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Multivalued Sehgal-Proinov type contraction mappings involving rational terms in modular metric spaces

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Abstract. This study intends to achieve new fixed point results, which will extend Seghal's results, considering the concept of $\alpha-$admissibility and Proinov type contraction comprising rational expressions for multivalued mappings in the frame of modular metric space. In addition, the viability of the outputs obtained here is demonstrated using a nonlinear integral equation.

1. Introduction and Preliminaries

Throughout the study, the symbol $\mathbb{N}$ represents the set of all positive natural numbers, and $\mathbb{R}_+$ represents the set of all non-negative real numbers.

Due to its straightforward applicability to many different areas of mathematics, the Banach fixed point theorem [6], announced by S. Banach, was a significant contributor to the development of metric fixed point theory a century ago. In light of this, there has been and still is a remarkable interest in and demand for this hypothesis. Besides guaranteeing the existence and uniqueness of a fixed point of contraction self-mapping, the theorem provides an effective technique to find the fixed point. In summary, the contraction self-mapping Q on a complete metric space $(\mathcal{U}, \mathsf{d})$, i.e., for all $\mathfrak{s}, \mathfrak{z} \in \mathcal{U}$, the expression

$$\mathsf{d}(\mathsf{Q}\mathfrak{s}, \mathsf{Q}\mathfrak{z}) \leq \mu \mathsf{d}(\mathfrak{s}, \mathfrak{z}), \quad where \quad \mu \in (0,1) \tag{1}$$

is satisfied, then Q owns a unique fixed point, and for every $\mathfrak{s}_0 \in \mathcal{U}$, the sequence $\{\mathsf{Q}^n \mathfrak{s}_0\}_{n\in\mathbb{N}}$ convergences to this fixed point.

There are several generalizations of the Banach fixed point theorem. One of them, at first, is Bryant's fixed point theorem [7], indicated by Bryant in 1968, as noted below.

Theorem 1.1. [7] *Let* $\mathsf{Q} : \mathcal{U} \to \mathcal{U}$ *be a self-mapping on a complete metric space* $(\mathcal{U}, \mathsf{d})$*. So, provided that* Q^N *satisfies the inequality* (1)*, which means that* Q^N *is a contraction map for some* $N \in \mathbb{N}$*, then* Q *has a unique fixed point.*

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That Q^N is continuous is a natural inference of Theorem 1.1. However, this result does not require that Q is continuous. Bryant gave an example illustrating this observation in [7].

In 1969, Sehgal [33] asserted a novel result, an extension of Theorem 1.1, with respect to "the contractive iteration of each point" in a complete metric space, as indicated below.

Theorem 1.2. [33] *Let a continuous self-mapping* $\mathsf{Q} : \mathcal{U} \to \mathcal{U}$ *and* $q \in [0, 1)$ *be given on a complete metric space* $(\mathcal{U}, \mathsf{d})$. *If for each* $\mathfrak{s} \in \mathcal{U}$, *there exists a positive integer* $n = n(\mathfrak{s})$ *such that*

$$\mathsf{d}\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}, \mathsf{Q}^{n(\mathfrak{z})}\mathfrak{z}\right) \leq q\mathsf{d}(\mathfrak{s}, \mathfrak{z}) \tag{2}$$

for all $\mathfrak{z} \in \mathcal{U}$, *then* Q *has a unique fixed point in* $\mathcal{U}$.

Furthermore, in [33], Sehgal's example uncovers that even if the inequality (1) is not satisfied, that is, not a contraction, it provides (2) and owns a fixed point. Next, Guseman [13] redefined the results by removing the continuity condition on the mapping. Also, for the latest study involving Sehgal's fixed point, see [2, 3, 12, 25, 32, 34, 35].

On the other hand, many authors have tried introducing new generalized metric space mainly by changing or adding the axioms of the studied metric space. Hereof, new forms have emerged, and many new topological forms have been contributed to the literature. One of these generalizations is modular metric space, introduced by Chistyakov [8–11] as a very attractive and stunning idea.

Initially, let $\mathbf{m} : (0, \infty) \times \mathcal{U} \times \mathcal{U} \to [0, \infty]$ be a function where $\mathcal{U}$ is a non-empty set. If so, for clarity, we will prefer the notion of $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z})$ rather than $\mathbf{m}(\mathfrak{b}, \mathfrak{s}, \mathfrak{z})$ for all $\mathfrak{b} > 0$ and $\mathfrak{s}, \mathfrak{z} \in \mathcal{U}$.

