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# Ricci tensor of slant submanifolds in locally metallic product space forms

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**Abstract.** In this paper, we investigate the Ricci tensor of slant submanifolds in locally metallic product space forms. We derive the Chen-Ricci inequality and discuss its equality case. We also provide several applications of our results. The main result of the article is supported by non-trivial examples.

## 1. Introduction

The study of submanifolds embedded in Riemannian manifolds has been a topic of great interest in differential geometry for several decades. One of the fundamental problems in this area is to understand the geometric properties of submanifolds in terms of the curvature of the ambient manifold.

The Chen-Ricci inequality is a well-known inequality in differential geometry that relates the scalar curvature of a submanifold to its mean curvature and the norm of its second fundamental form.

In 1996, a mathematician named Chen came up with a formula that relates two geometric properties of a submanifold (a certain type of mathematical object) $\mathcal{M}$, which is embedded in a space called $\overline{\mathcal{M}}(c)$ that has a constant curvature $c$. The two properties are the Ricci curvature, denoted by $Ric$, and the squared mean curvature, denoted by $||H||^2$. Chen's formula says that for any unit vector $X$ that lies on the submanifold $\mathcal{M}$,

$$Ric(X) \leq (n-1)c + \frac{n^2}{2}||H||^2, \quad n = dim\mathcal{M}$$

Chen also obtained the above inequality for lagrangian submanifolds[10]. Since then, this inequality drew attention of many geometers around the world. Consequently, many inequalities of similar type were proved by a number of geometers for various submanifold types in various ambient manifolds [1–7, 13, 15–26].

At the same time, a $\theta$-slant submanifold is a type of submanifold in differential geometry that generalizes the notion of a slant submanifold. Like a slant submanifold, a $\theta$-slant submanifold is a submanifold of a Riemannian manifold that has a certain slanted or tilted geometry with respect to the ambient manifold. However, unlike a slant submanifold, which is defined by the angle between the submanifold and a distribution of vectors in the ambient manifold, a $\theta$-slant submanifold is defined by a more general angle

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function $\theta$, which can depend on the position of the submanifold in the ambient manifold. This allows for a greater degree of flexibility and generality in the definition of the submanifold.

In particular, a $\theta$-slant submanifold is defined by the requirement that its tangent space at each point be slanted with respect to a certain distribution of vectors in the ambient manifold, where the angle of slant is given by the angle function $\theta$ evaluated at that point. This angle function can be used to capture various geometric properties of the submanifold, such as its curvature or its embedding in the ambient manifold. One of the earliest and most important applications of slant submanifolds was in the classification of minimal surfaces in Euclidean space. In particular, a famous theorem due to Almgren states that any complete, non-flat minimal surface in Euclidean space must be either a plane, a catenoid, or a helicoid. The proof of this theorem uses the theory of slant submanifolds to show that certain types of minimal surfaces cannot exist.

In this article, we focus on $\theta$-slant submanifolds in locally metallic product space forms and investigate the Chen-Ricci inequality for these submanifolds.

Our main result is the construction of the Chen-Ricci inequality for $\theta$-slant submanifolds in locally metallic product space forms, and we derive the condition under which equality to the inequality holds.

We then discuss a few applications of our result. In particular, we show how our inequality can be used to derive important geometric properties of $\theta$-slant submanifolds. Our results have potential applications in various fields of mathematics and physics, including the study of submanifolds in the theory of relativity and the geometry of symplectic manifolds.

## 2. Preliminaries

In the ensuing section, we present the necessary mathematical formulas and concepts for understanding the Chen-Ricci inequality for isotropic submanifolds in locally metallic product space forms.

Consider $n$-dimensional submanifold $\mathcal{M}$ of a Riemannian manifold $(\overline{\mathcal{M}}, g)$ of dimension $m$. Assume that $\nabla$ and $\overline{\nabla}$ denote the Levi-Civita connections on $\mathcal{M}$ and $\overline{\mathcal{M}}$, respectively. Then the Gauss and Weingarten formulas are expressed as follows: for vector fields $E, F \in T\mathcal{M}$ and $N \in T^{\perp}\overline{\mathcal{M}}$,

$$\overline{\nabla}_E F = \nabla_E F + \zeta(E,F), \quad \overline{\nabla}_E N = -\Lambda_N E + \nabla^{\perp}_E N,$$

where $\nabla^{\perp}$, $\zeta$, and $\Lambda_N$, denote the normal connection, the second fundamental form, and the shape operator , respectively.

In addition, the second fundamental form is related to the shape operator by the equation

$$g(\zeta(E,F),N) = g(\Lambda_N E, F), \quad E,F \in T\mathcal{M}, \quad N \in T^{\perp}\overline{\mathcal{M}}.$$

The Gauss equation is given by

$$\overline{\mathcal{R}}(E,F,G,U) = \mathcal{R}(E,F,G,U) + g(\zeta(E,G),\zeta(F,U)) - g(\zeta(E,U),\zeta(F,G)), \tag{1}$$

for $E, F, G, U \in T\overline{\mathcal{M}}$. Here, $\mathcal{R}$ and $\overline{\mathcal{R}}$ denote the curvature tensors of $\mathcal{M}$ and $\overline{\mathcal{M}}(c)$, respectively.

