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Interpolative contractive results for multivalued mappings over m-metric spaces with an application in homotopy

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Abstract. In this manuscript, we prove some new fixed point results for multivalued Kannan type interpolative contractive mappings over m-metric spaces. Our results cover all the possible cases of interpolative exponents: when their sum is equal to 1, less than 1 and greater than 1. We give examples to support our results. All the corresponding results in ordinary metric spaces are special cases of our new results for m-metric spaces. We also provide an application of homotopy to support our main result.

1. Introduction

Fixed point theory has bloomed in numerous dimensions after the proof of well known Banach contraction principle [1]. The technique of the Banach contraction principle has been exploited by many mathematicians to establish fixed point results for different types of contractions and various types of metric structures. Many researchers have worked on the technique of finding the fixed points for interpolative Kannan type contractions. To cite examples, Karapınar [2] developed some fixed point results for interpolative Kannan type contraction, noting that these mappings are more general than Kannan type contractions, Gaba and Karapınar [3] gave the result for the case when $\alpha + \beta < 1$ for interpolative Kannan type contractions, whereas Errai [4] proved his result for the case when $\alpha+\beta \geq 1$. All these results are proved in ordinary metric spaces and deal with single-valued mappings. For more results in this direction we will refer to see [5–9] and references mentioned therein. On the other hand, fixed point results for multivalued mappings have also been extended in many ways including different metric structures and contractions. Konwar et al. [10] provided some results for multivalued interpolative Kannan type contractions in ordinary metric spaces. Patle at el. [11] proved some fixed point results in m-metric spaces for Nadler and Kannan (multivalued) mappings. For more references, see [12, 13] and references therein.
Moreover, in the same direction many researchers have worked on interpolative contractions and have established quite interesting results, which are very useful in fixed point theory. To access these results, we refer to consult the references [14–19]. Additionally, the generalization of fixed point results in the structure of m-metric space is significant tool for addressing numerous problems in analysis. Therefore to explore notable results in m-metric spaces, we recommend referring to [20–30] and the references mentioned

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therein. Furthermore, the use of homotopy made a significant contribution in fixed point theory, especially in the study of existence and uniqueness of fixed points of continuous mappings. In this direction Frigon et al. [31] find the fixed points of homotopy over the structure of ordinary metric spaces. In [11] Patle et al. discuss the existence and uniqueness of fixed point of a homotopy mapping over m-metric spaces.
In this paper, we combine the ideas of multivalued interpolative contractions with the structure of m-metric and produce significant results for different possibilities of interpolative exponents like $\alpha+\beta=1$, $\alpha+\beta<1$ and $\alpha+\beta>1$. We have also verified all of our results with different examples. At the end, we gave an application of homotopy to support our main result.

2. Preliminaries

Firstly, we recall definition of m-metric spaces.

Definition 2.1. *[21] Let X be a nonempty set, a mapping $m: X\times X\to R^+$ is called the m-metric on X if it satisfies the following conditions*

(m_1) $m(x,y)=m(x,x)=m(y,y)\Leftrightarrow x=y$,

(m_2) $m_{xy}\leq m(x,y)$ *where* $m_{xy}:=\min\{m(x,x),m(y,y)\}$,

(m_3) $m(x,y)=m(y,x)$,

(m_4) $(m(x,y)-m_{xy})\leq(m(x,z)-m_{xz})+(m(z,y)-m_{zy})$,

for all $x,y,z\in X$. The pair (X,m) is called m-metric space.

Remark 2.2. *Every partial metric space is an m-metric space, but the converse may not be true, as we can see in the following example provided by Karpinar et al. in [32]*

Example 2.3. *Le $X=\{1,2,3\}$. Given m on $X\times X$ as follows:*
$m(1,1)=1,\ \ m(2,2)=3,\ \ m(3,3)=5,$
$m(1,2)=m(2,1)=10,\ \ m(1,3)=m(3,1)=7,\ \ m(2,3)=m(3,2)=7.$
So (X,m) is a m-metric space but it is not a partial metric space because $m(1,2)\nleq m(1,3)+m(3,1)-m(3,3)$.

Definition 2.4. *[21] Let (X,m) be a m-metric space. Then*
1. A sequence (x_n) in m-metric space converges to a point $x\in X$ iff

$$\lim_{n\to\infty}(m(x_n,x)-m_{x_nx})=0.$$

2. A sequence (x_n) in m-metric space (X,m) is called m-Cauchy sequence if the limits

$$\lim_{n,m\to\infty}(m(x_n,x_m)-m_{x_nx_m}),$$

and

$$\lim_{n,m\to\infty}(M_{x_n,x_m}-m_{x_nx_m}),$$

exists(and are finite), where $M_{x_nx_m}=\max(m(x_n,x_n),m(x_m,x_m))$.
3. A m-metric space (X,m) is said to be complete if every m-Cauchy sequence (x_n) in X converges to a point $x\in X$.

Lemma 2.5. *[21] Assume that $x_n\to x$ and $y_n\to y$ as $n\to\infty$ in m-metric space (X,m). Then*

$$\lim_{n\to\infty}(m(x_n,y_n)-m_{x_ny_n})=m(x,y)-m_{xy}.$$

Lemma 2.6. *[21] Let (x_n) be a sequence in m-metric space (X, m), if there exists $r \in [0, 1)$ such that*

$$m(x_{n+1}, x_n) \leq rm(x_n, x_{n-1}), \qquad \forall n \in \mathbb{N},$$

then
(A) $\lim_{n\to\infty} m(x_n, x_{n+1}) = 0$.
(B) $\lim_{n\to\infty} m(x_n, x_n) = 0$.
(C) $\lim_{m,n\to\infty} m_{x_m x_n} = 0$.
(D) (x_n) is m-Cauchy sequence.

Karapınar have defined the following Kannan type interpolative contraction in [2] and proved the corresponding fixed point result.

Definition 2.7. *[2] Let (X, d) be a metric space, a self mapping $T : X \to X$ is said to be an interpolative Kannan type contraction, if there exists $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$ such that*

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{1-\alpha}$$

for all $x, y \in X$ with $x \neq Tx$.

Theorem 2.8. *[2] Let (X, d) be a complete metric space and T be an interpolative Kannan type contraction. Then T has a unique fixed point in X.*

In [3] Gabba and Karapınar have defined the following Kannan type interpolative contraction for the case when $\alpha + \beta < 1$ and proved the corresponding fixed point theorem.

