(E_{1,R_1}^{i,+}) + I_2^i(E_{2,R_1}^{i,-})\right] =: \|P_n\|_p \sum_{i,k=1}^{2} \left[I_{k,R_1}^{i,+} + I_{k,R_1}^{i,-}\right], \tag{27}$$

from (26) where

$$\begin{aligned} I_{k,R_1}^{i,\pm} \quad &: \quad = I_{k,R_1}^i(E_{k,R_1}^{i,\pm}) := \int\limits_{F_{k,R_1}^{i,\pm}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{(q-1)\gamma_i}\,|\Psi(\tau) - \Psi(w)|^{qm}} \\ &\asymp \int\limits_{F_{k,R_1}^{i,\pm}} \frac{d(\Psi(\tau), L)\,|d\tau|}{|\Psi(\tau) - \Psi(w_i)|^{(q-1)\gamma_i}\,|\Psi(\tau) - \Psi(w)|^{qm}\,(|\tau| - 1)};\ i,\ k = 1,2. \end{aligned}$$

According to (26) and (27), it is sufficient to estimate the integrals $I_{n,k}^{i,\pm}$ for each $i = 1,2$ and $k = 1,2$.

Therefore, we consider the notations given below

$$F^{i,\pm}_{k,R_1,1} \quad : \quad = \left\{\tau \in F^{i,\pm}_{k,R_1} : |\Psi(\tau)-\Psi(w_i)| \geq |\Psi(\tau)-\Psi(w)|\right\},\ F^{i,\pm}_{k,R_1,2} := F^{i,\pm}_{k,R_1} \backslash F^{i,\pm}_{k,R_1,1},$$

$$I^{i,\pm}_{k,R_1,1} \quad : \quad = I\left(F^{i,\pm}_{k,R_1,1}\right) := \begin{cases} \int\limits_{F^{i,\pm}_{k,R_1,1}} \frac{d(\Psi(\tau),L)|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma_i(q-1)+qm}(|\tau|-1)}, & if\ \gamma_i > 0, \\ \int\limits_{F^{i,\pm}_{k,R_1,1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_i)(q-1)} d(\Psi(\tau),L)|d\tau|}{|\Psi(\tau)-\Psi(w)|^{qm}(|\tau|-1)}, & if\ \gamma_i \leq 0, \end{cases} \tag{28}$$

$$I^{i,\pm}_{k,R_1,2} \quad : \quad = I\left(F^{i,\pm}_{k,R_1,2}\right) := \int\limits_{F^{i,\pm}_{k,R_1,2}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{\gamma_i(q-1)+qm}\,(|\tau|-1)},$$

for any $k = 1, 2,$ and we will evaluate the integrals here. Given the possible values γ_i ($-1 < \gamma_i \leq 0$ and $\gamma_i > 0$), we will discuss the estimates for the $I^{i,\pm}_{n,k}$ separately. Now, we can start this estimations.

1. Let $z \in \Omega_R(\delta)$.

1.1. Let us calculate the integral $I^{1,+}_{k,R_1,j} + I^{1,-}_{k,R_1,j}$, for $k, j = 1, 2$ in case of $i = 1$.

1.1.1.1. For the integral $I^{1,+}_{1,R_1,1} + I^{1,-}_{1,R_1,1}$, we get that

$$I^{1,+}_{1,R_1,1} + I^{1,-}_{1,R_1,1} \leq n \cdot \int\limits_{F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma_1(q-1)+qm-1}} \leq n \cdot \int\limits_{F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}} \frac{|d\tau|}{|\tau - w|^{\left(\gamma_1(q-1)+qm-1\right)(1+\widetilde{\kappa})}} \tag{29}$$

$$\leq n \cdot n^{\left(\gamma_1(q-1)+qm-1\right)(1+\widetilde{\kappa})} \cdot mes\left(F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}\right) \leq n^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}$$

for $\gamma_1 > 0$ and

$$I^{1,+}_{1,R_1,1} + I^{1,-}_{1,R_1,1} \leq n \cdot d^{(-\gamma_1)(q-1)}_{1,R_1} d^{1-qm}_{1,R,R_1} \int\limits_{F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}} |d\tau| \tag{30}$$

$$\leq n^{1+\gamma_1(q-1)(1-\kappa)+\left(qm-1\right)(1+\widetilde{\kappa})} \cdot mes\left(F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}\right) \leq n^{\left(qm-1\right)(1+\widetilde{\kappa})+\gamma_1(q-1)(1-\kappa)}$$

for $-1 < \gamma_1 \leq 0$.

1.1.1.2. Analogously to the (29) and (30), for the integral $I^{1,+}_{1,R_1,2} + I^{1,-}_{1,R_1,2}$, we have

$$I^{1,+}_{1,R_1,2} + I^{1,-}_{1,R_1,2} \leq n \int\limits_{F^{1,+}_{1,R_1,2} \cup F^{1,-}_{1,R_1,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)+qm-1}} \tag{31}$$

$$\leq n \int\limits_{F^{1,+}_{2,R_1,2} \cup F^{1,-}_{2,R_1,2}} \frac{|d\tau|}{|\tau - w_1|^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}} \leq \begin{cases} n^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}, & \left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa}) > 1, \\ n \ln n, & \left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa}) = 1, \\ n, & \left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa}) < 1, \end{cases}$$

for $\gamma_1 > 0$ and

$$I^{1,+}_{1,R_1,2} + I^{1,-}_{1,R_1,2} \quad \leq \quad n \cdot \int\limits_{F^{1,+}_{1,R_1,2} \cup F^{1,-}_{1,R_1,2}} \frac{|d\tau|}{|\tau - w_1|^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}} \tag{32}$$

$$\leq \quad \begin{cases} n^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}, & \gamma_1(q-1)+qm-1 > \frac{1}{1+\widetilde{\kappa}}, \\ n \ln n, & \gamma_1(q-1)+qm-1 = \frac{1}{1+\widetilde{\kappa}}, \\ n, & \gamma_1(q-1)+qm-1 < \frac{1}{1+\widetilde{\kappa}}, \end{cases}$$

for $-1<\gamma_1\leq 0$.

1.1.2.1. In case $I^{1,+}_{2,R_1,1}+I^{1,-}_{2,R_1,1}$, we obtain

$$I^{1,+}_{2,R_1,1}+I^{1,-}_{2,R_1,1}\leq n\cdot\int\limits_{F^{1,+}_{2,R_1,1}\cup F^{1,-}_{2,R_1,1}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma_1(q-1)+qm-1}} \tag{33}$$

$$\leq n\cdot\int\limits_{F^{1,+}_{2,R_1,1}\cup F^{1,-}_{2,R_1,1}}\frac{|d\tau|}{|\tau-w|^{\left(\gamma_1(q-1)+qm-1\right)(1+\widetilde{\kappa})}}\preceq\begin{cases} n^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}, & \gamma_1(q-1)+qm-1>\frac{1}{1+\widetilde{\kappa}},\\ n\ln n, & \gamma_1(q-1)+qm-1=\frac{1}{1+\widetilde{\kappa}},\\ n, & \gamma_1(q-1)+qm-1<\frac{1}{1+\widetilde{\kappa}},\end{cases}$$

for $\gamma_1>0$ and

$$I^{1,+}_{2,R_1,1}+I^{1,-}_{2,R_1,1}\leq n\cdot\int\limits_{F^{1,+}_{2,R_1,1}\cup F^{1,-}_{2,R_1,1}}\frac{|d\tau|}{|\tau-w|^{\left[qm-1\right](1+\widetilde{\kappa})}}\preceq\begin{cases} n^{\left[qm-1\right](1+\widetilde{\kappa})}, & qm-1>\frac{1}{1+\widetilde{\kappa}},\\ n\ln n, & qm-1=\frac{1}{1+\widetilde{\kappa}},\\ n, & qm-1<\frac{1}{1+\widetilde{\kappa}},\end{cases} \tag{34}$$

for $-1<\gamma_1\leq 0$.

1.1.2.2. By analogy, for the integral $I^{1,+}_{2,R_1,2}+I^{1,-}_{2,R_1,2}$, we get

$$I^{1,+}_{2,R_1,2}+I^{1,-}_{2,R_1,2} \tag{35}$$

$$\leq n\cdot\int\limits_{F^{1,+}_{2,R_1,2}\cup F^{1,-}_{2,R_1,2}}\frac{|d\tau|}{|\tau-w_1|^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}}\preceq\begin{cases} n^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}, & \gamma_1(q-1)+qm-1>\frac{1}{1+\widetilde{\kappa}},\\ n\ln n, & \gamma_1(q-1)+qm-1=\frac{1}{1+\widetilde{\kappa}},\\ n, & \gamma_1(q-1)+qm-1<\frac{1}{1+\widetilde{\kappa}},\end{cases}$$

for $\gamma_1>0$ and

$$I^{1,+}_{2,R_1,2}+I^{1,-}_{2,R_1,2} \tag{36}$$

$$\leq n\cdot\int\limits_{F^{1,+}_{2,R_1,2}\cup F^{1,-}_{2,R_1,2}}\frac{|d\tau|}{|\tau-w_1|^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}}\preceq\begin{cases} n^{\left[\gamma_1(q-1)+qm-1\right](1+\widetilde{\kappa})}, & \gamma_1(q-1)+qm-1>\frac{1}{1+\widetilde{\kappa}},\\ n\ln n, & \gamma_1(q-1)+qm-1=\frac{1}{1+\widetilde{\kappa}},\\ n, & \gamma_1(q-1)+qm-1<\frac{1}{1+\widetilde{\kappa}},\end{cases}$$

for $-1<\gamma_1\leq 0$.