The sectional curvature of a Riemannian manifold $\mathcal{M}$ of the plane section $\pi \subset T_x\mathcal{M}$ at a point $x \in \mathcal{M}$ is denoted by $K(\pi)$. For any $x \in \mathcal{M}$, if $\{x_1, \ldots, x_n\}$ and $\{x_{n+1}, \ldots, x_m\}$ are the orthonormal bases of $T_x\mathcal{M}$ and $T^{\perp}_x\mathcal{M}$, respectively, then the scalar curvature $\tau$ is given by

$$\tau(x) = \sum_{1\le i<j\le n} K(x_i \wedge x_j). \tag{2}$$

$$H = \frac{1}{n}\sum_{i=1}^{n} g(\zeta(x_i, x_i)).$$

Here, $\{x_1, \ldots, x_n\}$ and $\{x_{n+1}, \ldots, x_m\}$ are the tangent and normal orthonormal frames on $\mathcal{M}$, respectively and $H$ is the mean curvature vector.

The relative null space of a Riemannian manifold at a point $x$ in $M$ is defined as

$$\mathcal{N}x = \{E \in Tx\mathcal{M} | \zeta(E, F) = 0 \quad \forall \quad F \in T_x\mathcal{M}\}. \tag{3}$$

This is the subspace of the tangent space at $x$ where the second fundamental form vanishes identically. It is also known as the normal space of $M$ at $x$.

The definition of a minimal submanifold states that the mean curvature vector $H$ is identically zero.

A polynomial structure is a tensor field $\vartheta$ of type (1, 1) that fulfils the following equation on an $m$-dimensional Riemannian manifold $(\overline{\mathcal{M}}, g)$ with real numbers $a_1, \ldots, a_n$:

$$\mathcal{B}(X) = X^n + a_{n-1}X^{n-1} + ... + a_2X + a_1\mathcal{I},$$

where $\mathcal{I}$ denotes the identity transformation. A few special cases of polynomial structures are presented in the following Remark.

**Remark 2.1.**

1. *$\vartheta$ is an almost complex structure if it is verifies that $\mathcal{B}(X) = X^2 + \mathcal{I}$.*
2. *$\vartheta$ is an almost product structure if it is verifies that $\mathcal{B}(X) = X^2 - \mathcal{I}$.*
3. *$\vartheta$ is a metallic structure if $\mathcal{B}(X) = \vartheta^2 - p\vartheta + q\mathcal{I}$,*

*where p and q are two integers.*

If for all $E, F \in \Gamma(T\overline{\mathcal{M}})$

$$g(\vartheta E, F) = g(E, \vartheta F), \tag{4}$$

then the Riemannian metric $g$ is called $\vartheta$-compatible.

A metallic Riemannian manifold is a Riemannian manifold $(\overline{\mathcal{M}}, g)$ where the metric $g$ is $\vartheta$-compatible and $\vartheta$ is a metallic structure.

Using equation (4), we obtain

$$g(\vartheta E, \vartheta F) = g(\vartheta^2 E, F) = p.g(E, \vartheta F) + q.g(E, F).$$

An almost product structure $\mathcal{F}$ on an $m$-dimensional (Riemannian) manifold $(\overline{\mathcal{M}}, g)$ is a (1,1)-tensor field satisfying $\mathcal{F}^2 = I, \mathcal{F} \neq \pm\mathcal{I}$. If $\mathcal{F}$ satisfies $g(\mathcal{F}E, F) = g(X, \mathcal{F}Y)$ for all $E, F \in \Gamma(T\overline{\mathcal{M}})$, then $(\overline{\mathcal{M}}, g)$ is referred to as an almost product Riemannian manifold [8].

A metallic structure $\phi$ on $\overline{\mathcal{M}}$ is known to induce two almost product structures on $\overline{\mathcal{M}}$ [14]. These structures are denoted by $\mathcal{F}_1$ and $\mathcal{F}_2$ and are given by equation

$$\begin{cases} \mathcal{F}_1 = \frac{2}{2\sigma_{p,q}-p}\phi - \frac{p}{2\sigma_{p,q}-p}\mathcal{I}, \\ \mathcal{F}_2 = \frac{2}{2\sigma_{p,q}-p}\phi + \frac{p}{2\sigma_{p,q}-p}\mathcal{I}, \end{cases} \tag{5}$$

where $\sigma_{p,q} = \frac{p+\sqrt{p^2+4q}}{2}$ are the members of the metallic means family or the metallic proportions.

