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# Geometric and physical characterizations of a spacetime concerning a novel curvature tensor

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**Abstract.** In this article, we introduce Ψ-concircular curvature tensor, a new tensor that generalizes the concircular curvature tensor. At first, we produce a few fundamental geometrical properties of Ψ-concircular curvature tensor and pseudo Ψ-concircularly symmetric manifolds and provide some interesting outcomes. Besides, we investigate Ψ-concircularly flat spacetimes and establish some significant results about Minkowski spacetime, RW-spacetime, and projective collineation. Moreover, we show that if a Ψ-concircularly flat spacetime admits a Ricci bi-conformal vector field, then it is either Petrov type N or conformally flat. Moreover, we consider pseudo Ψ concircularly symmetric spacetime with Codazzi type of Ricci tensor and prove that the spacetime is of Petrov types $I$, $D$ or $O$ and the spacetime turns into a $RW$ spacetime. Also, we establish that in a pseudo Ψ concircularly symmetric spacetime with harmonic Ψ-concircular curvature tensor, the semi-symmetric energy momentum tensor and Ricci semi-symmetry are equivalent. At last, we produce a non-trivial example to validate the existence of a $(PCS)_4$ manifold.

## 1. Introduction

The concircular curvature tensor $C$ plays a significant role in differential geometry and general relativity (briefly, $GR$). In semi-Riemannian manifolds, $C$ is demonstrated as [29]

$$C(U,V)G = K(U,V)G - \frac{R}{n(n-1)}[g(V,G)U - g(U,G)V], \tag{1}$$

in which $R$ and $K$ are scalar curvature and the (1,3) type Riemannian curvature tensor, respectively.

Investigating the nature of curvature features is one of the primary concerns in differential geometry. In light of the previously mentioned concept, we create a new curvature tensor in the present investigation that we call the Ψ-concircular curvature tensor.

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In a semi-Riemannian manifold $M^n$, we describe, the $\Psi$-concircular curvature tensor $C$ of type (1,3) in the following way:

$$\begin{aligned} C(U,V)G &= K(U,V)G \\ &\quad -\frac{\Psi R}{n(n-1)}[g(V,G)U - g(U,G)V], \end{aligned} \tag{2}$$

in which $\Psi$ is an arbitrary scalar function.

The aforesaid curvature tensor, in particular, simplifies to a concircular curvature tensor for $\Psi = 1$.

The preceding equation can also be written as

$$\begin{aligned} \widehat{C}(U,V,G,H) &= \widehat{K}(U,V,G,H) \\ &\quad -\frac{\Psi R}{n(n-1)}[g(V,G)g(U,H) - g(U,G)g(V,H)], \end{aligned} \tag{3}$$

in which $\widehat{C}(U,V,G,H) = g(C(U,V)G,H)$ and $\widehat{K}$ stands for the (0,4)type Riemannian curvature tensor given by

$$\widehat{K}(U,V,G,H) = g(K(U,V)G,H). \tag{4}$$

Suppose that $\{e_k\}$ $(1 \le k \le n)$ is an orthonormal basis of the tangent space at every point of the manifold. Then the equation (3) entails

$$\sum_1^n \widehat{C}(V,G,e_k,e_k) = 0 = \sum_1^n \widehat{C}(e_k,e_k,V,G) \tag{5}$$

and

$$\begin{aligned} \sum_1^n \widehat{C}(e_k,e_k,G,H) &= \sum_1^n \widehat{C}(G,e_k,e_k,H) \\ &= (1-\Psi)S(G,H), \end{aligned} \tag{6}$$

in which $R = \epsilon_k \sum_{k=1}^n S(e_k,e_k)$ ($\epsilon_k = g(e_k, e_k)$, that is, $\epsilon_1 = -1, \epsilon_2 = \cdots = \epsilon_n = 1$).

Subsequent equations can be simply obtained from the equation(3)

$$\widehat{C}(U, V, G, H) = -\widehat{C}(V, U, G, H), \tag{7}$$

$$\widehat{C}(U,V,G,H) = -\widehat{C}(U,V,H,G), \tag{8}$$

$$\widehat{C}(U,V,G,H) = \widehat{C}(G,H,U,V) \tag{9}$$

and

$$\widehat{C}(U,V,G,H) + \widehat{C}(V,G,U,H) + \widehat{C}(G,U,V,H) = 0. \tag{10}$$

Since symmetric spaces are so important to differential geometry, we are focusing on them in the present paper. These areas were developed in [3].