Definition 2.9. *Let (X, d) be a metric space, a self mapping $T : X \to X$ is called (λ, α, β)-interpolative Kannan type contraction if there exist $\lambda \in [0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, such that*

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{\beta}$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$.

Theorem 2.10. *[5] Let (X, d) be a complete metric space such that $d(x, y) \geq 1$ for all $x, y \in X$ and let $T : X \to X$ is an (λ, α, β)-interpolative Kannan type contraction, then T has a fixed point.*

In [4], Errai et al. proved the following fixed point result of Kannan type interpolative contractions for the case when $\alpha + \beta > 1$ with $\alpha, \beta \in (0, 1)$.

Theorem 2.11. *[4] Let (X, d) be a complete metric space and T is a self mapping on X, such that*

$$d(Tx, Ty) \leq \lambda[d(x, Tx)]^{\alpha}[d(y, Ty)]^{\beta},$$

for all $x, y \in X$ with $x \neq Tx$ and $y \neq Ty$, here $\lambda \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ such that $\alpha + \beta \geq 1$. If there exist $x \in X$ such that $d(x, Tx) \leq 1$, then T has a fixed point in X.

In [13], Konwar et al. proved the following fixed point result for multivalued interpolative Kannan type contractions.

Definition 2.12. *[13] Suppose that (X, d) is a metric space. A map $T : X \to CB(X)$ is called a multivalued interpolative Kannan type contraction if there exist $\lambda \in [0, 1)$ and $\alpha \in (0, 1)$, such that*

$$H(Tx, Ty) \leq \lambda[\triangle(x, Tx)]^{\alpha}[\triangle(y, Ty)]^{1-\alpha}$$

for all $x, y \in X$ with $x, y \notin Fix(T)$.

Theorem 2.13. *[13] Suppose that (X, d) be a complete metric space and T is a multivalued interpolative Kannan type contraction, such that Tx is compact for every $x \in X$. Then T has a fixed point.*

The concept of M-Pompeiu-Hausdorff metric is defined by Patle et al. in [11], following are the basic definition and results for such metric.

Definition 2.14. *[11] A subset "A" of a m-metric space (X, m) is said to be bounded if for any $x \in A$, there exist $x_0 \in X$ and $r \geq 0$ such that $x \in B_m(x_0, r)$, that is, $m(x, x_0) < m_{x_0x} + r$.*

Let $CB_m(X)$ be the collection of all non-empty, bounded and closed subsets of m-metric space X, then

$$H_m(P, Q) = \max\{\delta_m(P, Q), \delta_m(Q, P)\},$$

where $\delta_m(P, Q) = \sup\{\triangle_m(x, Q) : x \in P\}$ and $\triangle_m(x, Q) = \inf\{m(x, y) : y \in Q\}$.
In general, $H_m(A, A) \neq 0$, for $A \in CB_m(X)$.
Note that P is closed in (X, m) if and only if $\overline{P} = P$, where $\overline{P}$ denote the closure of P w.r.t. m-metric.

Lemma 2.15. *[11] Let P be any non-empty set in an m-metric space (X, m), then $x_0 \in \overline{P}$ if and only if*

$$\triangle_m(x_0, P) = \sup_{x \in P} m_{x_0x}.$$

Lemma 2.16. *[11] For any $P, Q, R \in CB_m(X)$ following are true*

1. $H_m(P, P) = \delta_m(P, P) = \sup_{x \in P}\{\sup_{y \in P} m_{xy}\}$.
2. $H_m(P, Q) = H_m(Q, P)$.
3. $H_m(P, Q) - \sup_{x \in P} \sup_{y \in Q} m_{xy} \leq (H_m(P, R) - \inf_{x \in P} \inf_{z \in R} m_{xz}) + (H_m(R, Q) - \inf_{z \in R} \inf_{y \in Q} m_{zy})$.

Lemma 2.17. *[11] Let $P, Q \in CB_m(X)$ and $q > 1$. Then for any $x \in P$, there is at least one $y \in Q$ such that*

$$m(x, y) \leq q H_m(P, Q).$$

Lemma 2.18. *[11] Let $P, Q \in CB_m(X)$ and $r > 0$. For any $x \in P$, there is at least one $y \in Q$, such that*

$$m(x, y) \leq H_m(P, Q) + r.$$

The author in [11] also proved the following Nadler's fixed point result for the structure of m-metric spaces.

Theorem 2.19. *[11] Let (X, m) be a m-complete m-metric space and $T : X \to CB_m(X)$ be a multivalued mapping, suppose there exist $\lambda \in (0, 1)$ such that*

$$H_m(Tx, Ty) \leq \lambda m(x, y)$$

for all $x, y \in X$. Then T admits a fixed point.

The following results are regarding to the fixed points of homotopy which are proved in [31] and [11].

Theorem 2.20. *[31] Let O be an open subset of a complete metric space (X, d). Consider a homotopy $G : [0, 1] \times \overline{O} \to X$ with closed bounded nonempty values, which satisfies the following conditions: for any two points $x, y \in \overline{O}$ and numbers $t, s \in I = [0, 1]$, there exists $\lambda \in (0, 1)$ such that*
(i). $d_G(G(t, x), G(t, y)) \leq \lambda d(x, y)$.
(ii). $d_G(G(t, x), G(s, x)) \leq |\phi(t) - \phi(s)|$, where $\phi : I \to \mathbb{R}$ is a continuous and increasing function.
(iii). $FixG(0, .) \cap \partial O = \emptyset$.
If $FixG(0, .) \neq \emptyset$, then $FixG(1, .) \neq \emptyset$.
Where $FixG(t, .)$ denotes the set of fixed points of homotopy for any fixed $t \in I$.