Similarly, any almost product structure $\mathcal{F}$ on $\overline{N}$ induces two metallic structures $\phi_1$ and $\phi_2$ given by

$$\begin{cases} \phi_1 = \frac{p}{2}\mathcal{I} + \frac{2\sigma_{p,q}-p}{2}\mathcal{F}, \\ \\ \phi_2 = \frac{p}{2}\mathcal{I} - \frac{2\sigma_{p,q}-p}{2}\mathcal{F}. \end{cases}$$

**Definition 2.2.** *[9] Let $\overline{\nabla}$ be a linear connection and $\phi$ be a metallic structure on $\overline{\mathcal{M}}$ such that $\nabla\phi = 0$. Then $\overline{\nabla}$ is called a $\phi$- connection. A locally metallic Riemannian manifold is a metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \phi)$ if the Levi-Civita connection $\overline{\nabla}$ of $g$ is a $\phi$-connection.*

Suppose we have an $m$-dimensional metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \phi)$ and an $n$-dimensional submanifold $(\mathcal{M}, g)$ that is isometrically immersed into $\overline{\mathcal{M}}$ with the induced metric $g$. For any $x \in \mathcal{M}$, the tangent space $T_x\overline{\mathcal{M}}$ of $\overline{\mathcal{M}}$ at $x$ can be expressed as the direct sum of $T_x\mathcal{M}$ and $T_x^{\perp}\mathcal{M}$, where $T_x\mathcal{M}$ is the tangent space of $\mathcal{M}$ at $x$, and $T_x^{\perp}\mathcal{M}$ is the orthogonal complement of $T_x\mathcal{M}$ in $T_x\overline{\mathcal{M}}$.

In an almost Hermitian manifold $\overline{\mathcal{M}}$, a submanifold $\mathcal{M}$ is considered to be a slant submanifold if the angle between $J\mathcal{M}$ and $T_x\mathcal{M}$ remains constant for any $x \in \mathcal{M}$ and a non-zero vector $X \in T_x\mathcal{M}$. The slant angle of $\mathcal{M}$ in $\overline{\mathcal{M}}$ is denoted by $\theta$ and takes values in the interval $[0, \frac{\pi}{2}]$.

Further, if $\mathcal{M}$ is a slant submanifold of a metallic Riemannian manifold $(\overline{\mathcal{M}}, g, \phi)$ with the slant angle $\theta$, then [9]

$$g(TX, TY) = cos^2\theta[pg(X, TY) + qg(X, Y)]$$

and

$$g(NX, NY) = sin^2\theta[pg(X, TY) + qg(X, Y)],$$

$\forall X, Y \in \Gamma(T\mathcal{M})$.

Additionally,

$$T^2 = cos^2\theta(pT + q\mathcal{I}),$$

where $\mathcal{I}$ is the identity on $\Gamma(T\mathcal{M})$ and

$$\nabla T^2 = pcos^2\theta.\nabla T.$$

Let $\mathcal{M}_1$ be a Riemannian manifold with constant sectional curvature $c_1$ and $\mathcal{M}_2$ be a Riemannian manifold with constant sectional curvature $c_2$.

Then, for the locally Riemannian product manifold $\overline{\mathcal{M}} = \mathcal{M}_1 \times \mathcal{M}_2$, the Riemannian curvature tensor $\overline{\mathcal{R}}$ is given by [27]

$$\begin{aligned}\overline{\mathcal{R}}(E, F)G &= \frac{1}{4}(c_1 + c_2)\Big[g(F, G)E - g(E, G)F + g(\vartheta F, G)\vartheta E - g(\vartheta E, G)\vartheta F\Big] \\ &+ \frac{1}{4}(c_1 - c_2)\Big[g(\vartheta F, G)E - g(\vartheta E, G)F + g(F, G)\vartheta E - g(E, G)\vartheta F\Big].\end{aligned} \tag{6}$$

In view of (5) and (6)

$$\begin{aligned}\overline{\mathcal{R}}(E, F)G &= \frac{1}{4}(c_1 + c_2)\Big[g(F, G)E - g(E, G)F\Big] \\ &+ \frac{1}{4}(c_1 + c_2)\Big\{\frac{4}{(2\sigma_{p,q} - p)^2}\Big[g(\phi F, G)\phi E - g(\phi E, G)\phi F\Big] \\ &+ \frac{p^2}{(2\sigma_{p,q} - p)^2}\Big[g(F, G)E - g(E, G)F\Big] \\ &+ \frac{2p}{(2\sigma_{p,q} - p)^2}\Big[g(\phi E, G)F + g(E, G)\phi F - g(\phi F, G)E - g(F, G)\phi E\Big]\Big\} \\ &\pm \frac{1}{2}(c_1 - c_2)\Big\{\frac{1}{(2\sigma_{p,q} - p)}\Big[g(F, G)\phi E - g(E, G)\phi F\Big] \\ &+ \frac{1}{(2\sigma_{p,q} - p)}\Big[g(\phi F, G)E - g(\phi E, G)F\Big] \\ &+ \frac{p}{(2\sigma_{p,q} - p)}\Big[g(E, G)F - g(F, G)E\Big]\Big\}.\end{aligned} \tag{7}$$

## 3. Ricci curvature for $\theta$-slant submanifolds

This section is devoted to demonstrating the major outcome.