For the Levi-Civita connection $\nabla$ of $(M^n, g)$, the Riemannian manifold $M^n$ is known as locally symmetric if $\nabla K = 0$ [3]. The aforementioned local symmetry criterion holds true for each point $U \in M^n$ where the local geodesic symmetry $F(U)$ is an isometry [24].

A non-flat semi-Riemannian manifold $(M, g), (n > 2)$ is called pseudo symmetric [4] if it's curvature tensor $K$ obeys

$$\begin{aligned} (\nabla_X \widehat{K})(U,V,G,H) &= 2D(X)\widehat{K}(U,V,G,H) + D(U)\widehat{K}(X,V,G,H) \\ &\quad + D(V)\widehat{K}(U,X,G,H) + D(G)\widehat{K}(U,V,X,H) \\ &\quad + D(H)\widehat{K}(U,V,G,X), \end{aligned} \tag{11}$$

in which $D$ denotes a non-zero 1-form and $\rho$ stands for the vector field described by

$$g(U, \rho) = D(U), \tag{12}$$

for all $U$. In the Cartan sense, $M$ turns into a symmetric manifold if $D = 0$. A n dimensional pseudo symmetric manifold is generally indicated by $(PS)_n$.

In this research, we looked at a semi-Riemannian manifold $(M^n, g), (n > 2)$ whose Ψ-concircular curvature tensor $\widehat{C}$ satisfies

$$\begin{aligned}(\nabla_X \widehat{C})(U, V, G, H) &= 2D(X)\widehat{C}(U, V, G, H) + D(U)\widehat{C}(X, V, G, H) \\ &+ D(V)\widehat{C}(U, X, G, H) + D(G)\widehat{C}(U, V, X, H) \\ &+ D(H)\widehat{C}(U, V, G, X).\end{aligned} \tag{13}$$

The manifold stated above is referred to as a pseudo Ψ-concircularly symmetric manifold and is designated by the symbol $(PCS)_n$.

The term Lorentzian manifold refers to a particular type of semi-Riemannian manifold that has a Lorentzian metric $g$. In $GR$, Spacetime is a connected time-oriented Lorentzian manifold $M^4$ having the signature $(-, +, +, +)$.

Perfect fluids (briefly, PFs) play a fascinating part in general relativity, lacking the heat conduction terms and the stress term related to viscosity [17]. In a PF, the energy momentum tensor (briefly, EMT) $T$ is described by

$$T(U, V) = (p + \sigma)B(U)B(V) + pg(U, V), \tag{14}$$

in which $B$ is a 1-form, $p$ and $\sigma$ stand for the isotropic pressure and the energy density, respectively [24] and $\rho$ is a unit time-like vector field $\{g(\rho, \rho) = -1\}$.

In absence of the cosmological constant in the theory of $GR$, Einstein's field equations are written as

$$S(U, V) - \frac{R}{2}g(U, V) = kT(U, V), \tag{15}$$

where $k$ is the gravitational constant.

In [31], Zengin and Tasci presented the investigation of pseudo-conharmonically symmetric manifolds. After that the authors have studied pseudo-conharmonically symmetric spacetimes [32]. Moreover, in [7], we have studied $\psi$-conharmonically symmetric spacetime. In [21], Mantica and Suh developed a new (1,3) type curvature, named $Q$ curvature tensor and pseudo $Q$-symmetric spacetimes have been investigated by Mallick and De [23], while $m$-Projectively flat spacetimes were explored by Ozen [30]. A condition for which a $(PS)_4$ spacetime would be a PF spacetime was recently discovered by Zhao et al.[33]. As well, many authors have looked at the spacetime of general relativity in various methods; for additional information, see ( [6], [15], [22]) and references therein.

Motivated by the above investigations here we characterize the $(PCS)_n$ manifold, Ψ-concircularly flat spacetimes and $(PCS)_4$ spacetimes.

The article is organized as:

The properties of Ψ-concircular curvature tensor are discussed in Section 2. In Section 3, we explore the curvature properties of $(PCS)_n$ $(n > 2)$ manifolds. Ψ-concircularly flat and $(PCS)_4$ spacetimes are investigated in Section 4. Finally, we produce a non-trivial example of a $(PCS)_4$ manifold.