Therefore, we have

$$\sum_{k,j=1}^{2}I^{1,+}_{k,R_1,j}+I^{1,-}_{k,R_1,j}\preceq\begin{cases} n^{\left[\gamma_1^*(q-1)+qm-1\right](1+\widetilde{\kappa})}, & \gamma_1^*(q-1)+qm-1>\frac{1}{1+\widetilde{\kappa}},\\ n\ln n, & \gamma_1^*(q-1)+qm-1=\frac{1}{1+\widetilde{\kappa}},\\ n, & \gamma_1^*(q-1)+qm-1<\frac{1}{1+\widetilde{\kappa}},\end{cases} \tag{37}$$

from (29)-(36) for any $\gamma_1>-1$ and $i=1$.

1.2. Let us estimate the integrals $I^{2,+}_{k,R_1,j}+I^{2,-}_{k,R_1,j}$, for $k,j=1,2$ in case of $i=2$.

1.2.1.1. Analogously to the previous cases, we obtain

$$I^{2,+}_{1,R_1,1}+I^{2,-}_{1,R_1,1}\leq n\int\limits_{F^{2,+}_{1,R_1,1}\cup F^{2,-}_{1,R_1,1}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{\gamma_2(q-1)+qm-1}}\leq n\int\limits_{F^{2,+}_{1,R_1,1}\cup F^{2,-}_{1,R_1,1}}\frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}}} \tag{38}$$

$$\leq n \int\limits_{F^{2,+}_{1,R_1,1}\cup F^{2,-}_{1,R_1,1}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}} \leq \begin{cases} n^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2} > \frac{1}{1+\kappa}, \\ n\ln n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2} = \frac{1}{1+\kappa}, \\ n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2} < \frac{1}{1+\kappa}, \end{cases}$$

for $\gamma_2 > 0$ and

$$I^{2,+}_{1,R_1,1} + I^{2,-}_{1,R_1,1} \leq n \int\limits_{F^{2,+}_{1,R_1,1}\cup F^{2,-}_{1,R_1,1}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w)|^{qm-1}} \tag{39}$$

$$\leq n \int\limits_{F^{2,+}_{1,R_1,1}\cup F^{2,-}_{1,R_1,1}} \frac{|d\tau|}{\left|\Psi(\tau) - \Psi(w_2^+)\right|^{\frac{qm-1}{1+\beta_2}}} \leq n \int\limits_{F^{2,+}_{1,R_1,1}\cup F^{2,-}_{1,R_1,1}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{qm-1}{1+\beta_2}(1+\kappa)}} \leq \begin{cases} n^{\frac{qm-1}{1+\beta_2}(1+\kappa)}, & \frac{qm-1}{1+\beta_2}(1+\kappa) > 1, \\ n\ln n, & \frac{qm-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{qm-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

for $\gamma_2 \leq 0$.

1.2.1.2. For the integral $I^{2,+}_{1,,R_1,2} + I^{2,-}_{1,R_1,2}$, we have

$$I^{2,+}_{1,,R_1,2} + I^{2,-}_{1,R_1,2} \leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)+qm-1}} \tag{40}$$

$$\leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{\left|\Psi(\tau) - \Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}} \leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}}$$

$$\leq \begin{cases} n^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa) > 1, \\ n\ln n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

for $\gamma_2 > 0$, and

$$I^{2,+}_{1,,R_1,2} + I^{2,-}_{1,R_1,,2} \leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{qm-1}} \tag{41}$$

$$\leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{qm-1}} \leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{\left|\Psi(\tau) - \Psi(w_2^+)\right|^{\frac{qm-1}{1+\beta_2}}}$$

$$\leq n \int\limits_{F^{2,+}_{1,R_1,2}\cup F^{2,-}_{1,R_1,2}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{qm-1}{1+\beta_2}(1+\kappa)}} \leq \begin{cases} n^{\frac{qm-1}{1+\beta_2}(1+\kappa)}, & \frac{qm-1}{1+\beta_2}(1+\kappa) > 1, \\ n\ln n, & \frac{qm-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{qm-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

for $-1 < \gamma_2 \leq 0$.

1.2.2.1. For the integral $I^{2,+}_{2,,R_1,1} + I^{2,-}_{2,R_1,1}$, we write

$$I^{2,+}_{2,,R_1,1} + I^{2,-}_{2,R_1,1} \leq n \int\limits_{F^{2,+}_{2,R_1,1}\cup F^{2,-}_{2,R_1,1}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)+qm-1}} \tag{42}$$

$$\leq n \int\limits_{F^{2,+}_{2,R_1,1}\cup F^{2,-}_{2,R_1,1}} \frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}} \leq n \int\limits_{F^{2,+}_{2,R_1,1}\cup F^{2,-}_{2,R_1,1}} \frac{|d\tau|}{\left|\tau-w_2^+\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}}$$

$$\leq \begin{cases} n^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)<1, \end{cases}$$

for $\gamma_2>0$ and

$$I^{2,+}_{2,,R_1,1}+I^{2,-}_{2,R_1,1} \leq n \int\limits_{F^{2,+}_{2,R_1,1}\cup F^{2,-}_{2,R_1,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{qm-1}} \leq n \int\limits_{F^{2,+}_{2,R_1,1}\cup F^{2,-}_{2,R_1,1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{qm-1}} \tag{43}$$

$$\leq n \int\limits_{F^{2,+}_{2,R_1,1}\cup F^{2,-}_{2,R_1,1}} \frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{qm-1}{1+\beta_2}}} \leq \begin{cases} n^{\frac{qm-1}{1+\beta_2}(1+\kappa)}, & \frac{qm-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{qm-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{qm-1}{1+\beta_2}(1+\kappa)<1, \end{cases}$$

for $-1<\gamma_2\leq 0$.

1.2.2.2. For the integral $I^{2,+}_{2,,R_1,2}+I^{2,-}_{2,R_1,,2}$, we get

$$I^{2,+}_{2,,R_1,2}+I^{2,-}_{2,R_1,,2} \leq n \int\limits_{F^{2,+}_{2,R_1,2}\cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}} \tag{44}$$

$$\leq n \int\limits_{F^{2,+}_{2,R_1,2}\cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{\left|\tau-w_2^+\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}} \leq \begin{cases} n^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)<1, \end{cases}$$

for $\gamma_2>0$ and

$$I^{2,+}_{2,,R_1,2}+I^{2,-}_{2,R_1,,2} \leq n \int\limits_{F^{2,+}_{2,R_1,2}\cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{qm-1}} \tag{45}$$

$$\leq n \int\limits_{F^{2,+}_{2,R_1,2}\cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{qm-1}} \leq n \int\limits_{F^{2,+}_{2,R_1,2}\cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{qm-1}{1+\beta_2}}}$$

$$\leq \begin{cases} n^{\frac{qm-1}{1+\beta_2}(1+\kappa)}, & \frac{qm-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{qm-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{qm-1}{1+\beta_2}(1+\kappa)<1, \end{cases}$$

for $-1<\gamma_2\leq 0$. So, we obtain

$$\sum_{k,j=1}^{2}\left[I^{2,+}_{k,R_1,j}+I^{2,-}_{k,R_1,j}\right] \leq \begin{cases} n^{\frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2}(1+\kappa)>1, \\ n\ln n, & \frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2}(1+\kappa)=1, \\ n, & \frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2}(1+\kappa)<1, \end{cases} \tag{46}$$

from (38) - (45) for any $\gamma_2 > -1$ and $i = 2$. Comparing (27), (37) and (46) for the $z \in \Omega_R(\delta)$, we have

$$A_n(z) \leq \|P_n\|_p \times$$

$$\times\left[\left\{\begin{array}{ll} n^{\frac{\left[\gamma_1^*(q-1)+qm-1\right]}{q}(1+\widetilde{\kappa})}, & \gamma_1^*(q-1)+qm-1 > \frac{1}{1+\widetilde{\kappa}}, \\ (n\ln n)^{\frac{1}{q}}, & \gamma_1^*(q-1)+qm-1 = \frac{1}{1+\widetilde{\kappa}}, \\ n^{\frac{1}{q}}, & \gamma_1^*(q-1)+qm-1 < \frac{1}{1+\widetilde{\kappa}}. \end{array}\right. + \left\{\begin{array}{ll} n^{\frac{\gamma_2^*(q-1)+qm-1}{q(1+\beta_2)}(1+\kappa)}, & \frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2} > \frac{1}{1+\kappa}, \\ (n\ln n)^{\frac{1}{q}}, & \frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2} = \frac{1}{1+\kappa}, \\ n^{\frac{1}{q}}, & \frac{\gamma_2^*(q-1)+qm-1}{1+\beta_2} < \frac{1}{1+\kappa}. \end{array}\right.\right] \tag{47}$$

2. Let $z \in \widehat{\Omega}_R(\delta)$. Then, it is clear that $|\Psi(\tau) - \Psi(w)| \geq c_1$ and so we must evaluate the above integrals for each $i,\ k = 1,2$.