Theorem 2.21. *[11] Let C be a closed set and O is an open subset of C in m-complete m-metric space (X,m). Let $G : C \times [a,b] \to CB_m(X)$ be a mapping which satisfies the following conditions:*
(i). $x \notin G(x,t)$ for all $x \in C \backslash O$ and for every $t \in [a,b]$.
(ii). There exists $\lambda \in (0,1)$ such that for every $t \in [a,b]$ and for all $x, y \in C$ we have

$$H_m(G(x,t), G(y,t)) \leq \lambda m(x,y).$$

(iii). There exists a continuous mapping $\phi : [a,b] \to \mathbb{R}$ satisfying

$$H_m(G(x,t), G(x,s)) \leq \lambda |\phi(t) - \phi(s)|.$$

(iv). If $c \in G(c,t)$ then $G(c,t) = \{c\}$.
If $G(.,t_1)$ admits a fixed point in C for at least one $t_1 \in [a,b]$, then $G(.,t)$ admits a fixed point in O for all $t \in [a,b]$. Moreover, the fixed point of $G(.,t)$ is unique for every fixed $t \in [a,b]$.

Now in the next section we prove some fixed point results in the frame work of m-metric spaces for interpolative Kannan type multivalued contractions.

3. Main results

Definition 3.1. *Let (X,m) be a m-metric space, a multivalued mapping $T : X \to CB_m(X)$ is called m-interpolative Kannan type contraction, if there exist $\lambda \in (0,1)$ and $\alpha \in (0,1)$ such that*

$$H_m(Tx, Ty) \leq \lambda [\triangle_m(x, Tx)]^{\alpha} [\triangle_m(y, Ty)]^{1-\alpha}, \tag{1}$$

for all $x, y \in X$ with $x, y \notin Fix(T)$ and $\triangle_m(x, Tx) \neq 0, \triangle_m(y, Ty) \neq 0$.

Theorem 3.2. *Let (X,m) be a m-complete m-metric space and $T : X \to CB_m(X)$ is m-interpolative Kannan type contraction. Then T has a fixed point.*

Proof. Choose $q = \frac{1}{\sqrt{\lambda}}$ and $r = \sqrt{\lambda}$, clearly we have $q > 1$ and $r < 1$. Let $x_0 \in X$, be an arbitrary element and $x_1 \in Tx_0$, since $q > 1$ so by Lemma 2.17 there exists $x_2 \in Tx_1$ such that

$$m(x_1, x_2) \leq q H_m(Tx_0, Tx_1). \tag{2}$$

By Definition 3.1, we have

$$H_m(Tx_0, Tx_1) \leq \lambda [\triangle_m(x_0, Tx_0)]^{\alpha} [\triangle_m(x_1, Tx_1)]^{1-\alpha}. \tag{3}$$

By combining (2) and (3),

$$m(x_1, x_2) \leq r [\triangle_m(x_0, Tx_0)]^{\alpha} [\triangle_m(x_1, Tx_1)]^{1-\alpha}.$$

Since $x_1 \in Tx_0$, so

$$\triangle_m(x_0, Tx_0) = \inf_{x \in Tx_0} m(x_0, x) \leq m(x_0, x_1).$$

Similarly, $\triangle_m(x_1, Tx_1) \leq m(x_1, x_2)$. Thus

$$m(x_1, x_2) \leq r [m(x_0, x_1)]^{\alpha} [m(x_1, x_2)]^{1-\alpha},$$

$$m(x_1, x_2)^{\alpha} \leq r [m(x_0, x_1)]^{\alpha},$$

$$m(x_1, x_2) \leq r^{1/\alpha} [m(x_0, x_1)] \leq r [m(x_0, x_1)].$$

Also for $x_2 \in Tx_1$ and $q > 1$, then by following Lemma 2.17 there exist $x_3 \in Tx_2$ such that

$$m(x_2, x_3) \leq q H_m(Tx_1, Tx_2). \tag{4}$$

By Definition 3.1, we have

$$H_m(Tx_1, Tx_2) \leq \lambda[\triangle_m(x_1, Tx_1)]^{\alpha}[\triangle_m(x_2, Tx_2)]^{1-\alpha}. \tag{5}$$

By combining (4) and (5),

$$m(x_2, x_3) \leq r[\triangle_m(x_1, Tx_1)]^{\alpha}[\triangle_m(x_2, Tx_2)]^{1-\alpha}.$$

Thus

$$m(x_2, x_3) \leq r^{1/\alpha}[m(x_1, x_2)] \leq r[m(x_1, x_2)].$$

Similarly, for any $x_n \in Tx_{n-1}$ and $q > 1$, by following Lemma 2.17 there exist $x_{n+1} \in Tx_n$ such that

$$m(x_n, x_{n+1}) \leq qH_m(Tx_{n-1}, Tx_n). \tag{6}$$

By Definition 3.1, we have

$$H_m(Tx_{n-1}, Tx_n) \leq \lambda[\triangle_m(x_{n-1}, Tx_{n-1})]^{\alpha}[\triangle_m(x_n, Tx_n)]^{1-\alpha}. \tag{7}$$

By combining (6) and (7),

$$m(x_n, x_{n+1}) \leq r[\triangle_m(x_{n-1}, Tx_{n-1})]^{\alpha}[\triangle_m(x_n, Tx_n)]^{1-\alpha}.$$

Thus

$$m(x_n, x_{n+1}) \leq r^{1/\alpha}[m(x_{n-1}, x_n)] \leq r[m(x_{n-1}, x_n)].$$

Hence the relation $m(x_n, x_{n+1}) \leq r[m(x_{n-1}, x_n)]$ holds for all $n \in \mathbb{N}$. Thus for any $n \in \mathbb{N}$, we have

$$m(x_n, x_{n+1}) \leq rm(x_{n-1}, x_n) \leq r^2 m(x_{n-2}, x_{n-1}) \cdots \leq r^n m(x_0, x_1),$$

or

$$m(x_n, x_{n+1}) \leq r^n m(x_0, x_1).$$

by taking limit as $n \to \infty$, we have

$$\lim_{n\to\infty} m(x_n, x_{n+1}) \leq \lim_{n\to\infty} r^n m(x_0, x_1) = 0,$$

because $r \in (0, 1)$. Also by definition of m-metric, we have

$$\lim_{n\to\infty} m_{x_n x_{n+1}} \leq \lim_{n\to\infty} m(x_n, x_{n+1}) = 0.$$

Thus

$$\lim_{n\to\infty} \min\{m(x_n, x_n), m(x_{n+1}, x_{n+1})\} = 0.$$

Hence

$$\lim_{n\to\infty} m(x_n, x_n) = 0.$$

Similarly, for any $n, p \in \mathbb{N}$ with $n \geq p$ we have

$$\lim_{n,p\to\infty} (M_{x_n x_p} - m_{x_n x_p}) = 0$$

also by triangle inequality of m-metric we have

$$\lim_{n,p\to\infty} (m(x_n, x_p) - m_{x_n x_p}) = 0.$$

Thus by definition (x_n) is m-Cauchy sequence in m-metric space (X, m) and the completeness of such metric space yields that there exist $x \in X$ such that (x_n) converges to x w.r.t. convergence in m-metric, i.e.