## 2. Ψ-concircular curvature tensor

**Proposition 2.1.** *A Ψ-concircularly flat semi-Riemannian manifold $M^n$ is an Einstein manifold with vanishing scalar curvature.*

*Proof.* Assuming $\widehat{C} = 0$, we obtain from the equation (3) that

$$\widehat{K}(U, V, G, H) = \frac{\Psi R}{n(n-1)}[g(V, G)g(U, H) - g(U, G)g(V, H)]. \tag{16}$$

In the above equation, contracting $V$ and $G$ gives

$$S(U, H) = \frac{\Psi R}{n} g(U, H). \tag{17}$$

Hence, the manifold is an Einstein manifold.

Again, contracting over $U$ and $H$ yields

$$R[1 - \Psi] = 0. \tag{18}$$

Thus, either $R = 0$ or $\Psi = 1$.

As we choose $\psi$ as an arbitrary scalar function, then ignoring the case $\Psi = 1$, we conclude with $R = 0$.

Hence, the proof. □

**Theorem 2.2.** *A $\Psi$-concircularly flat manifold is locally a semi Euclidean space.*

*Proof.* Since the scalar curvature $R = 0$ in a $\Psi$-concircularly flat manifold, from the equation (16) we assert that the manifold has vanishing sectional curvature. Hence, a $\Psi$-concircularly flat space is locally a semi Euclidean space.

This ends the proof. □

**Theorem 2.3.** *A $\Psi$-concircularly flat manifold is a conformally flat manifold.*

*Proof.* For a semi-Riemannian manifold, the Weyl conformal curvature tensor $W$ is described as[29]

$$W(U, V)H = K(U, V)H - \frac{1}{n-2}[g(V, H)QU - g(U, H)QV + S(V, H)U - S(U, H)V] + \frac{R}{(n-1)(n-2)}[g(V, H)U - g(U, H)V]. \tag{19}$$

If we consider a $\Psi$-concircularly flat manifold, then by Proposition 2.1, we acquire that the manifold is of zero scalar curvature. Therefore, from equation (17) we acquire $S = 0$ which entails that the manifold is Ricci flat. Hence, using the foregoing results in the equation (19) we say that the manifold is conformally flat.

Thus the proof is completed. □

## 3. $(PCS)_n$ $(n > 2)$ manifolds

**Proposition 3.1.** *In $(PCS)_n$ manifolds, Bianchi's Second Identity is satisfied by the $\Psi$-concircular curvature tensor, that is,*

$$(\nabla_X \widehat{C})(U, V, G, H) + (\nabla_G \widehat{C})(U, V, H, X) + (\nabla_H \widehat{C})(U, V, X, G) = 0. \tag{20}$$

*Proof.* Using (13) in the left hand side of the previous equation, we get the desired result. □

**Proposition 3.2.** *If a $(PCS)_n$ is of divergence-free $\Psi$-concircular curvature tensor, then the scalar curvature vanishes.*

*Proof.* From (3), taking a frame field and contracting over $V$ and $G$, we obtain

$$C^{\star}(U,H) = S(U,H) - \frac{R\Psi}{n} g(U,H), \tag{21}$$

in which $C^{\star}$ indicates the contracted $\Psi$-conformal curvature tensor.

Now contracting the equation(13) over $X$ and $H$, we infer

$$\begin{aligned}(divC)(U,V)G &= D(C(U,V)G)) \\ &\quad + D(U)C^{\star}(V,G) - D(V)C^{\star}(U,G).\end{aligned} \tag{22}$$

Making use of (3) and (21) in the preceding equation, we acquire

$$\begin{aligned}(divC)(U,V)G &= D(K(U,V)G) \\ &\quad + \frac{\Psi R}{n(n-1)}[g(V,G)D(U) - g(U,G)D(V)] \\ &\quad + D(U)[S(V,G) - \frac{R\Psi}{n} g(V,G)] \\ &\quad - D(V)[S(U,G) - \frac{R\Psi}{n} g(U,G)].\end{aligned} \tag{23}$$

Let us choose $divC = 0$. Now contracting the equation (23), we provide

$$R\{1 - \Psi\}D(U) = 0. \tag{24}$$

Hence, either $R = 0$ or $\Psi = 1$, a contradiction(discussed earlier).