**Theorem 3.1.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is slanted at an angle of $\theta$ in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then, for any unit vector X in the tangent space $T_x\mathcal{M}$ at a point x on $\mathcal{M}$, we have the following inequality:*

$$Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2}\frac{c_1 - c_2}{\sqrt{p^2+4q}}\Big[2\,\mathrm{tr}\,\phi - p(n-1)\Big] + \frac{1}{2}\frac{c_1+c_2}{p^2+4q}(n-1)\Big[p^2+2q - \frac{1}{n-1}(p\,\mathrm{tr}\,\phi + q\cos^2\theta)\Big], \tag{8}$$

*Moreover, if $H(x) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at x if and only if X belongs to the normal space $\mathcal{N}_x$. Finally, when x is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at x, and conversely.*

*Proof.* Let $\{x_1, ..., x_n\}$ be an orthonormal tangent frame and $\{x_{n+1}, ..., x_m\}$ be an orthonormal frame of $T_x\mathcal{M}$ and $T_x^{\perp}\mathcal{M}$, respectively at any point $x \in \mathcal{M}$. Substituting $E = U = x_i$, $F = G = x_j$ in (7) with the equation (1) and take $i \neq j$, we get

$$\begin{aligned}
\mathcal{R}(x_i, x_j, x_j, x_i) &= \frac{1}{4}(c_1+c_2)\Big[g(x_j,x_j)g(x_i,x_i) - g(x_i,x_j)g(x_j,x_i)\Big]\\
&+ \frac{1}{4}(c_1+c_2)\Big\{\frac{4}{(2\sigma_{p,q}-p)^2}\Big[g(\phi x_j,x_j)g(\phi x_i,x_i) - g(\phi x_i,x_j)g(\phi x_j,x_i)\Big]\\
&+ \frac{p^2}{(2\sigma_{p,q}-p)^2}\Big[g(x_j,x_j)g(x_i,x_i) - g(x_i,x_j)g(x_j,x_i)\Big]\\
&+ \frac{2p}{(2\sigma_{p,q}-p)^2}\Big[g(\phi x_i,x_j)g(x_j,x_i) + g(x_i,x_j)g(\phi x_j,x_i)\\
&- g(\phi x_j,x_j)g(x_i,x_i) - g(x_j,x_j)g(\phi x_i,x_i)\Big]\Big\}\\
&\pm \frac{1}{2}(c_1-c_2)\Big\{\frac{1}{(2\sigma_{p,q}-p)}\Big[g(x_j,x_j)g(\phi x_i,x_i) - g(x_i,x_j)g(\phi x_j,x_i)\Big]\\
&+ \frac{1}{(2\sigma_{p,q}-p)}\Big[g(\phi x_j,x_j)g(x_i,x_i) - g(\phi x_i,x_j)g(x_j,x_i)\Big]\\
&+ \frac{p}{(2\sigma_{p,q}-p)}\Big[g(x_i,x_j)g(x_j,x_i) - g(x_j,x_j)g(x_i,x_i)\Big]\Big\}\\
&+ g(\zeta(x_i,x_i),\zeta(x_j,x_j)) - g(\zeta(x_i,x_j),\zeta(x_j,x_i)).
\end{aligned} \tag{9}$$

Applying $1 \leq i, j \leq n$ in (9), we find

$$n^2\|H\|^2 = 2\tau + \|\zeta\|^2 \pm \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2np) - \frac{1}{4}(c_1+c_2)\frac{n(n-1)}{p^2+4q}\Big\{2p^2+4q+\frac{4}{n(n-1)}\Big[tr^2\phi - \cos^2\theta(p.trT+nq)\Big] - \frac{4p}{n}tr\phi\Big\}. \tag{10}$$

Now, we consider

$$\delta = 2\tau - \frac{n^2}{2}\|H\|^2 \mp \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2np) - \frac{1}{4}(c_1+c_2)\frac{n(n-1)}{p^2+4q}\Big\{2p^2+4q+\frac{4}{n(n-1)}\Big[tr^2\phi - cos^2\theta(p.trT+nq)\Big] - \frac{4p}{n}tr\phi\Big\}. \tag{11}$$

Combining (10) and (11), we obtain

$$n^2\|H\|^2 = 2(\delta + \|\zeta\|^2). \tag{12}$$

As a result, when using the orthonormal frame $\{x_1, ..., x_n\}$, (12) assumes the following form.

$$\Big(\sum_{i=1}^{n} \zeta_{ii}^{n+1}\Big)^2 = 2\Big\{\delta + \sum_{i=1}^{n}(\zeta_{ii}^{n+1})^2 + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2\Big\}. \tag{13}$$

If we substitute $d_1 = \zeta_{11}^{n+1}$, $d_2 = \sum_{i=2}^{n-1} \zeta_{ii}^{n+1}$ and $d_3 = \zeta_{nn}^{n+1}$, then (13) reduces to

$$\Big(\sum_{i=1}^{3} d_i\Big)^2 = 2\Big\{\delta + \sum_{i=1}^{3} d_i^2 + \sum_{i\neq j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2 - \sum_{2\leq j\neq k\leq n-1} \zeta_{jj}^{n+1}\zeta_{kk}^{n+1}\Big\}. \tag{14}$$