$$\lim_{n\to\infty} (m(x_n, x) - m_{x_n x}) = 0.$$

Also by Definition 3.1, we have

$$H_m(Tx_k, Tx) \leq \lambda[\triangle_m(x_k, Tx_k)]^\alpha[\triangle_m(x, Tx)]^{1-\alpha}$$

or

$$H_m(Tx_k, Tx) \leq \lambda[m(x_k, x_{k+1})]^\alpha[\triangle_m(x, Tx)]^{1-\alpha}.$$

Since $\triangle_m(x, Tx) < \infty$, so we have

$$\lim_{k\to\infty} H_m(Tx_k, Tx) = 0.$$

Since $x_{k+1} \in Tx_k$, so for $q > 1$ there exist $y \in Tx$ such that

$$m(x_{k+1}, y) \leq qH_m(Tx_k, Tx).$$

Thus

$$\lim_{k\to\infty} m(x_{k+1}, y) \leq q \lim_{n\to\infty} H_m(Tx_k, Tx) = 0.$$

Hence

$$\lim_{k\to\infty}[m(x_{k+1}, y) - m_{x_{k+1}y}] = 0. \tag{8}$$

Since

$$\triangle_m(x, Tx) - \sup_{y\in Tx} m_{xy} \leq \triangle_m(x, Tx) - m_{xy} \leq m(x, y) - m_{xy}.$$

Thus by triangular inequality of m-metric we have

$$m(x, y) - m_{xy} \leq [m(x, x_{k+1}) - m_{xx_{k+1}}] + [m(x_{k+1}, y) - m_{x_{k+1}y}],$$

by taking limit k tends to infinity on both sides and by using equation (8) along with the fact that (x_n) converges to x, we get

$$\triangle_m(x, Tx) - \sup_{y\in Tx} m_{xy} \leq m(x, y) - m_{xy} \leq 0,$$

or

$$\triangle_m(x, Tx) \leq \sup_{y\in Tx} m_{xy}.$$

Also, following is the consequence of properties of m-metric

$$\triangle_m(x, Tx) \geq \sup_{y\in Tx} m_{xy}.$$

Hence

$$\triangle_m(x, Tx) = \sup_{y\in Tx} m_{xy}.$$

Thus by Lemma 2.15, $x \in \overline{Tx} = Tx$. Hence x is the fixed point for T. □

Example 3.3. *Let $X = [0, \infty)$ and m-metric on X be defined as follows:*

$$m(x, y) = |x - y| + a,$$

where "a" is any non-negative real number. Let the mapping T on X be defined as:

$$T(x) = \begin{cases} [3, 4] & \text{if } x \in [0, 1/2), \\ [1/4, x] & \text{if } x \in [1/2, 200), \\ [1/x, 1] & \text{if } x \in [200, \infty). \end{cases}$$

Now for $\lambda = 3/4$ and $\alpha = 1/2$, we will show that T satisfies the m-interpolative condition used in Theorem 3.2.
(Case 1.) For $x, y \in [0, 1/2)$, we have

$$H_m(Tx, Ty) = H_m([3, 4], [3, 4]) \leq 1 + a.$$

Also

$$\begin{aligned} \lambda \triangle_m (x, Tx)^{1/2} \triangle_m (y, Ty)^{1/2} &= (3/4)m(x, 3)^{1/2} m(y, 3)^{1/2}, \\ &= (3/4)(|x - 3| + a)^{1/2}(|y - 3| + a)^{1/2}, \\ &\geq (3/4)(5/2 + a). \end{aligned}$$

Since $(3/4)(5/2 + a) \geq 1 + a$ holds, when $0 \leq a \leq 7/2$, thus in this case the m-interpolative condition holds for all $a \in [0, 7/2]$.
(Case 2). If $x \in [0, 1/2)$ and $y \in [200, \infty)$, then we have

$$H_m(Tx, Ty) = H_m([3, 4], [1/y, 1]) = 3 + a.$$

Also

$$\begin{aligned} \lambda \triangle_m (x, Tx)^{1/2} \triangle_m (y, Ty)^{1/2} &= (3/4)m(x, 3)^{1/2} m(y, 1)^{1/2}, \\ &= (3/4)(|x - 3| + a)^{1/2}(|y - 1| + a)^{1/2}, \\ &\geq (3/4)(5/2 + a)^{1/2}(200 - 1 + a)^{1/2}. \end{aligned}$$

Since relation

$$(3/4)(5/2 + a)^{1/2}(199 + a)^{1/2} \geq 3 + a,$$

holds, when $0 \leq a \leq 247$. Thus in this case the m-interpolative condition holds for all $a \in [0, 247]$.
(Case 3). If $x, y \in [200, \infty)$, then we have

$$\begin{aligned} H_m(Tx, Ty) &= H_m([1/x, 1], [1/y, 1]), \\ &\leq \max(|1 - 1/x| + a, |1 - 1/y| + a), \\ &\leq 1 + a. \end{aligned}$$

Also,

$$\begin{aligned} \lambda \triangle_m (x, Tx)^{1/2} \triangle_m (y, Ty)^{1/2} &= (3/4)m(x, 1)^{1/2} m(y, 1)^{1/2}, \\ &= (3/4)(|x - 1| + a)^{1/2}(|y - 1| + a)^{1/2}, \\ &\geq (3/4)(200 - 1 + a). \end{aligned}$$

Since the relation

$$(3/4)(199 + a) \geq 1 + a$$

holds, when $0 \leq a \leq 593$. Thus in this case the m-interpolative condition holds for all $a \in [0, 593]$.
Hence from all the above cases we conclude that the m-interpolative condition of Theorem 3.2 holds, whenever $a \in [0, 7/2]$. Thus for such values of "a" the multivalued mapping T have fixed points, which are for all $x \in [1/2, 200)$.