Therefore, the proof is finished. □

**Proposition 3.3.** *If a* $(PCS)_n$ $(n > 2)$ *manifold satisfies the Codazzi type of Ricci tensor, then the scalar curvature vanishes.*

*Proof.* Let in a $(PCS)_n$ $(n > 2)$ the Ricci tensor be of Codazzi type [13]. Therefore, we reveal

$$(\nabla_U S)(V,G) = (\nabla_V S)(U,G). \tag{25}$$

Equation (3) yields

$$\begin{aligned}(\nabla_X \widehat{C})(U,V,G,H) &= (\nabla_X \widehat{K})(U,V,G,H) \\ &\quad - \frac{(X\Psi)R + \Psi(XR)}{n(n-1)}[g(U,H)g(V,G) - g(V,H)g(U,G)].\end{aligned} \tag{26}$$

Using (26), we provide

$$\begin{aligned}&(\nabla_X \widehat{C})(U,V,G,H) + (\nabla_U \widehat{C})(V,X,G,H) + (\nabla_V \widehat{C})(X,U,G,H) \\ &= [(\nabla_X \widehat{K})(U,V,G,H) + (\nabla_U \widehat{K})(V,X,G,H) + (\nabla_V \widehat{K})(X,U,G,H)] \\ &- \frac{(X\Psi)R + \Psi(XR)}{n(n-1)}[g(U,H)g(V,G) - g(V,H)g(U,G)] \\ &- \frac{(U\Psi)R + \Psi(UR)}{n(n-1)}[g(V,H)g(X,G) - g(V,G)g(X,H)] \\ &- \frac{(V\Psi)R + \Psi(VR)}{n(n-1)}[g(U,G)g(X,H) - g(U,H)g(X,G)].\end{aligned} \tag{27}$$

Making use of the Second Bianchi identity for the Riemannian curvature tensor $\widehat{K}$, we acquire

$$\begin{aligned}
&(\nabla_X\widehat{C})(U,V,G,H) + (\nabla_U\widehat{C})(V,X,G,H) + (\nabla_V\widehat{C})(X,U,G,H) \\
&= -\frac{(X\Psi)R + \Psi(XR)}{n(n-1)}[g(U,H)g(V,G) - g(V,H)g(U,G)] \\
&-\frac{(U\Psi)R + \Psi(UR)}{n(n-1)}[g(V,H)g(X,G) - g(V,G)g(X,H)] \\
&-\frac{(V\Psi)R + \Psi(VR)}{n(n-1)}[g(U,G)g(X,H) - g(U,H)g(X,G)]. \qquad (28)
\end{aligned}$$

Now using (20) in the foregoing equation gives

$$\begin{aligned}
&\frac{(X\Psi)R + \Psi(XR)}{n(n-1)}[g(U,H)g(V,G) - g(V,H)g(U,G)] \\
&+\frac{(U\Psi)R + \Psi(UR)}{n(n-1)}[g(V,H)g(X,G) - g(V,G)g(X,H)] \\
&+\frac{(V\Psi)R + \Psi(VR)}{n(n-1)}[g(U,G)g(X,H) - g(U,H)g(X,G)] = 0. \qquad (29)
\end{aligned}$$

Contracting (29), we infer

$$\begin{aligned}
&\frac{(X\Psi)R + \Psi(XR)}{n}g(V,G) + \frac{(V\Psi)R + \Psi(VR)}{n(n-1)}g(X,G) \\
&-\frac{(X\Psi)R + \Psi(XR)}{n(n-1)}g(V,G) - \frac{(V\Psi)R + \Psi(VR)}{n}g(X,G) = 0. \qquad (30)
\end{aligned}$$

Again contracting the above equation, we provide

$$\frac{2-n}{n}[(X\Psi)R + \Psi(XR)] = 0. \qquad (31)$$

Since the Ricci tensor $S$ is of Codazzi type, then the scalar curvature $R$ = constant and therefore, $(XR) = 0$ for any $X$. Using this result in the previous equation, we get

$$(X\Psi)R = 0, \qquad (32)$$

which entails that either $R = 0$, or $\Psi$ = constant.

Since $\Psi$ is a scalar, it can not be constant and hence we acquire $R = 0$.

Hence the proof. □

## 4. Ψ-Concircularly flat spacetimes and $(PCS)_4$ spacetimes

In this article, we explore the 4 dimensional Ψ-concircularly flat and pseudo Ψ-concircularly symmetric spacetimes. The conclusions reached for the $(PCS)_n$ manifolds apply equally to the Lorentzian situation. Since we consider a spacetime, we may assume the associated vector field corresponding to the 1-form $D$ is a unit time-like vector field.