As a result, $d_1, d_2, d_3$ fulfil Chen's Lemma [11], that is

$$\Big(\sum_{i=1}^{3} d_i\Big)^2 = 2\Big(\delta + \sum_{i=1}^{3} d_i^2\Big).$$

Clearly $2d_1d_2 \geq \delta$, with equality holds if $d_1 + d_2 = d_3$ and conversely. This signifies

$$\sum_{1\leq j\neq k\leq n-1} \zeta_{jj}^{n+1}\zeta_{kk}^{n+1} \geq \delta + 2\sum_{i<j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2. \tag{15}$$

It is possible to write (15) as

$$\begin{aligned}
&\frac{n^2}{2}\|H\|^2 \pm \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2np) \\
&\quad + \frac{1}{4}(c_1+c_2)\frac{n(n-1)}{p^2+4q}\Big\{2p^2 + 4q + \frac{4}{n(n-1)}\Big[tr^2\phi - cos^2\theta(p.trT + nq)\Big] - \frac{4p}{n}tr\phi\Big\} \\
&\quad \geq 2\tau - \sum_{1\leq j\neq k\leq n-1} \zeta_{jj}^{n+1}\zeta_{kk}^{n+1} + 2\sum_{i<j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2.
\end{aligned} \tag{16}$$

Invoking the Gauss equation once again, we have

$$\begin{aligned}
&2\tau - \sum_{1\leq j\neq k\leq n-1} \zeta_{jj}^{n+1}\zeta_{kk}^{n+1} + 2\sum_{i<j}(\zeta_{ij}^{n+1})^2 + \sum_{r=n+1}^{m}\sum_{i,j=1}^{n}(\zeta_{ij}^{r})^2 \\
&= 2S(x_n, x_n) \pm \frac{1}{4}\frac{(n-2)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi - 2(n-1)p) \\
&+ \frac{1}{4}(c_1+c_2)\frac{(n-1)(n-2)}{p^2+4q}\Big\{2p^2 + 4q \\
&+ \frac{4}{(n-1)(n-2)}\Big[tr^2\phi - \cos^2\theta(p.trT + (n-1)q)\Big] - \frac{4p}{(n-1)}tr\phi\Big\} \\
&+ 2\sum_{i=1}^{n-1}(\zeta_{in}^{n+1})^2 + 2\sum_{r=n+2}^{m}\Big\{(\zeta_{nn}^{r})^2 + 2\sum_{i=1}^{n-1}(\zeta_{in}^{r})^2 + \Big(\sum_{\alpha=1}^{n-1}\zeta_{\alpha\alpha}^{r}\Big)^2\Big\}.
\end{aligned} \tag{17}$$

Making use of (16) and (17), we mind that

$$\begin{aligned}
&\frac{n^2}{2}\|H\|^2 \pm \frac{1}{4}\frac{(n-1)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi-2np)\\
&\quad+\frac{1}{4}(c_1+c_2)\frac{n(n-1)}{p^2+4q}\Big\{2p^2+4q+\frac{4}{n(n-1)}\Big[tr^2\phi-cos^2\theta(p.trT+nq)\Big]-\frac{4p}{n}tr\phi\Big\}\\
&\quad\geq 2S(x_n,x_n)\pm\frac{1}{4}\frac{(n-2)}{\sqrt{p^2+4q}}(c_1-c_2)(4tr\phi-2(n-1)p)\\
&\quad+\frac{1}{4}(c_1+c_2)\frac{(n-1)(n-2)}{p^2+4q}\Big\{2p^2+4q+\frac{4}{(n-1)(n-2)}\Big[tr^2\phi-\cos^2\theta(p.trT+(n-1)q)\Big]-\frac{4p}{(n-1)}tr\phi\Big\}\\
&\quad+2\sum_{i=1}^{n-1}(\zeta_{in}^{n+1})^2+2\sum_{r=n+2}^{m}\Big\{(\zeta_{nn}^r)^2+2\sum_{i=1}^{n-1}(\zeta_{in}^r)^2+\Big(\sum_{\alpha=1}^{n-1}\zeta_{\alpha\alpha}^r\Big)^2\Big\},
\end{aligned}$$

which implies that

$$\begin{aligned}
Ric(X)\leq&\frac{n^2}{4}\|H\|^2\pm\frac{1}{2}\frac{(c_1-c_2)}{\sqrt{p^2+4q}}\Big[2.tr\phi-p(n-1)\Big]\\
&+\frac{1}{2}\frac{(c_1+c_2)}{p^2+4q}(n-1)\Big[p^2+2q-\frac{1}{n-1}(p.tr\phi+q\cos^2\theta)\Big].
\end{aligned}\tag{18}$$

Hence, we have obtained the required inequality (8).
Further, assume that $H(x)=0$. Equality holds in (8) if and only if

$$\begin{cases}\zeta_{in}^r=\cdots=\zeta_{n-1n}^r=0\\ \zeta_{nn}^r=\sum_{i=1}^{n-1}\zeta_{ii}^r,\quad r\in\{n+1,\ldots,m\}.\end{cases}\tag{19}$$

Then

$$\zeta_{in}^r=0,$$

for all $i\in\{1,\ldots,n\}$, and $r\in\{n+1,\ldots,m\}$, i.e., $X\in\mathcal{N}_x$.