2.1. Let $i = 1$.

2.1.1.1. For the integral $I^{1,+}_{1,R_1,1} + I^{1,-}_{1,R_1,1}$,we obtain

$$I^{1,+}_{1,R_1,1} + I^{1,-}_{1,R_1,1} \leq n \cdot \left(\frac{1}{n}\right)^{(1-\kappa)} \cdot mes\left(F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}\right) \leq 1, \tag{48}$$

for $\gamma_1 > 0$ and

$$I^{1,+}_{1,R_1,1} + I^{1,-}_{1,R_1,1} \leq n \cdot d_{1,R_1}^{(-\gamma_1)(q-1)+1} \int\limits_{F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}} |d\tau| \leq n\left(\frac{1}{n}\right)^{(1-\gamma_1)(q-1)(1-\kappa)} \cdot mes\left(F^{1,+}_{1,R_1,1} \cup F^{1,-}_{1,R_1,1}\right) \leq 1, \tag{49}$$

for $-1 < \gamma_1 \leq 0$.

2.1.1.2. Similar to the (48) and (49), for the integral $I^{1,+}_{1,R_1,2} + I^{1,-}_{1,R_1,2}$, we find that

$$I^{1,+}_{1,R_1,2} + I^{1,-}_{1,R_1,2} \leq n \int\limits_{F^{1,+}_{1,R_1,2} \cup F^{1,-}_{1,R_1,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_1)|^{\gamma_1(q-1)-1}} \tag{50}$$

$$\leq n \int\limits_{F^{1,+}_{1,R_1,2} \cup F^{1,-}_{1,R_1,2}} \frac{|d\tau|}{|\tau - w_1|^{\left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa})}} \leq \left\{\begin{array}{ll} n^{\left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa})}, & \left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa}) > 1, \\ n\ln n, & \left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa}) = 1, \\ n, & \left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa}) < 1, \end{array}\right.$$

for $\gamma_1 > 0$ and

$$I^{1,+}_{1,R_1,2} + I^{1,-}_{1,R_1,2} \leq n \cdot \int\limits_{F^{1,+}_{1,R_1,2} \cup F^{1,-}_{1,R_1,2}} d(\Psi(\tau), L)\,|d\tau| \leq n\left(\frac{1}{n}\right)^{(1-\kappa)} \leq n^{\kappa},$$

for $-1 < \gamma_1 \leq 0$.

2.1.2.1. For the integral $I^{1,+}_{2,R_1,1} + I^{1,-}_{2,R_1,1}$, we get

$$I^{1,+}_{2,R_1,1} + I^{1,-}_{2,R_1,1} \leq n \cdot \int\limits_{F^{1,+}_{2,R_1,1} \cup F^{1,-}_{2,R_1,1}} d(\Psi(\tau), L)\,|d\tau| \leq n^{\kappa}, \tag{51}$$

for $\gamma_1 > 0$ and

$$I^{1,+}_{2,R_1,1} + I^{1,-}_{2,R_1,1} \leq n \cdot \left(\frac{1}{n}\right)^{(1-\kappa)} \leq n^{\kappa},$$

for $-1 < \gamma_1 \leq 0$.

2.1.2.2. For the integral $I_{2,R_1,2}^{1,+}+I_{2,R_1,2}^{1,-}$, we obtain

$$I_{2,R_1,2}^{1,+}+I_{2,R_1,2}^{1,-}\preceq n\cdot\left(\frac{1}{n}\right)^{(1-\kappa)}\preceq n^{\kappa}, \tag{52}$$

for $\gamma_1>0$ and

$$I_{2,R_1,2}^{1,+}+I_{2,R_1,2}^{1,-}\preceq n\cdot\left(\frac{1}{n}\right)^{(1-\kappa)}\preceq n^{\kappa},$$

for $-1<\gamma_1\leq 0$.

Taking into account (48)-(52), we have

$$\sum_{k,j=1}^{2} I_{k,R_1,j}^{1,+}+I_{k,R_1,j}^{1,-}\preceq\begin{cases} n^{[\gamma_1(q-1)-1](1+\widetilde{\kappa})}, & [\gamma_1(q-1)-1](1+\widetilde{\kappa})>1,\\ n\ln n, & [\gamma_1(q-1)-1](1+\widetilde{\kappa})=1,\\ n, & [\gamma_1(q-1)-1](1+\widetilde{\kappa})<1,\end{cases}$$

for $\gamma_1>0$ and

$$\sum_{k,j=1}^{2} I_{k,R_1,j}^{1,+}+I_{k,R_1,j}^{1,-}\preceq n^{\kappa},$$

for $-1<\gamma_1\leq 0$.

Therefore, we reach

$$\sum_{k,j=1}^{2} I_{k,R_1,j}^{1,+}+I_{k,R_1,j}^{1,-}\preceq\begin{cases} n^{[\gamma_1(q-1)-1](1+\widetilde{\kappa})}, & [\gamma_1(q-1)-1](1+\widetilde{\kappa})>1;\gamma_1>0,\\ n\ln n, & [\gamma_1(q-1)-1](1+\widetilde{\kappa})=1;\gamma_1>0,\\ n, & [\gamma_1(q-1)-1](1+\widetilde{\kappa})<1;\gamma_1>0,\\ n^{\kappa}, & -1<\gamma_1\leq 0,\end{cases} \tag{53}$$

for any $\gamma_1>-1$ and $i=1$.

2.2. Now, let us calculate the integrals $I_{k,R_1,j}^{2,+}+I_{k,R_1,j}^{2,-}$, for $k,j=1,2$ in case of $i=2$.

2.2.1.1. Similar to previous evaluations for the integral $I_{1,R_1,1}^{2,+}+I_{1,R_1,1}^{2,-}$ $k,j=1,2$, we obtain

$$I_{1,R_1,1}^{2,+}+I_{1,R_1,1}^{2,-}\preceq n\cdot\left(\frac{1}{n}\right)^{(1-\kappa)}\left[mes(I_{1,R_1,1}^{2,+}+I_{1,R_1,1}^{2,-})\right]\preceq n^{\kappa-\frac{1-\kappa}{1+\beta_2}}, \tag{54}$$

for $\gamma_2>0$ and

$$I_{1,R_1,1}^{2,+}+I_{1,R_1,1}^{2,-}\preceq n\cdot\left(\frac{1}{n}\right)^{(1-\kappa)} d^{(-\gamma_2)(q-1)}(z_2,L_{R_1})\left[mes(I_{1,R_1,1}^{2,+}+I_{1,R_1,1}^{2,-})\right]\preceq n^{\kappa},$$

for $-1<\gamma_2\leq 0$.

2.2.1.2. For the integral $I_{1,,R_1,2}^{2,+}+I_{1,R_1,,2}^{2,-}$, we have

$$I_{1,,R_1,2}^{2,+}+I_{1,R_1,,2}^{2,-}\preceq n\int\limits_{F_{1,R_1,2}^{2,+}\cup F_{1,R_1,2}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)-1}}\preceq n\int\limits_{F_{1,R_1,2}^{2,+}\cup F_{1,R_1,2}^{2,-}}\frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}} \tag{55}$$

$$\preceq n\int\limits_{F_{1,R_1,2}^{2,+}\cup F_{1,R_1,2}^{2,-}}\frac{|d\tau|}{\left|\tau-w_2^+\right|^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}}\preceq\begin{cases} n^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)>1,\\ n\ln n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)=1,\\ n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)<1,\end{cases}$$

for $\gamma_2 > 0$ and

$$I^{2,+}_{1,,R_1,2} + I^{2,-}_{1,R_1,2} \le n \int\limits_{F^{2,+}_{1,R_1,2} \cup F^{2,-}_{1,R_1,2}} d(\Psi(\tau), L)\,|d\tau| \le n \left(\frac{1}{n}\right)^{(1-\kappa)} \left[mes(I^{2,+}_{1,R_1,2} + I^{2,-}_{1,R_1,2})\right] \le n^{\kappa - \frac{1}{1+\beta_2}}$$

for $-1 < \gamma_2 \le 0$.