Definition 3.4. *Let (X, m) be a m-metric space, a multivalued mapping $T : X \to CB_m(X)$ is called (λ, α, β)-type m-interpolative Kannan contraction, if there exist $\lambda \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha + \beta < 1$, such that*

$$H_m(Tx, Ty) \leq \lambda[\triangle_m(x, Tx)]^\alpha [\triangle_m(y, Ty)]^\beta$$

for all $x, y \in X$ with $x, y \notin Fix(T)$ and $\triangle_m(x, Tx) \geq 1, \triangle_m(y, Ty) \neq 0$.

Theorem 3.5. *Let (X, m) be a m-complete m-metric space and $T : X \to CB_m(X)$ is a (λ, α, β)-type m-interpolative Kannan contraction, then T has a fixed point.*

Proof. Choose $q = \frac{1}{\sqrt{\lambda}}$ and $r = \sqrt{\lambda}$, clearly we have $q > 1$ and $r < 1$. Let $x_0 \in X$, be an arbitrary element and $x_1 \in Tx_0$, then for $q > 1$ by using Lemma 2.17, there exist $x_2 \in Tx_1$, such that

$$m(x_1, x_2) \leq qH_m(Tx_0, Tx_1). \tag{9}$$

By Definition 3.4, we have

$$H_m(Tx_0, Tx_1) \leq \lambda[\triangle_m(x_0, Tx_0)]^{\alpha}[\triangle_m(x_1, Tx_1)]^{\beta},$$

or

$$H_m(Tx_0, Tx_1) \leq \lambda[\triangle_m(x_0, Tx_0)]^{1-\beta}[\triangle_m(x_1, Tx_1)]^{\beta}, \tag{10}$$

because $\alpha < 1 - \beta$ and $\triangle_m(x_0, Tx_0) \geq 1$. Thus by combining (9) and (10),

$$m(x_1, x_2) \leq r[\triangle_m(x_0, Tx_0)]^{1-\beta}[\triangle_m(x_1, Tx_1)]^{\beta}.$$

Since $x_1 \in Tx_0$ so $\triangle_m(x_0, Tx_0) = \inf_{x \in Tx_0} m(x_0, x) \leq m(x_0, x_1)$, similarly $\triangle_m(x_1, Tx_1) \leq m(x_1, x_2)$. Thus

$$m(x_1, x_2) \leq r[m(x_0, x_1)]^{1-\beta}[m(x_1, x_2)]^{\beta},$$

$$m(x_1, x_2)^{1-\beta} \leq r[m(x_0, x_1)]^{1-\beta},$$

$$m(x_1, x_2) \leq r^{1/1-\beta}[m(x_0, x_1)] \leq r[m(x_0, x_1)].$$

Also for $x_2 \in Tx_1$ and $q > 1$, by following Lemma 2.17 there exists $x_3 \in Tx_2$ such that

$$m(x_2, x_3) \leq qH_m(Tx_1, Tx_2). \tag{11}$$

By Definition 3.4 and (11), we get

$$m(x_2, x_3) \leq r^{1/1-\beta}[m(x_1, x_2)] \leq r[m(x_1, x_2)]. \tag{12}$$

Similarly, for any $x_n \in Tx_{x_{n-1}}$ and $q > 1$ by following Lemma 2.17, there exist $x_{n+1} \in Tx_n$ such that

$$m(x_n, x_{n+1}) \leq qH_m(Tx_{n-1}, Tx_n). \tag{13}$$

By Definition 3.4 and using the fact that $\alpha + \beta < 1$, we have

$$H_m(Tx_{n-1}, Tx_n) \leq \lambda[\triangle_m(x_{n-1}, Tx_{n-1})]^{1-\beta}[\triangle_m(x_n, Tx_n)]^{\beta}. \tag{14}$$

By combining (13) and (14),

$$m(x_n, x_{n+1}) \leq r[\triangle_m(x_{n-1}, Tx_{n-1})]^{1-\beta}[\triangle_m(x_n, Tx_n)]^{\beta},$$

or

$$m(x_n, x_{n+1}) \leq r^{1/1-\beta}[m(x_{n-1}, x_n)] \leq r[m(x_{n-1}, x_n)].$$

Hence the relation

$$m(x_n, x_{n+1}) \leq r[m(x_{n-1}, x_n)]$$

holds for all $n \in \mathbb{N}$. Next by adopting the similar procedure as in Theorem 3.2 we get the fixed point. □

Now we give an example to justify the Theorem 3.5.

Example 3.6. *Let $X = [0, \infty)$ and m-metric on X be defined as follows:*

$$m(x,y) = \begin{cases} x & \text{if } x = y, \\ x+y & \text{if } x \neq y. \end{cases}$$

Let the mapping T on X be defined as:

$$T(x) = \begin{cases} \{x, x+1\} & \text{if } x \in [0,5], \\ \{3\} & \text{if } x \in (5,\infty). \end{cases}$$

Now for $\alpha = 1/2, \beta = 1/4$ and $\lambda = 3/4$, we will show that T is $(3/4, 1/2, 1/4)$-type m-nterpolative Kannan contraction and satisfies the condition used in Theorem 3.5.
For $x, y \in (5, \infty)$, we have

$$H_m(Tx,Ty) = H_m(\{3\},\{3\}) = 3.$$

Also,

$$\begin{aligned} \lambda \triangle_m (x,Tx)^{1/2} \triangle_m (y,Ty)^{1/4} &= (3/4)m(x,3)^{1/2}m(y,3)^{1/4}, \\ &= (3/4)(x+3)^{1/2}(y+3)^{1/4}, \\ &\geq (3/4)(8)^{0.75} = 3.57 \geq 3. \end{aligned}$$

Hence T satisfies the interpolative condition of Definition 3.4 and also $\triangle_m(x,Tx) = \triangle_m(x,\{3\}) \geq 1$ for all $x \notin Fix(T)$. Thus by Theorem 3.5 T has fixed point which are infinite many, i.e.$\forall x \in [0,5]$.

Theorem 3.7. *Let (X, m) be a m-complete m-metric space and $T : X \to CB_m(X)$ be a multivalued mapping such that*

$$H_m(Tx,Ty) \leq \lambda[\triangle_m(x,Tx)]^{\alpha}[\triangle_m(y,Ty)]^{\beta} \tag{15}$$

for all $x, y \in X$ with $x, y \notin Fix(T)$ and $\triangle_m(x,Tx) \neq 0, \triangle_m(y,Ty) \neq 0$, where $\lambda \in (0, 1)$ and $\alpha, \beta \in (0, 1)$ with $\alpha+\beta > 1$. If there exist $a \in X$ such that $\triangle_m(a, Ta) \leq 1$. Then T has a fixed point.