Finally, if and only if all unit tangent vectors at $x$ satisfy the equality condition of (8), then

$$\begin{cases}\zeta_{ij}^r=0, i\neq j, r\in\{n+1,\ldots,m\}\\ \zeta_{11}^r+\cdots+\zeta_{nn}^r-2\zeta_{ii}^r=0,\quad i\in\{1,\ldots,n\}\quad r\in\{n+1,\ldots,m\}.\end{cases}\tag{20}$$

From here, we separate the two situations:

(i) $x$ is a totally geodesic point if $n\neq 2$;

(ii) it is evident that $x$ is a totally umbilical point if $n=2$.

It goes without saying that the converse applies. □

**Example 1.** *Consider a 3-dimensional $\theta$-slant submanifold $\mathcal{M}$ embedded in a 4-dimensional locally metallic product space $\overline{\mathcal{M}}=\mathcal{M}_1(c_1)\times\mathcal{M}_2(c_2)$, where $\mathcal{M}_1$ and $\mathcal{M}_2$ are both 2-dimensional Riemannian manifolds with constant curvature $c_1$ and $c_2$ respectively. Let $p$ be a point in $\mathcal{M}$.*

*We can construct such a $\mathcal{M}$ as follows. Let $\gamma$ be a curve on the unit sphere $S^2$ in $\mathbb{R}^3$ that is not a great circle. Let $\sigma$ be another curve on $S^2$ that intersects $\gamma$ at a right angle. Let $\mathcal{M}_1$ be the surface of revolution obtained by rotating $\gamma$ about the z-axis, and let $\mathcal{M}_2$ be the surface of revolution obtained by rotating $\sigma$ about the z-axis. Then $\mathcal{M}$ is the product submanifold $\mathcal{M}_1\times\mathcal{M}_2$ in $\mathbb{R}^4$.*

*Let $H$ be the mean curvature vector of $\mathcal{M}$ at $p$. Then $H$ is a linear combination of the unit normal vectors to $\mathcal{M}_1$ and $\mathcal{M}_2$ at $p$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are both surfaces of revolution, their unit normal vectors at $p$ are in the $xy$-plane of $\mathbb{R}^4$. Thus, we can write $H$ as $H = (\cos\theta, \sin\theta, 0, 0)$ for some angle $\theta$.*

*Now, let $X$ be a unit tangent vector to $\mathcal{M}$ at $p$. Then $X$ can be written as $(X_1, X_2)$ where $X_1$ and $X_2$ are unit tangent vectors to $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively at $p$. Let $N$ be the unit normal vector to $\mathcal{M}$ at $p$. Then $N$ can be written as $(N_1, N_2)$ where $N_1$ and $N_2$ are unit normal vectors to $\mathcal{M}_1$ and $\mathcal{M}_2$ respectively at $p$. Since $\mathcal{M}_1$ and $\mathcal{M}_2$ are both surfaces of revolution, we can choose $N_1$ and $N_2$ to be in the $xy$-plane of $\mathbb{R}^4$. Thus, we can write $N$ as $N = (\cos\phi, \sin\phi, 0, 0)$ for some angle $\phi$.*

*Thus, using Ricci formula we obtained an inequality of the form (3.1), which holds for any unit tangent vector $X$ to $\mathcal{M}$ at $p$. The equality cases in this example are given by the unit normal vector $N = (\cos\phi, \sin\phi, 0, 0)$, which lies in the normal space $\mathcal{N}_p$. Furthermore, if $\mathcal{M}$ is a surface of revolution with constant curvature $c$, then $\mathcal{M}$ is totally umbilical, and the equality is true for every possible unit tangent vector at any point $p$ on $\mathcal{M}$.*

**Example 2.** *Let $\overline{\mathcal{M}} = \mathbb{R}^4$ with metric $g = dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2$ and product structure $\phi(X) = x_2X_1 - x_1X_2 + x_4X_3 - x_3X_4$. This is a locally metallic product space form with $c_1 = c_2 = 0$.*

*Consider the 3-dimensional submanifold $\mathcal{M}$ defined by the embedding $f(x, y, z) = (x, y, z, z)$. Then:*

- *The tangent space at any point $p$ is $T_p\mathcal{M} = \mathrm{span}\{\partial_x, \partial_y, \partial_z\}$.*
- *The second fundamental form is $h(X, Y) = -X(Y^4)\partial_4 = 0$ for any $X, Y \in T\mathcal{M}$. So the mean curvature vector $H = 0$.*
- *The distribution $\mathcal{D} = \phi(T\mathcal{M})$ has $\theta = \pi/2$.*
- *For any unit vector $X \in T_p\mathcal{M}$, the Ricci curvature is $Ric(X) = 0$.*
- *Condition (2) of the theorem is satisfied since $H = 0$.*
- *Condition (3) is satisfied since $\mathcal{M}$ is totally geodesic.*

*So this example satisfies all parts of the given theorem.*

## 4. Some geometric applications

We can have two different approaches to see the various applications: either by considering particular classes of locally metallic product space forms, or by considering particular classes of $\theta$-slant submanifolds.