2.2.2.1. For the integral $I^{2,+}_{2,,R_1,1} + I^{2,-}_{2,R_1,1}$, we get

$$I^{2,+}_{2,,R_1,1} + I^{2,-}_{2,R_1,1} \le n \int\limits_{F^{2,+}_{2,R_1,1} \cup F^{2,-}_{2,R_1,1}} d(\Psi(\tau), L)\,|d\tau| \le n \left(\frac{1}{n}\right)^{(1-\kappa)} \le n^{\kappa} \tag{56}$$

for $\gamma_2 > 0$ and

$$I^{2,+}_{2,,R_1,1} + I^{2,-}_{2,R_1,1} \le n \left(\frac{1}{n}\right)^{(1-\kappa)} \le n^{\kappa}$$

for $-1 < \gamma_2 \le 0$.

2.2.2.2. For the integral $I^{2,+}_{2,,R_1,2} + I^{2,-}_{2,R_1,,2}$, we find that

$$I^{2,+}_{2,,R_1,2} + I^{2,-}_{2,R_1,,2} \le n \int\limits_{F^{2,+}_{2,R_1,2} \cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{|\Psi(\tau) - \Psi(w_2)|^{\gamma_2(q-1)-1}} \le n \int\limits_{F^{2,+}_{2,R_1,2} \cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{\left|\Psi(\tau) - \Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}} \tag{57}$$

$$\le n \int\limits_{F^{2,+}_{2,R_1,2} \cup F^{2,-}_{2,R_1,2}} \frac{|d\tau|}{\left|\tau - w_2^+\right|^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}} \le \begin{cases} n^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) > 1, \\ n \ln n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) < 1, \end{cases}$$

for $\gamma_2 > 0$ and

$$I^{2,+}_{2,,R_1,2} + I^{2,-}_{2,R_1,,2} \le n^{\kappa},$$

for $-1 < \gamma_2 \le 0$.

Thus, we reach

$$\sum_{k,j=1}^{2} I^{2,+}_{k,R_1,j} + I^{2,-}_{k,R_1,j} \le \begin{cases} n^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) > 1; \gamma_2 > 0, \\ n \ln n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) = 1; \gamma_2 > 0, \\ n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) < 1; \gamma_2 > 0, \\ n^{\kappa}, & -1 < \gamma_2 \le 0, \end{cases} \tag{58}$$

from (54) - (57) for any $\gamma_2 > -1$ and $i = 2$. Taking into consideration (27), (53) and (58) for the $z \in \widehat{\Omega}_R(\delta)$, we obtain

$$A_n(z) \le \|P_n\|_p \left[\begin{cases} n^{\frac{[\gamma_1(q-1)-1](1+\widetilde{\kappa})}{q}}, & [\gamma_1(q-1)-1](1+\widetilde{\kappa}) > 1; \gamma_1 > 0, \\ (n \ln n)^{\frac{1}{q}}, & [\gamma_1(q-1)-1](1+\widetilde{\kappa}) = 1; \gamma_1 > 0, \\ n^{\frac{1}{q}}, & [\gamma_1(q-1)-1](1+\widetilde{\kappa}) < 1; \gamma_1 > 0, \\ n^{\frac{\kappa}{q}}, & -1 < \gamma_1 \le 0, \end{cases} + \begin{cases} n^{\frac{\gamma_2(q-1)-1}{q(1+\beta_2)}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) > 1; \gamma_2 > 0, \\ (n \ln n)^{\frac{1}{q}}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) = 1; \gamma_2 > 0, \\ n^{\frac{1}{q}}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa) < 1; \gamma_2 > 0, \\ n^{\frac{\kappa}{q}}, & -1 < \gamma_2 \le 0. \end{cases} \right] \tag{59}$$

B) By Cauchy integral representation of the $v-th$ derivatives $\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(v)}$, we have

$$\left(\frac{1}{\Phi^{n+1}(z)}\right)^{(v)}=-\frac{1}{2\pi i}\int\limits_{L_{R_1}}\frac{1}{\Phi^{n+1}(\zeta)}\frac{d\zeta}{(\zeta-z)^{v+1}}\ ,\ z\in\Omega_R.$$

Then, using notation for $L_R^{\pm}$, $E^{i,\pm}_{1,R_1}$, $E^{i,\pm}_{2,R_1}$ and $F^{i,\pm}_{j,R_1}:=\Phi(E^{i,\pm}_{j,R_1})$ from (25), we obtain

$$\begin{aligned} B_{n,v}(z)\quad &: =\left|\left(\Phi^{-n-1}(z)\right)^{(v)}\right|\le\frac{1}{2\pi}\int\limits_{L_{R_1}}\left|\frac{1}{\Phi^{n+1}(\zeta)}\right|\frac{|d\zeta|}{|\zeta-z|^{v+1}}\le\frac{1}{2\pi}\int\limits_{L_{R_1}}\frac{|d\zeta|}{|\zeta-z|^{v+1}}\\ &=\sum_{i,j=1}^{2}\int\limits_{F^{i,+}_{j,R_1}\cup F^{i,-}_{j,R_1}}\frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v+1}}\asymp\sum_{i,j=1}^{2}\int\limits_{F^{i,+}_{j,R_1}\cup F^{i,-}_{j,R_1}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v+1}\,(|\tau|-1)}. \end{aligned}$$

Let us calculate the integrals in the final sum for the cases $z\in\Omega_R(\delta)$ and $z\in\widehat{\Omega}_R(\delta)$.
Let $z\in\Omega_R(\delta)$.
3.1. For the integrals over the $F^{1,+}_{j,R_1}\cup F^{1,-}_{j,R_1}$, $j=1,2$, in case of $i=1$ we get

$$\int\limits_{F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v+1}\,(|\tau|-1)}\le n\cdot\int\limits_{F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v}}\le n\cdot\int\limits_{F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}}\frac{|d\tau|}{|\tau-w|^{v(1+\widetilde{\kappa})}}\le n^{v(1+\widetilde{\kappa})}$$

and

$$\int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v+1}\,(|\tau|-1)}\le n\cdot\int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v}}\le n\cdot\int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}}\frac{|d\tau|}{|\tau-w|^{v(1+\kappa)}}\le n^{v(1+\kappa)}.$$

3.2. For the integrals over the $F^{2,+}_{j,R_1}\cup F^{2,-}_{j,R_1}$, $j=1,2$, in case of $i=2$ we find

$$\int\limits_{F^{2,+}_{1,R_1}\cup F^{2,-}_{1,R_1}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v+1}\,(|\tau|-1)}\le n\cdot\int\limits_{F^{2,+}_{1,R_1}\cup F^{2,-}_{1,R_1}}\frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{v}{1+\beta_2}}}\le n\cdot\int\limits_{F^{2,+}_{1,R_1}\cup F^{2,-}_{1,R_1}}\frac{|d\tau|}{\left|\tau-w_2^+\right|^{\frac{v}{1+\beta_2}(1+\kappa)}}$$

$$\le\begin{cases} n^{\frac{v}{1+\beta_2}(1+\kappa)}, & \frac{v}{1+\beta_2}(1+\kappa)>1,\\ n\ln n, & \frac{v}{1+\beta_2}(1+\kappa)=1,\\ n, & \frac{v}{1+\beta_2}(1+\kappa)<1,\end{cases}$$

and

$$\int\limits_{F^{2,+}_{2,R_1}\cup F^{2,-}_{2,R_1}}\frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v+1}\,(|\tau|-1)}\le n\cdot\int\limits_{F^{2,+}_{2,R_1}\cup F^{2,-}_{2,R_1}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w)|^{v}}\le n\cdot\int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}}\frac{|d\tau|}{|\tau-w|^{v(1+\kappa)}}\le n^{v(1+\kappa)}.$$

Therefore, we have

$$B_{n,v}(z)\le n^{v(1+\widetilde{\kappa})}+\begin{cases} n^{\frac{v}{1+\beta_2}(1+\kappa)}, & \frac{v}{1+\beta_2}(1+\kappa)>1,\\ n\ln n, & \frac{v}{1+\beta_2}(1+\kappa)=1,\\ n, & \frac{v}{1+\beta_2}(1+\kappa)<1,\end{cases}+n^{v(1+\kappa)}. \tag{60}$$

for the $z\in\Omega_R(\delta)$.