**Theorem 4.1.** *A Ψ-concircularly flat spacetime is locally isometric to a Minkowski spacetime and of Petrov classification O.*

*Proof.* From Proposition 2.1, we find that the scalar curvature $R = 0$ in a Ψ-concircularly flat spacetime. Then from the equation (3) we assert that the spacetime has vanishing sectional curvature. Hence, a Ψ-concircularly flat spacetime and Minkowski spacetime are locally isometric ([10], p. 67).

From Theorem 2.3, we say that a Ψ-concircularly flat spacetime is conformally flat and therefore belongs to Petrov classification $O$.

Hence, the proof is completed. □

**Theorem 4.2.** *A $\Psi$-concircularly flat PF spacetime with irrotational velocity vector field is a RW-spacetime.*

*Proof.* Now we choose a $\Psi$-concircularly flat PF-spacetime.

From Theorem 2.3, we say that a $\Psi$-concircularly flat spacetime is conformally flat, and hence the conformal curvature tensor is divergence free. In [20], it is established that a PF-spacetime with irrotational velocity vector field and with divergence free conformal curvature tensor reduces to a GRW-spacetime. Also, since the spacetime is conformally flat, then the GRW-spacetime is a RW spacetime [2].

Therefore, the proof is finished. □

**Theorem 4.3.** *A dust fluid spacetime satisfying EFE without cosmological constant with vanishing $\Psi$-concircular curvature tensor does not exist.*

*Proof.* For a spacetime of dust fluid [27], $T$ is described by

$$T(U,V) = \mu B_2(U)B_2(V), \tag{33}$$

in which $B_2$ stands for the velocity vector field and $\mu$ is the dust-like matter's energy density.

Using the equations (15) and (33), we obtain

$$k\mu B_2(U)B_2(V) = 0. \tag{34}$$

Taking a frame field and contraction the previous equation, we get

$$k\mu = 0, \tag{35}$$

Therefore, the equation (33) yields

$$T(U,V) = 0. \tag{36}$$

Hence, the fluid is vacuum. This is not a physically significant scenario bearing in mind that the universe contains matter.

This ends the proof. □

### 4.1. *Projective collineation*

If a continuous group of local diffeomorphism of $M$ maps geodesics into geodesics, it is referred to as projective collineation (PC) [1] and its generator is referred to as a projective vector field. A vector field $V$ is a PC if and only if

$$L_V\Gamma^i_{jk} = \delta^i_j q_k + \delta^i_k q_j,$$

where $L_V$ denotes the Lie derivative operator along $V$ and $q_j = q_{,j}$ in which $q$ is a 1-form. Therefore, locally $q_j$ is an exact form. Specifically, if $L_V\Gamma^i_{jk} = 0$, then the PC turns into the affine collineation or affine motion. We know that the projective vector field $V$ obeys

$$L_V K^l_{ijk} = \delta^l_k q_{i,j} - \delta^l_j q_{i,k}, \tag{37}$$

$$L_V K_{ij} = (1-n)q_{i,j}, \tag{38}$$

$$L_V P^l_{ijk} = 0, \tag{39}$$

where $K^l_{ijk}$, $K_{ij}$ and $P^l_{ijk}$ are the components of the curvature tensor, Ricci tensor, and projective curvature tensor, respectively.

**Theorem 4.4.** *Let a Ψ-concircularly flat spacetime admits a projective collineation $V$. Then*

*(i) the projective collineation is proper and, for dimension 4, the metric is either a pp-wave or flat, provided $q_i \neq 0$.*

*(ii) An affine collineation is generated by $V$ and the metric is a pp-wave or decomposable, or a homothetic Killing vector field, provided $q_i = 0$.*

*Proof.* Let us choose the PC in a Ψ-concircularly flat spacetime. We prove earlier that the Ψ-concircularly flat spacetime is Ricci flat and hence from (38), we acquire $q_{i,j} = 0$. If $q_i \neq 0$, then the PC is proper and for dimension 4, the metric must be either a pp-wave [12] or flat. If $q_i = 0$, then an affine collineation is generated by $V$ and for the spacetime the metric is a pp-wave or decomposable, or $V$ is a homothetic Killing vector field.