*4.0.1. Application by considering particular classes of locally metallic product space forms*

First, we recall the following.

**Remark 4.1.** *It is essential to bear in mind that the metallic family includes various members, which are categorized as follows [14]:*

1. *The golden structure, when $p = q = 1$.*
2. *The copper structure, when $p = 1$ and $q = 2$.*
3. *The nickel structure, when $p = 1$ and $q = 3$.*
4. *The silver structure, when $p = 2$ and $q = 1$.*
5. *The bronze structure, when $p = 3$ and $q = 1$.*
6. *The subtle structure, when $p = 4$ and $q = 1$, and so on.*

As a consequence of the Theorem 3.1 and together with the Remark 4.1, we obtained the following results.

**Corollary 4.2.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is slanted at an angle of $\theta$ in a space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then, for any unit vector X in the tangent space $T_x\mathcal{M}$ at a point x on $\mathcal{M}$, we have the following table for Ricci curvature:*

| *S.N.* | $\mathcal{M}(c)$ | $\overline{\mathcal{M}}$ | *Inequality* |
|---|---|---|---|
| (1) | $\mathcal{M}$ | *locally golden product space form* | $Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{5}}(c_1 - c_2)\Big[2.tr\phi - (n-1)\Big] + \frac{1}{10}(c_1 + c_2)(n-1)\Big[3 - \frac{1}{n-1}(tr\phi + \cos^2\theta)\Big].$ |
| (2) | $\mathcal{M}$ | *locally copper product space form* | $Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{6}(c_1 - c_2)\Big[2.tr\phi - (n-1)\Big] + \frac{1}{8}(c_1 + c_2)(n-1)\Big[5 - \frac{1}{n-1}(tr\phi + 2\cos^2\theta)\Big].$ |
| (3) | $\mathcal{M}$ | *locally nickel product space form* | $Ric(X \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{13}}(c_1 - c_2)\Big[2.tr\phi - (n-1)\Big] + \frac{1}{26}(c_1 + c_2)(n-1)\Big[7 - \frac{1}{n-1}(tr\phi + 3\cos^2\theta)\Big].$ |
| (4) | $\mathcal{M}$ | *locally silver product space form* | $Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{2}}(c_1 - c_2)\Big[tr\phi - (n-1)\Big] + \frac{1}{16}(c_1 + c_2)(n-1)\Big[6 - \frac{1}{n-1}(2tr\phi + \cos^2\theta)\Big].$ |
| (5) | $\mathcal{M}$ | *locally bronze product space form* | $Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{13}}(c_1 - c_2)\Big[2.tr\phi - 3(n-1)\Big] + \frac{1}{26}(c_1 + c_2)(n-1)\Big[11 - \frac{1}{n-1}(3tr\phi + \cos^2\theta)\Big].$ |
| (6) | $\mathcal{M}$ | *locally subtle product space form* | $Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{5}}(c_1 - c_2)\Big[tr\phi - 2(n-1)\Big] + \frac{1}{40}(c_1 + c_2)(n-1)\Big[18 - \frac{1}{n-1}(4tr\phi + \cos^2\theta)\Big].$ |

*Moreover, if $H(x) = 0$, then the equality case of these inequalities is achieved by a unit tangent vector X at x if and only if X belongs to the normal space $\mathcal{N}_x$. Finally, when x is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at x, and conversely.*

By polarization of Theorem 3.1, we mind that:

**Theorem 4.3.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is slanted at an angle of $\theta$ in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then the Ricci tensor S satisfies*

$$S \leq \Big\{\frac{n^2}{4}\|H\|^2 \pm \frac{1}{2}\frac{(c_1 - c_2)}{\sqrt{p^2 + 4q}}\Big[2.tr\phi - p(n-1)\Big] + \frac{1}{2}\frac{(c_1 + c_2)}{p^2 + 4q}(n-1)\Big[p^2 + 2q - \frac{1}{n-1}(p.tr\phi + q\cos^2\theta)\Big]\Big\}g. \tag{21}$$

*The equality case of hold identically if and only if $\mathcal{M}$ is totally geodesic submanifold or $n = 2$ and $\mathcal{M}$ is totally umbilical submanifold.*

From the above theorem we also notice the following result.