3.3. Let $z \in \widehat{\Omega}_R(\delta)$. Then, we obtain that

$$B_{n,v}(z) \preceq \sum_{i,j=1}^{2} \int\limits_{F_{j,R_1}^{i,+} \cup F_{j,R_1}^{i,-}} |\Psi'(\tau)| |d\tau| \asymp \sum_{i,j=1}^{2} \int\limits_{F_{j,R_1}^{i,+} \cup F_{j,R_1}^{i,-}} \frac{d(\Psi(\tau), L) |d\tau|}{|\tau| - 1} \preceq n^{\kappa}. \tag{61}$$

Now, combining (19), (20), (21), (47), (59), (60) and (61), we reach

$$\left|P_n^{(m)}(z)\right| \preceq \left|\Phi^{n+1}(z)\right| \left[\frac{\|P_n\|_p}{d(z,L)} A_{n,p}(z) + \sum_{v=1}^{m} C_m^v B_{n,v}(z) \left|P_n^{(m-v)}(z)\right| \right], \tag{62}$$

where

$$A_{n,p}(z) \preceq$$

$$\preceq \begin{cases} n^{\left(\frac{\gamma_1+1}{p}+m-1\right)(1+\widetilde{\kappa})}, & p>1, \quad m \geq 2, \\ n^{\frac{\gamma_1+1}{p}(1+\widetilde{\kappa})}, & p<p_4^1, \quad m=1, \\ (n \ln n)^{1-\frac{1}{p}}, & p=p_4^1, \quad m=1, \\ n^{1-\frac{1}{p}}, & p>p_4^1, \quad m=1, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_2+1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_2}}, & p>1, \quad \begin{cases} m \geq 2, \\ \beta_2 < (m-1)(1+\kappa)-1, \end{cases} \\ n^{\left(\frac{\gamma_2+1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_2}}, & 1<p<p_6^2(m), \quad \begin{cases} m \geq 2, \\ \beta_2 \geq (m-1)(1+\kappa)-1, \end{cases} \\ n^{\frac{\gamma_2+1}{p}\frac{1+\kappa}{1+\beta_2}}, & p<p_5^2, \quad m=1, \\ (n \ln n)^{1-\frac{1}{p}}, & p=p_5^2, \quad m=1, \\ n^{1-\frac{1}{p}}, & p>p_5^2, \quad m=1, \end{cases}$$

if $\gamma_1, \gamma_2 > 0$ and

$$A_{n,p}(z) \preceq$$

$$\preceq \begin{cases} n^{\left(\frac{1}{p}+m-1\right)(1+\widetilde{\kappa})}, & p>1, \quad m \geq 2, \\ n^{\frac{1}{p}(1+\widetilde{\kappa})}, & p<2+\widetilde{\kappa}, \quad m=1, \\ (n \ln n)^{1-\frac{1}{p}}, & p=2+\widetilde{\kappa}, \quad m=1, \\ n^{1-\frac{1}{p}}, & p>2+\widetilde{\kappa}, \quad m=1, \end{cases} + \begin{cases} n^{\left(\frac{1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_2}}, & p>1, \quad \begin{cases} m \geq 2, \\ \beta_2 < (m-1)(1+\kappa)-1, \end{cases} \\ n^{\left(\frac{1}{p}+m-1\right)\frac{1+\kappa}{1+\beta_2}}, & 1<p<p_8^2, \quad \begin{cases} m \geq 2, \\ \beta_2 \geq (m-1)(1+\kappa)-1, \end{cases} \\ n^{\frac{1}{p}\frac{1+\kappa}{1+\beta_2}}, & p<1+\frac{1+\kappa}{1+\beta_2}, \quad m=1, \\ (n \ln n)^{1-\frac{1}{p}}, & p=1+\frac{1+\kappa}{1+\beta_2}, \quad m=1, \\ n^{1-\frac{1}{p}}, & p>1+\frac{1+\kappa}{1+\beta_2}, \quad m=1, \end{cases}$$

if $-1 < \gamma_1, \gamma_2 \leq 0$,

$$B_{n,v}(z) \preceq n^{v(1+\widetilde{\kappa})} + \begin{cases} n^{\frac{v}{1+\beta_2}(1+\kappa)}, & \frac{v}{1+\beta_2}(1+\kappa) > 1, \\ n \ln n, & \frac{v}{1+\beta_2}(1+\kappa) = 1, \\ n, & \frac{v}{1+\beta_2}(1+\kappa) < 1, \end{cases} + n^{v(1+\kappa)}.$$

for $z \in \Omega_R(\delta)$ and

$$A_{n,p}(z) \preceq \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & p<p_1^1, \quad \gamma_1>0, \\ (n \ln n)^{1-\frac{1}{p}}, & p=p_1^1, \quad \gamma_1>0, \\ n^{1-\frac{1}{p}}, & p>p_1^1, \quad \gamma_1>0, \\ n^{\kappa(1-\frac{1}{p})}, & p>1, \quad -1<\gamma_1 \leq 0, \end{cases} + \begin{cases} n^{\frac{\gamma_2+1-p}{p(1+\beta_2)}(1+\kappa)}, & p<p_2^2, \quad \gamma_2>0, \\ (n \ln n)^{1-\frac{1}{p}}, & p=p_2^2, \quad \gamma_2>0, \\ n^{1-\frac{1}{p}}, & p>p_2^2, \quad \gamma_2>0, \\ n^{\kappa(1-\frac{1}{p})}, & p>1, \quad -1<\gamma_2 \leq 0, \end{cases}$$

and

$$B_{n,v}(z) \preceq n^{\kappa}$$

for $z \in \widehat{\Omega}_R(\delta)$.

Therefore, it remains to show that $d(z, L_{R_1}) \geq d(z, L)$ for the complete the proof of the Theorem 2.1. Let us show that $d(z, L_{R_1}) \geq d(z, L)$ holds for all $z \in \Omega_R$. Really, we have $d(z, L_{R_1}) \geq \delta \geq d(z, L)$ for the points $z \in \widehat{\Omega}_R(\delta)$. Now, let $z \in \Omega(L_{R_1}, d(L_{R_1}, L_R))$. Then, we find that $|w - w_1| \geq \|w - w_2| - |w_2 - w_1\| \geq \left\| w - w_2| - \frac{1}{2}|w - w_2 \right\| \geq \frac{1}{2}|w - w_2|$ where $d(z, L_{R_1}) = |z - \xi_1|$ for $\xi_1 \in L_{R_1}$ and $d(z, L) = |z - \xi_2|$ for $\xi_2 \in L$ and $w = \Phi(z), w_i = \Phi(\xi_i)$, $i = 1, 2$. According to Lemma 3.1, we obtain the inequality $d(z, L_{R_1}) \geq d(z, L)$ that ends the proof of Theorem 2.1.

Proof of Theorem 2.3. Suppose that $G \in \widetilde{PQ}\left(\kappa; f_1,\ g_2\right)$, for some $0 < \kappa < 1$, $f_1(x) = C_1 x^{1+\alpha_1}$, $\alpha_1 \geq 0$ and $g_2(x) = C_2 x^{1+\beta_2}$, $\beta_2 > 0$ and $h(z)$ is defined as in (1). By applying Cauchy integral formula to $H_n(z) = \frac{P_n(z)}{\Phi^{n+1}(z)}$ for $z \in \Omega_R$, we write

$$\left| \frac{P_n(z)}{\Phi^{n+1}(z)} \right| \leq \frac{1}{2\pi} \int\limits_{L_{R_1}} \left| \frac{P_n(\zeta)}{\Phi^{n+1}(\zeta)} \right| \frac{|d\zeta|}{|\zeta - z|} \leq \frac{1}{d(z, L_{R_1})} \int\limits_{L_{R_1}} |P_n(\zeta)|\, |d\zeta| =: \frac{1}{d(z, L_{R_1})} A_n. \tag{63}$$

Therefore, according to the above reason $d(z, L_{R_1}) \geq d(z, L)$ for $z \in \Omega_R$, we obtain

$$|P_n(z)| \leq \frac{\left|\Phi^{n+1}(z)\right|}{d(z, L)} A_n, \text{ where } A_n = \sum_{i=1}^{2} \int\limits_{L^i_{R_1}} |P_n(\zeta)|\, |d\zeta|\,. \tag{64}$$

Let us evaluate the integrals in A_n. By first multiplying the numerator and denominator of the integrand by $h^{1/p}(\zeta)$ and then applying the Hölder inequality, we get

$$A_n \leq \sum_{i=1}^{2} \left(\int\limits_{L^i_{R_1}} h(\zeta)\, |P_n(\zeta)|^p\, |d\zeta| \right)^{1/p} \left(\int\limits_{L^i_{R_1}} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} \left|\zeta - z_j\right|^{\frac{q}{p}\gamma_j}} \right)^{1/q} =: \sum_{i=1}^{2} \left(\widetilde{J}^i_{1,R_1} \cdot \widetilde{J}^i_{2,R_1} \right), \frac{1}{p} + \frac{1}{q} = 1. \tag{65}$$