This finishes the proof. □

**Definition 4.5.** *On a semi Riemannian manifold a vector field $X$ is called Ricci bi-conformal vector field [8] if it obeys the subsequent equations*

$$L_X g = \alpha g + \beta S \tag{40}$$

*and*

$$L_X S = \alpha S + \beta g \tag{41}$$

*for non-zero smooth functions $\alpha$ and $\beta$.*

**Theorem 4.6.** *If a Ψ-concircularly flat spacetime admits a Ricci bi-conformal vector field, then it is either Petrov type N or conformally flat.*

*Proof.* Since in a Ψ-concircularly flat spacetime $S = 0$, then equation (40) implies $L_X g = \alpha g$ which entails that $X$ is a conformal vector field.

In [26], Sharma has established that " If a spacetime with divergence-free conformal curvature tensor permits a conformal Killing vector field, then it is either Petrov type N or conformally flat."

Obviously, in a Ψ-concircularly flat spacetime the divergence of the conformal curvature tensor vanishes.

Hence, the proof is completed. □

**Theorem 4.7.** *A $(PCS)_4$ spacetime satisfying Codazzi type of Ricci tensor is of Petrov type I, D or O and the spacetime turns into a RW spacetime.*

*Proof.* Let us assume a $(PCS)_4$ spacetime with the Codazzi type of Ricci tensor. Then by Proposition 3.3, we get the scalar curvature $R = 0$, and hence $(PCS)_4$ spacetime reduces to a $(PS)_4$ spacetime. We know that the 1-form $D$ is closed in a $(PS)_4$ spacetime [25] and hence $D$ is closed in a $(PCS)_4$ spacetime.

We know that

$$\begin{aligned}(divW)(U,V)X \;=\; & \frac{n-3}{n-2}[\{(\nabla_U S)(V,X) - (\nabla_V S)(U,X)\} \\ & -\frac{1}{2(n-1)}\{g(V,X)dR(U) - g(U,X)dR(V)\}].\end{aligned} \tag{42}$$

Since the Ricci tensor is of Codazzi type, then the conformal curvature tensor is divergence free. From [19] it is to be noted that in a $GRW$ spacetime, $W(U,V)\rho = 0$ if and only if $(div\, W)(U,V)Z = 0$. Also, $W(U,V)\rho = 0$ entails $W$ is purely electric[16]. The spacetimes are of Petrov types $I$, $D$ or $O$ if $W$ is purely electric ([28], p. 73).

In 4-dimension, $W(U,V)\rho = 0$ is identical to ([18], p. 128)

$$\begin{aligned}&\eta(U)\overline{W}(X,Y,Z,V) + \eta(X)\overline{W}(Y,U,Z,V) \\ &\quad +\eta(Y)\overline{W}(U,X,Z,V) = 0,\end{aligned} \tag{43}$$

where $\eta(X) = g(X,\rho)$ and $\overline{W}(X,Y,Z,V) = g(W(X,Y)Z,V)$ for all $X$, $Y$, $Z$, $V$, $U$.

Replacing $U$ by $\rho$ yields

$$\overline{W}(X,Y,Z,V)=0, \tag{44}$$

that is, the spacetime is conformally flat.

A $GRW$ spacetime is conformally flat if and only if it is a $RW$ spacetime [2].

Thus, we have the proof. □

**Definition 4.8.** *A semi Riemannian manifold is called Ricci semi-symmetric if the Ricci tensor S fulfills*

$$K(U,V)\cdot S=0,$$

*for all $U,V\in\chi(M)$, where $K(U,V)$ acts as a derivation on the curvature tensor $K$.*

**Theorem 4.9.** *In a $(PCS)_4$ spacetime with harmonic Ψ-concircular curvature tensor, the semi-symmetric EMT and Ricci semi-symmetry are equivalent.*

*Proof.* Consider a $(PCS)_4$ spacetime with harmonic Ψ-concircular curvature tensor. Then, using equation (15), we acquire

$$S(U,V)=kT(U,V). \tag{45}$$

From the last relation, we infer $K\cdot S=K\cdot T$. □

Also, from equation (45), we obtain $\nabla S=\nabla T$. Since $\nabla T=0$ entails $K\cdot T=0$, hence we write the subsequent:

**Corollary 4.10.** *A $(PCS)_4$ spacetime with harmonic Ψ-concircular curvature tensor and covariant constant EMT is Ricci semi-symmetric.*

For a general relativistic spacetime the foregoing result has been established in [5].