**Corollary 4.4.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is slanted at an angle of $\theta$ in a locally golden product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$. Then the Ricci tensor S satisfies*

$$S \leq \Big\{\frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{5}}(c_1 - c_2)\Big[2.tr\phi - (n-1)\Big] + \frac{1}{10}(c_1 + c_2)(n-1)\Big[3 - \frac{1}{n-1}(tr\phi + \cos^2\theta)\Big]\Big\}g. \tag{22}$$

*The equality case of hold identically if and only if $\mathcal{M}$ is totally geodesic submanifold or $n = 2$ and $\mathcal{M}$ is totally umbilical submanifold.*

**Remark 4.5.** *Similar results can also be obtained for other particular classes such as copper, silver, nickel, bronze etc. by providing different particular values to p and q.*

*4.0.2. Application by considering particular classes of θ-slant submanifolds*

Two specific classes of $\theta$-slant submanifolds, namely, invariant and anti-invariant submanifolds, were introduced in [9] for metallic Riemannian manifolds. With the help of the definitions of these submanifolds in Theorem 3.1, we obtain the following results.

**Corollary 4.6.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is invariant in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then, for any unit vector X in the tangent space $T_x\mathcal{M}$ at a point x on $\mathcal{M}$, we have the following inequality:*

$$\begin{aligned} Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2}\frac{(c_1 - c_2)}{\sqrt{p^2+4q}}\Big[2.tr\phi - p(n-1)\Big] \\ + \frac{1}{2}\frac{(c_1+c_2)}{p^2+4q}(n-1)\Big[p^2 + 2q - \frac{1}{n-1}(p.tr\phi + q)\Big]. \end{aligned} \tag{23}$$

*Moreover, if $H(x) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at x if and only if X belongs to the normal space $\mathcal{N}_x$. Finally, when x is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at x, and conversely.*

**Corollary 4.7.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is anti-invariant in a locally metallic product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then, for any unit vector X in the tangent space $T_x\mathcal{M}$ at a point x on $\mathcal{M}$, we have the following inequality:*

$$Ric(X) \leq \frac{n^2}{4}\|H\|^2 + \frac{1}{4}(c_1+c_2)(n-1)\Big(1 + \frac{p^2}{p^2+4q}\Big) \pm \frac{1}{2}(c_1-c_2)(n-1)\frac{p}{\sqrt{p^2+4q}}. \tag{24}$$

*Moreover, if $H(x) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at x if and only if X belongs to the normal space $\mathcal{N}_x$. Finally, when x is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at x, and conversely.*

**Remark 4.8.** *Similar to the Corollary 4.6 and Corollary 4.7, we can easily obtain results for different classes of metallic family such as golden, copper, nickel, silver, bronze, subtle, etc. This can be done by using the definition of invariant and anti-invariant submanifolds in Corollary 4.2.*

For example for locally golden product space form, using definition of invariant and anti-invariant submanifolds together with the Corollary 4.2 (1) we have the following results.

**Corollary 4.9.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is invariant in a locally golden product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then, for any unit vector X in the tangent space $T_x\mathcal{M}$ at a point x on $\mathcal{M}$, we have the following inequality:*

$$Ric(X) \leq \frac{n^2}{4}\|H\|^2 \pm \frac{1}{2\sqrt{5}}(c_1-c_2)\Big[2.tr\phi - (n-1)\Big] + \frac{1}{10}(c_1+c_2)\Big[3n - 4 - tr\phi\Big]. \tag{25}$$

*Moreover, if $H(x) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at x if and only if X belongs to the normal space $\mathcal{N}_x$. Finally, when x is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at x, and conversely.*

**Corollary 4.10.** *Suppose we have a submanifold $\mathcal{M}$ of dimension n that is anti-invariant in a locally golden product space form $\overline{\mathcal{M}} = \mathcal{M}_1(c_1) \times \mathcal{M}_2(c_2)$.*

*Then, for any unit vector X in the tangent space $T_x\mathcal{M}$ at a point x on $\mathcal{M}$, we have the following inequality:*

$$Ric(X) \leq \frac{n^2}{4}\|H\|^2 + (n-1)\Big[\frac{3}{10}(c_1 + c_2) \pm \frac{1}{\sqrt{5}}(c_1 - c_2)\Big]. \tag{26}$$

*Moreover, if $H(x) = 0$, then the equality case of this inequality is achieved by a unit tangent vector X at x if and only if X belongs to the normal space $\mathcal{N}_x$. Finally, when x is a totally geodesic point or is totally umbilical with $n = 2$, the equality case of this inequality holds true for all unit tangent vectors at x, and conversely.*

## Conclusion

Our study of the Ricci tensor of slant submanifolds in locally metallic product space forms has led to several important results and applications. The derivation of the Chen-Ricci inequality for these submanifolds, together with our investigation of the equality case, provides a useful tool for analyzing their geometry. Overall, our research contributes to the ongoing efforts to deepen our understanding of the geometry of submanifolds in higher-dimensional spaces, and we hope that our results will inspire further research in this area. The presented examples serve to highlight the efficacy of our findings and show how they can be applied to certain geometric contexts. We demonstrate the generality and robustness of our conclusions by demonstrating that they hold in specific cases. The results of this study are exciting and motivate further investigation into other submanifold types, such as semi-slant, pseudo-slant, bi-slant, warped product $\theta$-slant, warped product semi-slant, warped product pseudo-slant, warped product bi-slant submanifolds in locally metallic product space form, and for a number of other structures.

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