According to Lemma 3.3, we obtain

$$\widetilde{J}^i_{1,R_1} \leq \|P_n\|_p\,,\ i = 1, 2 \tag{66}$$

for the integral $\widetilde{J}^i_{1,R_1}$. Then, from (65) and (66) we have

$$A_n \leq \|P_n\|_p \sum_{i=1}^{2} \left(\widetilde{J}^i_{2,R_1} \right).$$

For the integral $\widetilde{J}^i_{2,R_1}$, we find that

$$\widetilde{J}^i_{2,R_1} := \int\limits_{L^i_{R_1}} \frac{|d\zeta|}{\prod\limits_{j=1}^{l} \left|\zeta - z_j\right|^{\frac{q}{p}\gamma_i}} \asymp \int\limits_{L^i_{R_1}} \frac{|d\zeta|}{|\zeta - z_i|^{\frac{q}{p}\gamma_i}},\ i = 1, 2. \tag{67}$$

Then, from (67), we write

$$A_n \leq \|P_n\|_p \sum_{i=1}^{2} \left(\widetilde{J}^i_{2,R_1} \right), \text{ where } \widetilde{J}^i_{2,R_1} = \int\limits_{L^i_{R_1}} \frac{|d\zeta|}{|\zeta - z_i|^{(q-1)\gamma_i}}\ , i = 1, 2. \tag{68}$$

Taking into consideration above notations, replacing the variable $\tau = \Phi(\zeta)$, we get

$$\begin{aligned}\widetilde{J}^{i}_{2,R_1} &= \int\limits_{L^i_{R_1}} \frac{|d\zeta|}{|\zeta - z_i|^{\gamma_i}} = \int\limits_{\Phi(L^i_{R_1})} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{(q-1)\gamma_i}} \asymp \sum_{i,j=1}^{2} \int\limits_{F^{i,+}_{j,R_1}\cup F^{i,-}_{j,R_1}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{(q-1)\gamma_i}} \\ &= : \sum_{i,j=1}^{2} \left[\widetilde{J}(F^{i,+}_{j,R_1}) + \widetilde{J}(F^{i,-}_{j,R_1})\right],\end{aligned}$$

and herefrom we obtain

$$A_n \le \|P_n\|_p \sum_{i=1}^{2} \left[\widetilde{J}^{i}_{2,R_1}\right] =: \|P_n\|_p \sum_{i=1}^{2} \left[\widetilde{I}^{i}_{1}(E^{i,+}_{1,R_1}) + \widetilde{I}^{i}_{2}(E^{i,-}_{2,R_1})\right] =: \|P_n\|_p \sum_{i,k=1}^{2} \left[\widetilde{I}^{i,+}_{k,R_1} + \widetilde{I}^{i,-}_{k,R_1}\right] \tag{69}$$

owing to (68) where

$$\widetilde{I}^{i,\pm}_{k,R_1} := \widetilde{I}^{i}_{k,R_1}(E^{i,\pm}_{k,R_1}) := \int\limits_{F^{i,\pm}_{k,R_1}} \frac{|\Psi'(\tau)|\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{(q-1)\gamma_i}} \asymp \int\limits_{F^{i,\pm}_{k,R_1}} \frac{d(\Psi(\tau),L)\,|d\tau|}{|\Psi(\tau)-\Psi(w_i)|^{(q-1)\gamma_i}\,(|\tau|-1)};\ i,\ k=1,2,$$

from (18) and (25). According to (63) and (64), it is sufficient to estimate the integrals $\widetilde{I}^{i,\pm}_{k,R_1}$ for each $i = 1,2$ and $k = 1,2$.

1. Let us estimate the integrals $\widetilde{I}^{1,+}_{k,R_1} + \widetilde{I}^{1,-}_{k,R_1}$, for $k = 1,2$ in case of $i = 1$.

1.1. For the integral $\widetilde{I}^{1,+}_{1,R_1} + \widetilde{I}^{1,-}_{1,R_1}$, we have

$$\begin{aligned}\widetilde{I}^{1,+}_{1,R_1} + \widetilde{I}^{1,-}_{1,R_1} &\le n\cdot \int\limits_{F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)-1}} \le n\cdot \int\limits_{F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1(q-1)-1)(1+\widetilde{\kappa})}} \\ &\le n\cdot n^{(\gamma_1(q-1)-1)(1+\widetilde{\kappa})}\cdot mes\left(F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}\right) \le n^{[\gamma_1(q-1)-1](1+\widetilde{\kappa})}\end{aligned} \tag{70}$$

for $\gamma_1 > 0$ and

$$\begin{aligned}\widetilde{I}^{1,+}_{1,R_1} + \widetilde{I}^{1,-}_{1,R_1} &\le n\cdot d^{(-\gamma_1)(q-1)}_{1,R_1} d_{1,R,R_1} \int\limits_{F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}} |d\tau| \\ &\le n\left(\frac{1}{n}\right)^{(-\gamma_1)(q-1)+1)(1-\kappa)}\cdot mes\left(F^{1,+}_{1,R_1}\cup F^{1,-}_{1,R_1}\right) \le 1\end{aligned} \tag{71}$$

for $-1 < \gamma_1 \le 0$.

1.2. For the integral $\widetilde{I}^{1,+}_{2,R_1} + \widetilde{I}^{1,-}_{2,R_1}$, we find

$$\widetilde{I}^{1,+}_{2,R_1} + \widetilde{I}^{1,-}_{2,R_1} \le n\cdot \int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}} \frac{|d\tau|}{|\Psi(\tau)-\Psi(w_1)|^{\gamma_1(q-1)-1}} \tag{72}$$

$$\le n\cdot \int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}} \frac{|d\tau|}{|\tau - w_1|^{(\gamma_1(q-1)-1)(1+\widetilde{\kappa})}} \le \begin{cases} n^{[\gamma_1(q-1)-1](1+\widetilde{\kappa})}, & [\gamma_1(q-1)-1](1+\widetilde{\kappa}) > 1, \\ n\ln n, & [\gamma_1(q-1)-1](1+\widetilde{\kappa}) = 1, \\ n, & [\gamma_1(q-1)-1](1+\widetilde{\kappa}) < 1, \end{cases}$$

for $\gamma_1 > 0$ and

$$\widetilde{I}^{1,+}_{2,R_1} + \widetilde{I}^{1,-}_{2,R_1} \asymp \int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}} \frac{|\Psi(\tau)-\Psi(w_1)|^{(-\gamma_1)(q-1)}\,d(\Psi(\tau),L)\,|d\tau|}{(|\tau|-1)} \le n^{\kappa} \int\limits_{F^{1,+}_{2,R_1}\cup F^{1,-}_{2,R_1}} |d\tau| \le n^{\kappa} \tag{73}$$

for $-1<\gamma_1\leq 0$. Therefore, we have

$$\sum_{k=1}^{2}\left[\widetilde{I}_{k,R_1}^{1,+}+\widetilde{I}_{k,R_1}^{1,-}\right]\leq\begin{cases} n^{\left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa})}, & \left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa})>1,\\ n\ln n, & \left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa})=1,\\ n, & \left[\gamma_1(q-1)-1\right](1+\widetilde{\kappa})<1,\\ n^{\kappa} & -1<\gamma_1\leq 0,\end{cases} \tag{74}$$

for any $\gamma_1>-1$.

2. Let us estimate the integrals $\widetilde{I}_{k,R_1}^{2,+}+\widetilde{I}_{k,R_1}^{2,-}$, for $k=1,2$ in case of $i=2$.

2.1. For the integral $\widetilde{I}_{1,R_1}^{2,+}+\widetilde{I}_{1,R_1}^{2,-}$, we get

$$\widetilde{I}_{1,R_1}^{2,+}+\widetilde{I}_{1,R_1}^{2,-}\leq n\int\limits_{F_{1,R_1}^{2,+}\cup F_{1,R_1}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)-1}} \tag{75}$$

$$\leq n\int\limits_{F_{1,R_1}^{2,+}\cup F_{1,R_1}^{2,-}}\frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)-1}{1+\beta_2}}}\leq\begin{cases} n^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)>1,\\ n\ln n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)=1,\\ n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)<1,\end{cases}$$

for $\gamma_2>0$ and

$$\widetilde{I}_{1,R_1}^{2,+}+\widetilde{I}_{1,R_1}^{2,-}\leq n\int\limits_{F_{1,R_1}^{2,+}\cup F_{1,R_1}^{2,-}}|\Psi(\tau)-\Psi(w_2)|^{(-\gamma_2)(q-1)+1}\,|d\tau| \tag{76}$$

$$\leq n\int\limits_{F_{1,R_1}^{2,+}\cup F_{1,R_1}^{2,-}}\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{(-\gamma_2)(q-1)+1}{1+\beta_2}}|d\tau|\leq n\cdot d_{2,R_1}^{\frac{(-\gamma_2)(q-1)+1}{1+\beta_2}}\,mes(F_{1,R_1}^{2,+}\cup F_{1,R_1}^{2,-})\leq 1$$

for $-1<\gamma_2\leq 0$.