Definition 1.3. [9, 10] *Let* $\mathbf{m} : (0, \infty) \times \mathcal{U} \times \mathcal{U} \to [0, \infty]$ *be a function where* $\mathcal{U}$ *is a non-empty set. Thereby,* $\mathbf{m}$ *is entitled to modular metric provided that for all* $\mathfrak{s}, \mathfrak{z}, \mathsf{z} \in \mathcal{U}$, *the circumstances*

($\mathbf{m}_1$) $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z}) = 0$ *for all* $\mathfrak{b} > 0$ *if and only if* $\mathfrak{s} = \mathfrak{z}$,

($\mathbf{m}_2$) $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z}) = \mathbf{m}_{\mathfrak{b}}(\mathfrak{z}, \mathfrak{s})$ *for all* $\mathfrak{b} > 0$,

($\mathbf{m}_3$) $\mathbf{m}_{\mathfrak{b}+\mu}(\mathfrak{s}, \mathfrak{z}) \leq \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathsf{z}) + \mathbf{m}_{\mu}(\mathsf{z}, \mathfrak{z})$ *for all* $\mathfrak{b}, \mu > 0$,

are satisfied. Thereupon, $(\mathcal{U}, \mathbf{m})$ *is a modular metric space abbreviated as MMS.*

Instead of ($\mathbf{m}_1$), *if we consider the condition*

($\mathbf{m}_1{}'$) $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{s}) = 0$ *for all* $\mathfrak{b} > 0$,

then $\mathbf{m}$ *is a (metric) pseudo-modular on* $\mathcal{U}$. *Moreover, a modular metric* $\mathbf{m}$ *on* $\mathcal{U}$ *has the property of regular if the new condition, which is a weaker version of* ($\mathbf{m}_1$),

($\mathbf{m}_1{}''$) $\mathfrak{s} = \mathfrak{z}$ *if and only if* $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{s}) = 0$, *for some* $\mathfrak{b} > 0$,

is provided. Lastly, $\mathbf{m}$ *is called convex modular if for* $\mathfrak{b}, \mu > 0$ *and* $\mathfrak{s}, \mathfrak{z}, \mathsf{z} \in \mathcal{U}$,

$$\mathbf{m}_{\mathfrak{b}+\mu}(\mathfrak{s}, \mathfrak{z}) \leq \frac{\mathfrak{b}}{\mathfrak{b}+\mu}\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathsf{z}) + \frac{\mu}{\mathfrak{b}+\mu}\mathbf{m}_{\mu}(\mathsf{z}, \mathfrak{z}).$$

On the other hand, the function $\mathfrak{b} \to \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z})$ is non-increasing on $(0, \infty)$ for any $\mathfrak{s}, \mathfrak{z} \in \mathcal{U}$, where $\mathbf{m}$ is a metric pseudo-modular on a set $\mathcal{U}$. Undoubtedly, for $0 < \mu < \mathfrak{b}$, it is verified by

$$\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z}) \leq \mathbf{m}_{\mathfrak{b}-\mu}(\mathfrak{s}, \mathfrak{s}) + \mathbf{m}_{\mu}(\mathfrak{s}, \mathfrak{z}) = \mathbf{m}_{\mu}(\mathfrak{s}, \mathfrak{z}).$$

Definition 1.4. [9, 10] *Let* $\mathbf{m}$ *be a pseudo-modular on* $\mathcal{U}$ *and* $\mathfrak{s}_0 \in \mathcal{U}$ *be fixed. Thereby, the following sets are mentioned as modular spaces (around* $\mathfrak{s}_0$*):*

- $\mathcal{U}_{\mathbf{m}} = \mathcal{U}_{\mathbf{m}}(\mathfrak{s}_0) = \{\mathfrak{s} \in \mathcal{U} : \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{s}_0) \to 0\}$ *as* $\mathfrak{b} \to \infty$, *and*

- $\mathcal{U}^*_{\mathbf{m}} = \mathcal{U}^*_{\mathbf{m}}(\mathfrak{s}_0) = \{\mathfrak{s} \in \mathcal{U} : \exists \mathfrak{b} = \mathfrak{b}(\mathfrak{s}) > 0 \ \textit{such that} \ \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{s}_0) < \infty\}$.