A spacetime is called conformally semi-symmetric [14] if it obeys $K\cdot W=0$ . A $(PCS)_4$ spacetime with harmonic Ψ-concircular curvature tensor is obviously conformally semi-symmetric. In [11], it is established that conformally semi-symmetric spacetimes and semi-symmetric spacetimes are equivalent and a semi-symmetric spacetimes is of Petrov types **D**, $N$, or $O$. Thus, we write:

**Corollary 4.11.** *A $(PCS)_4$ spacetime with harmonic Ψ-concircular curvature tensor is of Petrov types* **D**, *N or O.*

In [9], De and Velimirovic established the subsequent outcomes:

**Theorem 4.12.** *In a PF spacetime let the EMT be semi-symmetric. Then the spacetime is characterized by the subsequent cases:*

*(i) The PF behaves as a cosmological constant and the spacetime represents inflation. Also, it is named a phantom barrier.*

*(ii) The PF will start to behave as exotic matter or, equivalently it represents the quintessence barrier.*

**Remark 4.13.** *The above theorem holds in a $(PCS)_4$ spacetime with harmonic Ψ-concircular curvature tensor if the spacetime is Ricci semi-symmetric.*

## 5. Example

In a four-dimensional Lorentzian manifold $\mathbb{R}^4$, we define a Lorentzian metric $g$, which is written as

$$\begin{aligned} ds^2 &= g_{ij}d\mathrm{w}^i d\mathrm{w}^j \\ &= (d\mathrm{w}_1)^2+(\mathrm{w}_1)^2(d\mathrm{w}_2)^2+(\mathrm{w}_2)^2(d\mathrm{w}_3)^2-(d\mathrm{w}_4)^2, \end{aligned} \tag{46}$$

in which $i, j = 1, 2, 3, 4$.

Using (46), the Lorentzian metric's non-vanishing components are expressed as

$$g_{44} = -1, \;\; g_{33} = (w_2)^2, \;\; g_{22} = (w_1)^2, \;\; g_{11} = 1. \tag{47}$$

The components of the Christoffel symbols, the curvature tensors, and the Ricci tensor are provided by

$$\Gamma^1_{22} = -w_1, \;\; \Gamma^2_{12} = \frac{1}{w_1}, \;\; \Gamma^3_{23} = \frac{1}{w_2}, \;\; \Gamma^2_{33} = -\frac{w_2}{(w_1)^2},$$

$$K_{1332} = -\frac{w_2}{w_1}, \quad S_{12} = -\frac{1}{w_1 w_2},$$

in which equation (47) is used.

We shall prove that $(\mathbb{R}^4, g)$ is a $(PCS)_4$ manifold.

Now we choose $\Psi$ (the scalar) as

$$\Psi = \left\{ \frac{2}{w_2}, \quad \text{for any } w \in \mathbb{R}^4 \right.$$

The only components of $\Psi$-concircular curvature tensor that is not vanishing, along with its covariant derivatives, are expressed by

$$C_{1332} = -\frac{w_2}{w_1}, \tag{48}$$

$$C_{1332,1} = \frac{w_2}{(w_1)^2} \tag{49}$$

and

$$C_{1332,2} = -\frac{1}{w_1}. \tag{50}$$

Choose the following 1-forms

$$D_i = \begin{cases} -\frac{1}{3w_1}, & \text{when} \quad i = 1 \\ \frac{1}{3w_2}, & \text{for} \quad i = 2 \\ 0, & \text{for} \quad i = 3, 4. \end{cases}$$

In light of these 1-forms, the equation (13) becomes

$$C_{1332,1} = 3D_1 C_{1332}, \tag{51}$$

$$C_{1332,2} = 3D_2 C_{1332}, \tag{52}$$

since (13) trivially holds in all other situations. Equations (48) and (49) are used to acquire

$$\begin{aligned} \text{R.H.S. of (51)} &= 3D_1 C_{1332} \\ &= 3\left(-\frac{1}{3w_1}\right)\left(-\frac{w_2}{w_1}\right) \\ &= \frac{w_2}{(w_1)^2} \\ &= \text{L.H.S. of (51).} \end{aligned}$$

Utilizing the equations (48) and (50) yield

$$\begin{aligned} \text{R.H.S. of (52)} &= 3D_2C_{1332} \\ &= 3\frac{1}{3w_2}(-\frac{w_2}{w_1}) \\ &= -\frac{1}{w_1} \\ &= \text{L.H.S. of (52).} \end{aligned}$$

It follows that the manifold is a $(PCS)_4$ manifold.

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