2.2. For the integral $\widetilde{I}_{2,,R_1}^{2,+}+\widetilde{I}_{2,R_1}^{2,-}$, we obtain

$$\widetilde{I}_{2,,R_1}^{2,+}+\widetilde{I}_{2,R_1}^{2,-}\leq n\int\limits_{F_{2,R_1}^{2,+}\cup F_{2,R_1}^{2,-}}\frac{|d\tau|}{|\Psi(\tau)-\Psi(w_2)|^{\gamma_2(q-1)-1}}$$

$$\leq n\int\limits_{F_{2,R_1}^{2,+}\cup F_{2,R_1}^{2,-}}\frac{|d\tau|}{\left|\Psi(\tau)-\Psi(w_2^+)\right|^{\frac{\gamma_2(q-1)+qm-1}{1+\beta_2}(1+\kappa)}}\leq\begin{cases} n^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)>1,\\ n\ln n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)=1,\\ n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)<1,\end{cases}$$

for $\gamma_2>0$ and

$$\widetilde{I}_{2,,R_1}^{2,+}+\widetilde{I}_{2,R_1}^{2,-}\leq\int\limits_{F_{2,R_1}^{2,+}\cup F_{2,R_1}^{2,-}}\frac{d(\Psi(\tau),L)\,|d\tau|}{(|\tau|-1)}\leq n^{\kappa} \tag{77}$$

for $-1<\gamma_2\leq 0$. Therefore, we find

$$\sum_{k=1}^{2}\left[\widetilde{I}_{k,R_1}^{2,+}+\widetilde{I}_{k,R_1}^{2,-}\right]\leq\begin{cases} n^{\frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)>1,\\ n\ln n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)=1,\\ n, & \frac{\gamma_2(q-1)-1}{1+\beta_2}(1+\kappa)<1,\\ n^{\kappa}, & -1<\gamma_2\leq 0,\end{cases} \tag{78}$$

from (75) - (77) for any $\gamma_2 > -1$ and $i = 2$.

By combining the results (69), (74) and (78), we write

$$A_n \leq \|P_n\|_p \left[\begin{cases} n^{\frac{[\gamma_1(q-1)-1]}{q}(1+\widetilde{\kappa})}, & \gamma_1(q-1)-1 > \frac{1}{1+\widetilde{\kappa}}, \\ (n\ln n)^{\frac{1}{q}}, & \gamma_1(q-1)-1 = \frac{1}{1+\widetilde{\kappa}}, \\ n^{\frac{1}{q}}, & \gamma_1(q-1)-1 < \frac{1}{1+\widetilde{\kappa}}. \\ n^{\frac{\kappa}{q}} & -1 < \gamma_1 \leq 0. \end{cases} + \begin{cases} n^{\frac{\gamma_2(q-1)-1}{q(1+\beta_2)}(1+\kappa)}, & \frac{\gamma_2(q-1)-1}{1+\beta_2} > \frac{1}{1+\kappa}, \\ (n\ln n)^{\frac{1}{q}}, & \frac{\gamma_2(q-1)-1}{1+\beta_2} = \frac{1}{1+\kappa}, \\ n^{\frac{1}{q}}, & \frac{\gamma_2(q-1)-1}{1+\beta_2} < \frac{1}{1+\kappa}, \\ n^{\frac{\kappa}{q}} & -1 < \gamma_2 \leq 0, \end{cases} \right] \tag{79}$$

for the $p > 1$ and $z \in \Omega_R$. Taking (64) and (79) together, we have

$$|P_n(z)| \leq c \frac{|\Phi(z)|^{n+1}}{d(z,L)} B_{n,1} \|P_n\|_p, \tag{80}$$

for any $p > 1$ and $z \in \Omega_R$ where $c = c(L, p, \gamma_i) > 0,\ i = 1, 2,$ is the constant independent from n and z and

$$B_{n,1} := \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & p < p_1^1, \quad \gamma_1 > 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 > 0, \\ n^{1-\frac{1}{p}}, & p > p_1^1, \quad \gamma_1 > 0, \\ n^{\left(1-\frac{1}{p}\right)\kappa} & p > 1, \quad -1 < \gamma_1 \leq 0, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_2+1}{p}-1\right)\frac{(1+\kappa)}{1+\beta_2}}, & p < p_2^2, \quad \gamma_2 > 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad \gamma_2 > 0, \\ n^{1-\frac{1}{p}}, & p > p_2^2, \quad \gamma_2 > 0, \\ n^{\left(1-\frac{1}{p}\right)\kappa} & p > 1, \quad -1 < \gamma_2 \leq 0. \end{cases}$$

Therefore, the proof of Theorem 2.3 is completed.

Proof of Theorem 2.4. From Corollary 2.2 and Theorem 2.3, we get

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{n+1}(z)\right|}{d(z,L)} \left[A_{n,p}^2(z,1) + |P_n(z)| \begin{cases} n^{1+\widetilde{\kappa}}, & \text{if } z \in \Omega_R(\delta), \\ n^{\kappa}, & \text{if } z \in \widehat{\Omega}_R(\delta), \end{cases} \right] \tag{81}$$

where for $m = 1$ and any $\gamma_1, \gamma_2 > -1$, $\beta_2 > 0$.

Taking into account estimates (12) for $A_{n,p}^2(z,1)$ and (13) for $|P_n(z)|$ and then substituting them into (81), we have

$$\left|P_n'(z)\right| \leq \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p \begin{cases} \begin{cases} n^{\frac{\gamma_1+1}{p}(1+\widetilde{\kappa})}, & 1 < p < p_4^1, & \gamma_1 \geq \widetilde{\gamma}_5, & \gamma_2 > 0, \\ n^{\frac{\gamma_2+1}{p}\frac{1+\kappa}{1+\beta_2}}, & 1 < p < p_5^2, & 0 < \gamma_1 < \widetilde{\gamma}_5, & \gamma_2 > 0, \\ (n\ln n)^{1-\frac{1}{p}} & p = \max\left\{p_4^1; p_5^2\right\}, & \gamma_1 > 0, & \gamma_2 > 0, \\ n^{1-\frac{1}{p}} & p > \max\left\{p_4^1; p_5^2\right\}, & \gamma_1 > 0, & \gamma_2 > 0, \\ n^{\frac{1}{p}(1+\widetilde{\kappa})}, & p < 2+\widetilde{\kappa}, & -1 < \gamma_1 \leq 0, & -1 < \gamma_2 \leq 0, \\ (n\ln n)^{1-\frac{1}{p}}, & p = 2+\widetilde{\kappa}, & -1 < \gamma_1 \leq 0, & -1 < \gamma_2 \leq 0, \\ n^{1-\frac{1}{p}}, & p > 2+\widetilde{\kappa} & -1 < \gamma_1 \leq 0, & -1 < \gamma_2 \leq 0, \end{cases} \\ + \\ \begin{cases} n^{\frac{\gamma_1+1}{p}(1+\widetilde{\kappa})}, & 1 < p < p_1^1, & \gamma_1 \geq \widetilde{\gamma}_3, & \gamma_2 > 0, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+1+\widetilde{\kappa}}, & 1 < p < p_2^2, & 0 < \gamma_1 < \widetilde{\gamma}_3, & \gamma_2 > 0, \\ n^{1+\widetilde{\kappa}}(n\ln n)^{1-\frac{1}{p}}, & p = p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, & \gamma_2 > 0, \\ n^{1+\widetilde{\kappa}}(n\ln n)^{1-\frac{1}{p}}, & p = p_2^2, & 0 < \gamma_1 < \widetilde{\gamma}_2, & \gamma_2 > 0, \\ n^{2-\frac{1}{p}+\widetilde{\kappa}}, & p > p_1^1, & \gamma_1 \geq \widetilde{\gamma}_2, & \gamma_2 > 0, \\ n^{2-\frac{1}{p}+\widetilde{\kappa}}, & p > p_2^2, & 0 < \gamma_1 < \widetilde{\gamma}_2, & \gamma_2 > 0, \\ n^{\kappa(1-\frac{1}{p})+1+\widetilde{\kappa}} & p > 1, & -1 < \gamma_1 \leq 0, & -1 < \gamma_2 \leq 0, \end{cases} \end{cases}$$

for the $z \in \Omega_R(\delta)$;