Proof. Choose $q = \frac{1}{\sqrt{\lambda}}$ and $r = \sqrt{\lambda}$, if $a = x_0$ such that $\triangle_m(x_0,Tx_0) \leq 1$. Let $x_1 \in Tx_0$, then for $q > 1$ by using Lemma 2.17 there exist $x_2 \in Tx_1$, such that

$$m(x_1,x_2) \leq qH_m(Tx_0,Tx_1). \tag{16}$$

By (15), we have

$$H_m(Tx_0,Tx_1) \leq \lambda[\triangle_m(x_0,Tx_0)]^{\alpha}[\triangle_m(x_1,Tx_1)]^{\beta}. \tag{17}$$

Thus by combining (16) and (17),

$$m(x_1,x_2) \leq r[\triangle_m(x_0,Tx_0)]^{\alpha}[\triangle_m(x_1,Tx_1)]^{\beta}.$$

Also

$$m(x_1,x_2) \leq r[\triangle_m(x_0,Tx_0)]^{\alpha}[m(x_1,x_2)]^{\beta},$$

or

$$\begin{gathered} m(x_1,x_2)^{1-\beta} \leq r[\triangle_m(x_0,Tx_0)]^{\alpha}, \\ m(x_1,x_2) \leq r^{1/1-\beta}[\triangle_m(x_0,Tx_0)]^{\alpha/1-\beta} \leq r, \end{gathered}$$

because $\alpha/1 - \beta > 1$ and $\triangle_m(x_0,Tx_0) \leq 1$. Similarly, we get

$$m(x_2,x_3) \leq r^2.$$

Thus by mathematical induction and interpolative condition, following relation holds for all $n \in \mathbb{N}$,

$$m(x_n, x_{n+1}) \leq r^n.$$

Now by taking limit, we get

$$\lim_{n\to\infty} m(x_n, x_{n+1}) = 0.$$

Also by second condition of m-metric, we have

$$m_{x_n x_{n+1}} \leq m(x_n, x_{n+1}).$$

Thus

$$\lim_{n\to\infty} m_{x_n x_{n+1}} = 0.$$

This implies $\lim_{n\to\infty} m(x_n, x_n) = 0$ or $\lim_{n\to\infty} m(x_{n+1}, x_{n+1}) = 0$. If one is zero then obviously other is zero.

$$\lim_{n,p\to\infty} [M_{x_n,x_p} - m_{x_n x_p}] = 0. \tag{18}$$

Also if $n \geq p$, then by triangular inequality of m-metric and by using above relations, we have

$$\lim_{n,p\to\infty} [m(x_n, x_p) - m_{x_n,x_p}] = 0. \tag{19}$$

It follows from (18) and (19) that (x_n) is m-Cauchy sequence. Since (X, m) is m-complete m-metric space, so the sequence (x_n) converges to a point $x \in X$. Also by (15), we have

$$H_m(Tx_k, Tx) \leq \lambda[\triangle_m(x_k, Tx_k)]^\alpha[\triangle_m(x, Tx)]^\beta,$$

or

$$H_m(Tx_k, Tx) \leq \lambda[m(x_k, x_{k+1})]^\alpha[\triangle_m(x, Tx)]^\beta,$$

by taking limit $k \to \infty$, we get

$$\lim_{k\to\infty} H_m(Tx_k, Tx) = 0.$$

Since $x_{k+1} \in Ta_k$, so for $q > 1$ there exist $y \in Tx$, such that

$$m(x_{k+1}, y) \leq qH_m(Tx_k, Tx).$$

Thus

$$\lim_{k\to\infty} m(x_{k+1}, y) \leq q \lim_{n\to\infty} H_m(Tx_k, Tx) = 0.$$

Hence

$$\lim_{k\to\infty}[m(x_{k+1}, y) - m_{x_{k+1}y}] = 0. \tag{20}$$

Since

$$\triangle_m(x, Tx) - \sup_{y\in Tx} m_{xy} \leq \triangle_m(x, Tx) - m_{xy} \leq m(x, y) - m_{xy}.$$

Thus for $x, y \in X$, by using triangular inequality of m-metric we have

$$m(x, y) - m_{xy} \leq [m(x, x_{k+1}) - m_{xx_{k+1}}] + [m(x_{k+1}, y) - m_{x_{k+1}y}].$$

By taking limit k tends to infinity on both sides and by using equation (20) along with the fact that (x_n) converges to x, we get

$$\triangle_m(x, Tx) - \sup_{y\in Tx} m_{xy} \leq m(x, y) - m_{xy} \leq 0,$$

because the right hand side is independent of k, so no limit involve there. Hence

$$\triangle_m(x, Tx) \leq \sup_{y \in Tx} m_{xy}.$$

Also by following properties of m-metric, we have

$$\triangle_m(x, Tx) \geq \sup_{y \in Tx} m_{xy}.$$

Hence

$$\triangle_m(x, Tx) = \sup_{y \in Tx} m_{xy}.$$

Thus by Lemma 2.15, $x \in \overline{Tx} = Tx$. Hence x is the fixed point for T. □

Example 3.8. *Let $X = [0, \infty)$ and m-metric on X be defined as follows:*

$$m(x, y) = \begin{cases} x & \text{if } x = y, \\ x + y & \text{if } x \neq y. \end{cases}$$

Let the mapping T on X be defined as:

$$T(x) = \begin{cases} [0, x + 1] & \text{if } x \in [0, 4], \\ [1/x, 1/4] & \text{if } x \in (4, \infty). \end{cases}$$

Now for $\alpha = 1/2, \beta = 3/4$ and $\lambda = 98/101$, we will show that T satisfies the interpolative condition used in Theorem 3.7.
If $x, y \in (4, \infty)$, then we have

$$\begin{aligned} H_m(Tx, Ty) &= H_m([1/x, 1/4], [1/y, 1/4]), \\ &= \max(m(1/4, 1/x), m(1/4, 1/y)) \leq 1/2. \end{aligned}$$

Also,

$$\begin{aligned} \lambda \triangle_m (x, Tx)^{1/2} \triangle_m (y, Ty)^{3/4} &= (98/101) m(x, 1/x)^{1/2} m(y, 1/y)^{3/4}, \\ &= (98/101)(x + 1/x)^{1/2}(y + 1/y)^{3/4}, \\ &\geq 5.92 > 1/2. \end{aligned}$$

Hence the required interpolative condition of Theorem (3.7) holds. Thus T have infinite many fixed points which are for all $x \in [0, 4]$.