Note that $\mathcal{U}_{\mathbf{m}} \subset \mathcal{U}^*_{\mathbf{m}}$, but the inverse is generally not valid. Accordingly, from [9, 10], a (nontrivial) metric $\mathsf{d}_{\mathbf{m}}$, which is presented in follows and generated by the modular $\mathbf{m}$, for any $\mathfrak{s}, \mathfrak{z} \in \mathcal{U}_{\mathbf{m}}$

$$\mathsf{d}_{\mathbf{m}}(\mathfrak{s}, \mathfrak{z}) = \inf\{\mathfrak{b} > 0 : \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z}) \leq \mathfrak{b}\},$$

is identified on $\mathcal{U}_{\mathbf{m}}$. Furthermore, if we consider a convex modular $\mathbf{m}$ on $\mathcal{U}$, then $\mathcal{U}_{\mathbf{m}} = \mathcal{U}^*_{\mathbf{m}}$ thereupon, these sets are endowed with the metric

$$\mathsf{d}^*_{\mathbf{m}}(\mathfrak{s}, \mathfrak{z}) = \inf\{\mathfrak{b} > 0 : \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}, \mathfrak{z}) \leq 1\},$$

which is referred to as the Luxembourg distance, for any $\mathfrak{s}, \mathfrak{z} \in \mathcal{U}_{\mathbf{m}}$.

Definition 1.5. [9, 10] *Let $\mathcal{U}^*_{\mathbf{m}}$ be an MMS, $\{\mathfrak{s}_n\}_{n\in\mathbb{N}} \in \mathcal{U}^*_{\mathbf{m}}$ be a sequence, and $\mathfrak{M}$ be a subset of $\mathcal{U}^*_{\mathbf{m}}$.*

1. *$\{\mathfrak{s}_n\}_{n\in\mathbb{N}}$ is an $\mathbf{m}-$convergent sequence to $\mathfrak{s} \in \mathcal{U}^*_{\mathbf{m}}$ if and only if $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}_n, \mathfrak{s}) \to 0$, as $n \to \infty$ for all $\mathfrak{b} > 0$ and $\mathfrak{s}$ is called the $\mathbf{m}-$limit of $\{\mathfrak{s}_n\}_{n\in\mathbb{N}}$.*
2. *If $\lim_{n,m\to\infty} \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}_n, \mathfrak{s}_m) = 0$, for all $\mathfrak{b} > 0$, then the sequence $\{\mathfrak{s}_n\}_{n\in\mathbb{N}}$ in $\mathcal{U}^*_{\mathbf{m}}$ is named as an $\mathbf{m}-$Cauchy sequence.*
3. *If any $\mathbf{m}-$Cauchy sequence in $\mathcal{U}^*_{\mathbf{m}}$ is $\mathbf{m}-$convergent to the point of $\mathcal{U}^*_{\mathbf{m}}$, then $\mathcal{U}^*_{\mathbf{m}}$ is called $\mathbf{m}-$complete space.*
4. *The set $\mathfrak{M}$ is $\mathbf{m}-$closed, provided that the $\mathbf{m}-$limit of an $\mathbf{m}-$convergent sequence of $\mathfrak{M}$ all the time belongs to $\mathfrak{M}$.*
5. *$\mathsf{Q} : \mathcal{U}^*_{\mathbf{m}} \to \mathcal{U}^*_{\mathbf{m}}$ is an $\mathbf{m}-$continuous mapping if $\mathbf{m}_{\mathfrak{b}}(\mathfrak{s}_n, \mathfrak{s}) \to 0$, provided to $\mathbf{m}_{\mathfrak{b}}(\mathsf{Q}\mathfrak{s}_n, \mathsf{Q}\mathfrak{s}) \to 0$ as $n \to \infty$.*
6. *$\mathfrak{M}$ is an $\mathbf{m}-$bounded set provided that*

$$\delta_{\mathbf{m}}(\mathfrak{M}) = \sup\{\mathbf{m}_1(\mathfrak{s}, \mathfrak{z}) : \mathfrak{s}, \mathfrak{z} \in \mathfrak{M}\} < \infty.$$

7. *$\mathfrak{M}$ is an $\mathbf{m}-$compact set if, for any $\{\mathfrak{s}_n\}_{n\in\mathbb{N}}$ in $\mathfrak{M}$, there exists a subset sequence $\{\mathfrak{s}_{n_k}\}$ and $\mathfrak{s} \in \mathfrak{M}$ such that $\mathbf{m}_1(\mathfrak{s}_{n_k}, \mathfrak{s}) \to 0$.*
8. *$\mathbf{m}$ holds the Fatou property $\Leftrightarrow$ for any sequence $\{\mathfrak{s}_n\}_{n\in\mathbb{N}}$ in $\mathcal{U}^*_{\mathbf{m}}$ $\mathbf{m}-$convergent to $\mathfrak{s}$, then*

$$\mathbf{m}_1(\mathfrak{s}, \mathfrak{z}) \leq \liminf_{n\to\infty} \mathbf{m}_1(\mathfrak{s}_n, \mathfrak{z})$$

for any $\mathfrak{z} \in \mathcal{U}^*_{\mathbf{m}}$.