$$\left|P_n'(z)\right| \le \frac{\left|\Phi^{2(n+1)}(z)\right|}{d(z,L)} \|P_n\|_p \left\{ \begin{array}{l} \left\{ \begin{array}{llll} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \gamma_1 \ge \widetilde{\gamma}_3, & \gamma_2>0, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}}, & 1<p<p_2^2, & 0<\gamma_1<\widetilde{\gamma}_3, & \gamma_2>0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=p_1^1, & \gamma_1 \ge \widetilde{\gamma}_2, & \gamma_2>0, \\ (n\ln n)^{1-\frac{1}{p}}, & p=p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, & \gamma_2>0, \\ n^{1-\frac{1}{p}}, & p>p_1^1, & \gamma_1 \ge \widetilde{\gamma}_2, & \gamma_2>0, \\ n^{1-\frac{1}{p}}, & p>p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, & \gamma_2>0, \\ n^{\kappa(1-\frac{1}{p})}, & p>1, & -1<\gamma_1\le 0 & -1<\gamma_2\le 0, \end{array} \right. \\ + \\ \left\{ \begin{array}{llll} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \gamma_1 \ge \widetilde{\gamma}_3, & \gamma_2>0, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+\widetilde{\kappa}}, & 1<p<p_2^2, & 0<\gamma_1<\widetilde{\gamma}_3, & \gamma_2>0, \\ n^{\widetilde{\kappa}}(n\ln n)^{1-\frac{1}{p}}, & p=p_1^1, & \gamma_1 \ge \widetilde{\gamma}_2, & \gamma_2>0, \\ n^{\widetilde{\kappa}}(n\ln n)^{1-\frac{1}{p}}, & p=p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, & \gamma_2>0, \\ n^{1-\frac{1}{p}+\widetilde{\kappa}}, & p>p_1^1, & \gamma_1 \ge \widetilde{\gamma}_2, & \gamma_2>0, \\ n^{1-\frac{1}{p}+\widetilde{\kappa}}, & p>p_2^2, & 0<\gamma_1<\widetilde{\gamma}_2, & \gamma_2>0, \\ n^{\kappa(1-\frac{1}{p})+\widetilde{\kappa}}, & p>1, & -1<\gamma_1\le 0, & -1<\gamma_2\le 0, \end{array} \right. \end{array} \right.$$

if $z \in \widehat{\Omega}_R(\delta)$. Performing addition on the right-hand sides, after simple calculations we find the corresponding estimate for the modulus of the first-order derivative.

Proof of Theorem 2.5. From Corollary 2.2, we know that

$$P_n''(z) \le \left|\Phi^{n+1}(z)\right| \left[\frac{\|P_n\|_p}{d(z,L)} A_{n,p}^2(z,2)) + C_2^1 B_{n,1}^1 \left|P_n'(z)\right| + C_2^2 B_{n,2}^1 \left|P_n(z)\right| \right].$$

In this last inequality, by considering the estimates (12) for $A_{n,p}^2(z,2)$, (13) for $|P_n(z)|$, (14) for $\left|P_n'(z)\right|$ and previously given related for $B_{n,v}^1$, $v=1,2$, we obtain

$$\left|P_n''(z)\right| \le \frac{\left|\Phi^{3(n+1)}(z)\right| \|P_n\|_p}{d(z,L)} A_{n,p}^6(z)$$

with the simple calculations where

$$A_{n,p}^6(z) := \left\{ \begin{array}{llll} n^{\left(\frac{\gamma_1+1}{p}+1\right)(1+\widetilde{\kappa})}, & 1<p<p_3, & \gamma_1 \ge \widetilde{\gamma}_1(2), & \beta_2 \le \kappa, \\ n^{\left(\frac{\gamma_1+1}{p}+1\right)(1+\widetilde{\kappa})}, & p \ge p_3, & \gamma_1>0, & \beta_2 \le \kappa, \\ n^{\left(\frac{\gamma_2+1}{p}+1\right)\frac{1+\kappa}{1+\beta_2}}, & p_3 \le p < p_6(2), & \gamma_1<\widetilde{\gamma}_1(2), & \beta_2>\kappa, \\ + & & & \\ n^{\left(\frac{\gamma_1+1}{p}+2\right)(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \gamma_1 \ge \widetilde{\gamma}_3, & \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+3(1+\widetilde{\kappa})}, & 1<p<p_1^1, & \widetilde{\gamma}_5 \le \gamma_1<\widetilde{\gamma}_3, & \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}(\ln n)^{1-\frac{1}{p}}, & p=p_1^1, & \gamma_1 \ge \widetilde{\gamma}_2, & \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}, & p>p_1^1, & \gamma_1 \ge \widetilde{\gamma}_2, & \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}, & p>q_1, & \widetilde{\gamma}_5 \le \gamma_1<\widetilde{\gamma}_2, & \\ n^{\left(1-\frac{1}{p}\right)+2(1+\widetilde{\kappa})}(\ln n)^{1-\frac{1}{p}}, & p=p_2^2, & \widetilde{\gamma}_5 \le \gamma_1<\widetilde{\gamma}_2, & \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+3(1+\widetilde{\kappa})}, & 1<p<p_2^2, & \gamma_1<\widetilde{\gamma}_5, & \\ n^{\left(1-\frac{1}{p}\right)+3(1+\widetilde{\kappa})}, & p>p_2^2, & \gamma_1<\widetilde{\gamma}_2, & \end{array} \right.$$

if $\gamma_1,\gamma_2 > 0$,

$$A^6_{n,p}(z) = \begin{cases} n^{\kappa(1-\frac{1}{p})+3(1+\widetilde{\kappa})}, & p > 1, \quad \beta_2 \le \kappa, \\ n^{\kappa(1-\frac{1}{p})+3(1+\widetilde{\kappa})}, & 1 < p < \frac{(1+\kappa)+(1+\beta_2)}{(1+\beta_2)-(1+\kappa)}, \quad \beta_2 > \kappa, \end{cases}$$

if $-1 < \gamma_1,\gamma_2 \le 0$, for the $z \in \Omega_R(\delta)$;

$$A^6_{n,p}(z) = \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})}, & 1 < p < p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}}, & 1 < p < p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_3, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \\ (n\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}}, & p > p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}}, & p > p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \end{cases} + \begin{cases} n^{\left(\frac{\gamma_1+1}{p}-1\right)(1+\widetilde{\kappa})+2\kappa}, & 1 < p < p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_3, \\ n^{\left[\frac{\gamma_2+1}{p}-1\right]\frac{1+\kappa}{1+\beta_2}+2\kappa}, & 1 < p < p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_3, \\ n^{1-\frac{1}{p}+2\kappa}(\ln n)^{1-\frac{1}{p}}, & p = p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+2\kappa}(\ln n)^{1-\frac{1}{p}}, & p = p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+2\kappa}, & p > p_1^1, \quad \gamma_1 \ge \widetilde{\gamma}_2, \\ n^{1-\frac{1}{p}+2\kappa}, & p > p_2^2, \quad 0 < \gamma_1 < \widetilde{\gamma}_2, \end{cases}$$

if $\gamma_1, \gamma_2 > 0$,

$$A^6_{n,p}(z) := n^{\kappa(1-\frac{1}{p})+2\kappa},$$

if $-1 < \gamma_1,\gamma_2 \le 0$ for the $z \in \widehat{\Omega}_R(\delta)$. Therefore, the proof of Theorem 2.5 is completed.

Proof of Theorem 2.6. Assume that $G \in \widetilde{PQ}\left(\kappa\,; f_i,\ g_i\right)$, for some $0 \le \kappa < 1$, $f_i(x) = c_i x^{1+\alpha_i}$, $\alpha_i \ge 0$, $i = \overline{1, l_1}$, and $g_i(x) = c_i x^{1+\beta_i}, \beta_i > 0$, $i = \overline{l_1+1, l}$. By the Cauchy integral formula for $m-th$ derivatives, we have

$$P_n^{(m)}(z) = \frac{m!}{2\pi i} \int\limits_{\partial B(z,d(z,L_R))} \frac{P_n(t)}{(t-z)^{m+1}} dt, \ m = 1, 2, ...$$

According to (3), we get

$$\left|P_n^{(m)}(z)\right| \le \frac{m!}{2\pi} \max_{z\in\partial B(z,d(z,L_R))} |P_n(t)| \int\limits_{\partial B(z,d(z,L_R))} \frac{|dt|}{|t-z|^{m+1}} \le \max_{t\in\overline{G}} |P_n(t)| \frac{1}{d^m(z,L_R)}.$$

Applying [14, Th.2.1] and using Lemma 3.3, we find that

$$\left|P_n^{(m)}(z)\right| \le \|P_n\|_p \cdot \left\{ n^{m\left(1+\widetilde{\kappa}\right)} + n^{m\frac{1+\kappa}{1+\beta_2}} \right\} \cdot \left\{ n^{\frac{(\gamma_1^*+1)(1+\widetilde{\kappa})}{p}} + n^{\left(\frac{\gamma_2^*}{1+\beta_2}+1\right)\frac{1+\kappa}{p}} \right\}.$$

So, we complete the proof of Theorem 2.6, since $z \in L$ is arbitrary.

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