4. Application

In this section we will give an application of our main result and prove the existence of fixed point for the homotopy in the structure of m-metric spaces. To start the main result of this section first we will prove the following lemma.

Lemma 4.1. *Let (X, m) be a m-metric space, $T : X \to CB_m(X)$ be a multivalued m-interpolative Kannan type contraction satisfying (1) for all $x, y \in X$. If $c \in Tc$ for some $c \in X$, then $m(x, x) = 0$ for all $x \in Tc$.*

Proof. Let $c \in Tc$. Then by Lemma 2.15, we have

$$\triangle_m(c, Tc) = \sup_{x \in Tc} m_{cx},$$

or

$$\triangle_m(c, Tc) = \sup_{x \in Tc} m_{xx}.$$

Also we know that

$$H_m(Tc, Tc) = \delta_m(Tc, Tc) = \sup_{x \in Tc} \triangle_m(x, Tc) = \sup_{x \in Tc} m_{xx}.$$

Let's assume that $m(c, c) > 0$, then we have

$$\sup_{x \in Tc} m_{xx} = H_m(Tc, Tc) \leq \lambda \triangle_m (c, Tc)^{\alpha} \triangle_m (c, Tc)^{1-\alpha} = \lambda \triangle_m (c, Tc) \leq \lambda m(c, c).$$

Thus we get $\sup_{x \in Tc} m_{x,x} \leq \lambda m(c, c)$. A contradiction, because $\lambda \in (0, 1)$. Hence our first supposition is wrong and we have $m(c, c) = 0$. Also $\sup_{x \in Tc} m_{xx} = 0$, hence $m(x, x) = 0$ for all $x \in Tc$. □

Theorem 4.2. *Let C be a closed set and O is an open subset of C in m-complete m-metric space (X, m). Let $G : C \times [a, b] \to CB_m(X)$ be a mapping which satisfies the following axioms:*
(i). $x \notin G(x, t)$ for all $x \in C \backslash O$ and for every $t \in [a, b]$.
(ii). For any $x, y \in C$ and for every $t \in [a, b]$ there exist $\lambda \in (0, 1)$ and $\alpha \in (0, 1)$ such that

$$H_m(G(x, t), G(y, t)) \leq \lambda \triangle_m(x, G(x, t))^{\alpha} \triangle_m(y, G(y, t))^{1-\alpha}.$$

(iii). There exists a continuous function $\psi : [a, b] \to R$ satisfying

$$H_m(G(x, t), G(x, s)) \leq \lambda |\psi(t) - \psi(s)|.$$

(iv). If $c \in G(c, t)$ then $G(c, t) = \{c\}$.
If $G(., t_1)$ admits a fixed point in C for at least one $t_1 \in [a, b]$, then $G(., t)$ admits a fixed point in O for all $t \in [a, b]$.

Proof. Consider, the set

$$W = \{t \in [a, b] : a \in G(a, t) \text{ for some } a \in O\}.$$

Then W is non empty because $G(., t_1)$ admits a fixed point in C for atleast one $t_1 \in [a, b]$, then by (i) that fixed point belongs to O thus W is non empty.
Now to prove our result, we only show that W is both open and closed in $[a, b]$, then by connectedness of $[a, b]$, it follows that $W = [a, b]$.
First we show that W is open. Let $t_0 \in W$ and $a_0 \in O$ with $a_0 \in G(a_0, t_0)$. Since O is open and a_0 is an interior element in O, so there exists an open ball $B_m(a_0, r)$ such that $a_0 \in B_m(a_0, r) \subseteq O$, where $r > 0$. Since ψ is a continuous function so for $\epsilon = \frac{1}{\lambda}(r + m_{aa_0}) > 0$ there exists $\delta > 0$ such that for any $t \in (t_0 - \delta, t_0 + \delta) = S_\delta(t_0)$, we have

$$|\psi(t) - \psi(t_0)| < \epsilon.$$

Since $a_0 \in G(a_0, t_0)$, by Lemma 4.1, we have $m(x, x) = 0$ for every $x \in G(a_0, t_0)$. Thus by definition of m-metric we have, $m_{xa_0} = 0$, for all $x \in X$. Now by (ii), (iii), (iv) and Lemma 2.16, we have

$$\begin{aligned}
m(G(a, t), a_0) &= H_m(G(a, t), G(a_0, t_0)), \\
&\leq H_m(G(a, t), G(a, t_0)) + H_m(G(a, t_0), G(a_0, t_0)) - \inf_{p \in G(a,t)} \inf_{q \in G(a,t_0)} m_{pq} \\
&\quad - \inf_{q \in G(a,t_0)} \inf_{c \in G(a_0,t_0)} m_{qc} + \sup_{p \in G(a,t)} \sup_{c \in G(a_0,t_0)} m_{pc}, \\
&= H_m(G(a, t), G(a, t_0)) + H_m(G(a, t_0), G(a_0, t_0)) - \inf_{p \in G(a,t)} \inf_{q \in G(a,t_0)} m_{pq}, \\
&\leq H_m(G(a, t), G(a, t_0)) + H_m(G(a, t_0), G(a_0, t_0)), \\
&\leq \lambda |\psi(t) - \psi(t_0)| + \lambda \triangle_m (a, G(a, t_0))^{\alpha} \triangle_m (a_0, G(a_0, t_0))^{1-\alpha}, \\
&\leq \lambda \epsilon + \lambda \triangle_m (a, G(a, t_0))^{\alpha} m(a_0, a_0)^{1-\alpha}.
\end{aligned}$$

Since $m(a_0, a_0) = 0$ and $\epsilon = \frac{1}{\lambda}[r + m_{a,a_0}]$ thus we have

$$m(G(a,t), a_0) \leq r + m_{a,a_0}.$$

Hence for for every fixed $t \in S_\delta(t_0)$, the mapping $G(.,t) : \overline{B_m(a_0,r)} \to CB^m(\overline{B_m(a_0,r)})$ satisfies all the conditions of Theorem 3.2, therefore $G(.,t)$ has a fixed point in $\overline{B_m(a_0,r)} \subseteq C$ for every $t \in S_\delta(t_0)$. By following (i), that fixed point should be in O. Thus by construction of W we conclude that $S_\delta(t_0) \subseteq W$, because for every $t \in S_\delta(t_0)$ we got $a \in O$ such that $a \in G(a,t)$. Since $t_0 \in W$ is an arbitrary element of W, so W is open in $[a,b]$. Now we show that W is closed. Let $\{t_k\}$ be a sequence in W which converges to some $t^* \in [a,b]$. Since for every natural number $k \geq 1$ we have $t_k \in W$, also by construction of W for every such $t_k \in W$ there exist an element $a_k \in O$ such that $a_k \in G(a_k, t_k)$. Thus we have

$$\begin{aligned} m(a_k, a_j) &= H_m(G(a_k,t_k), G(a_j,t_j)), \\ &\leq H_m(G(a_k,t_k), G(a_k,t_j)) + H_m(G(a_k,t_j), G(a_j,t_j)), \\ &\leq \lambda|\psi(t_k) - \psi(t_j)| + \lambda \bigtriangleup_m (a_k, G(a_k,t_j))^\alpha \bigtriangleup_m (a_j, G(a_j,t_j))^{1-\alpha}, \\ &\leq \lambda|\psi(t_k) - \psi(t_j)| + \lambda \bigtriangleup_m (a_k, G(a_k,t_j))^\alpha m(a_j,a_j)^{1-\alpha}, \\ &= \lambda|\psi(t_k) - \psi(t_j)|. \end{aligned}$$

By using the continuity and the fact that $\{t_k\}$ converges to t^*, we get $\psi(t_k) \to \psi(t^*)$, $\psi(t_j) \to \psi(t^*)$ as $k, j \to \infty$. Thus

$$\lim_{k,j\to\infty} m(a_k, a_j) = 0.$$

Thus $\lim_{k,j\to\infty} m_{a_k a_j} = 0$ and $\lim_{k\to\infty} m(a_k,a_k) = 0 = \lim_{j\to\infty} m(a_j,a_j)$. Hence

$$\lim_{k,j\to\infty} (m(a_k,a_j) - m_{a_k a_j}) = 0 \text{ and } \lim_{k,j\to\infty} (M_{a_k,a_j} - m_{a_k a_j}) = 0.$$

Hence $\{a_k\}$ is a m-Cauchy sequence. Thus by completeness of X, it yields that there exist $a^* \in X$ such that $\{a_k\}$ converges to a^*. By definition of convergence in m-metric. We have

$$\lim_{k\to\infty}(m(a_k,a^*) - m_{a_k a^*}) = 0 \text{ and } \lim_{k\to\infty}(M_{a_k,a^*} - m_{a_k a^*}) = 0.$$

Since $\lim_{k\to\infty} m(a_k,a_k) = 0$, so we have

$$\lim_{k\to\infty} m(a_k,a^*) = 0 \text{ and } \lim_{k\to\infty} M_{a_k,a^*} = 0.$$

Thus we get $m(a^*,a^*) = 0$. Now we show that $a^* \in G(a^*,t^*)$. Since for any $z \in G(a_k,t_k)$ by Lemma 4.1 we have $m(z,z) = 0$, thus $m_{xy} = 0$ for all $x \in X$, hence

$$\begin{aligned} \bigtriangleup_m(a_k, G(a^*,t^*)) &\leq H_m(G(a_k,t_k), G(a^*,t^*)), \\ &= H_m(G(a^*,t^*), G(a_k,t_k)), \\ &\leq H_m(G(a^*,t^*), G(a^*,t_k)) + H_m(G(a^*,t_k), G(a_k,t_k)) - \inf_{x\in G(a^*,t^*)} \inf_{y\in G(a^*,t_k)} m_{xy} \\ &\quad - \inf_{y\in G(a^*,t_k)} \inf_{z\in G(a_k,t_k)} m_{yz} + \sup_{y\in G(a^*,t_k)} \sup_{z\in G(a_k,t_k)} m_{yz}, \\ &= H_m(G(a^*,t^*), G(a^*,t_k)) + H_m(G(a^*,t_k), G(a_k,t_k)) - \inf_{x\in G(a^*,t^*)} \inf_{y\in G(a^*,t_k)} m_{xy}, \\ &\leq H_m(G(a^*,t^*), G(a^*,t_k)) + H_m(G(a^*,t_k), G(a_k,t_k)), \\ &\leq \lambda|\psi(t^*) - \psi(t_k)| + \lambda \bigtriangleup_m (a^*, G(a^*,t_k))^\alpha \bigtriangleup_m (a_k, G(a_k,t_k))^{1-\alpha}, \\ &\leq \lambda|\psi(t_k) - \psi(t^*)| + \lambda \bigtriangleup_m (a^*, G(a^*,t_k))^\alpha m(a_k,a_k)^{1-\alpha}. \end{aligned}$$

Since $a_k \in G(a_k, t_k)$, so $m(a_k, a_k) = 0$, by Lemma 4.1. Also if we take $k \to \infty$ we get

$$\lim_{k\to\infty} \triangle_m(a_k, G(a^*, t^*)) = 0.$$

Hence

$$\triangle_m(a^*, G(a^*, t^*)) = 0.$$

Since $m(a^*, a^*) = 0$, we have

$$\sup_{b\in G(a^*,t^*)} m_{a^*b} = \sup_{b\in G(a^*,t^*)} \min\{m(a^*, a^*), m(b, b)\} = 0.$$

Thus by combining above two expressions, we get

$$\triangle_m(a^*, G(a^*, t^*)) = \sup_{b\in G(a^*,t^*)} m_{a^*b}.$$

Therefore, by Lemma 2.15, we have $a^* \in G(a^*, t^*)$. Thus $a^* \in O$. Hence $t^* \in W$ and W is closed in $[a, b]$. Hence W is both open and closed in a connected space $[a, b]$, so $W = [a, b]$. Thus $G(., t)$ admits a fixed point in O for all $t \in [a, b]$. □

5. Conclusions

The fixed point results for multivalued interpolative Kannan type contractions has been proved for all possible cases of sum of interpolative exponents and for every case significant examples are discussed. Also at the end, the homotopy result verified the validity of the main result.

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