Definition 1.6. [1] *Let $\mathbf{m}$ be modular. $\mathbf{m}$ is said to satisfy the Δ_2-condition if the following expression holds:*

($\mathcal{D}$) *$\lim_{n\to\infty} \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}_n, \mathfrak{s}) = 0$ for some $\mathfrak{b} > 0$ implies $\lim_{n\to\infty} \mathbf{m}_{\mathfrak{b}}(\mathfrak{s}_n, \mathfrak{s}) = 0$, for all $\mathfrak{b} > 0$.*

Nevertheless, the converse of condition ($\mathcal{D}$) is not always valid.

Consider the subsequent sets.

- $\mathcal{CB}(\mathfrak{M}) = \{\mathcal{X} : \mathcal{X} \text{ is non}-\text{void}, \ \mathbf{m}-\text{closed}, \text{ and } \mathbf{m}-\text{bounded subset of } \mathfrak{M}\}$.
- $\mathcal{K}(\mathfrak{M}) = \{\mathcal{X} : \mathcal{X} \text{ is non}-\text{void}, \ \mathbf{m}-\text{compact subset of } \mathfrak{M}\}$.
- The Hausdorff-Pompei modular metric is defined on $\mathcal{CB}(\mathfrak{M})$ by

$$\mathcal{H}_{\mathbf{m}}(\mathcal{R}, \mathcal{S}) = \max\left\{\sup_{\mathfrak{s}\in\mathcal{R}} \mathbf{m}_1(\mathfrak{s}, \mathcal{S}), \sup_{\mathfrak{z}\in\mathcal{S}} \mathbf{m}_1(\mathcal{R}, \mathfrak{z})\right\}$$

for $\mathbf{m}_1(\mathfrak{s}, \mathcal{S}) = \inf_{\mathfrak{z}\in\mathcal{S}} \mathbf{m}_1(\mathfrak{s}, \mathfrak{z})$.

In 1969, Nadler [24] expanded the Banach fixed point theorem for multivalued mappings in a metric space by handling the notion of the Hausdorff-Pompei metric. Moreover, this concept is also discussed in modular metric spaces. As noted in [1], Abdou and Khamsi characterized the multivalued Lipschitzian mapping in this space.

Definition 1.7. [1] *Let $(\mathcal{U}, \mathbf{m})$ be an MMS, $\mathsf{Q} : \mathfrak{M} \to C\mathfrak{B}(\mathfrak{M})$ be a mapping, and $\mathfrak{M}$ be a non-void subset of $\mathcal{U}_{\mathbf{m}}$. Then, for any $\mathfrak{s}, \mathfrak{z} \in \mathfrak{M}$ and $\gamma \geq 0$, if the inequality*

$$\mathcal{H}_{\mathbf{m}}(\mathsf{Q}(\mathfrak{s}), \mathsf{Q}(\mathfrak{z})) \leq \gamma \mathbf{m}_1(\mathfrak{s}, \mathfrak{z})$$

is provided, then the mapping Q is entitled to a multivalued Lipschitzian.

The following lemmas are essential for multivalued mappings in MMS.

Lemma 1.8. [1] *Let $(\mathcal{U}, \mathbf{m})$ be an MMS and $\mathfrak{M}$ be a non-void subset of $\mathcal{U}_{\mathbf{m}}$. Let $\mathcal{R}, \mathcal{S} \in C\mathfrak{B}(\mathfrak{M})$; then for each $\varepsilon > 0$ and $\mathfrak{s} \in \mathcal{R}$, there exists $\mathfrak{z} \in \mathcal{S}$ such that*

$$\mathbf{m}_1(\mathfrak{s}, \mathfrak{z}) \leq \mathcal{H}_{\mathbf{m}}(\mathcal{R}, \mathcal{S}) + \varepsilon.$$

Furthermore, provided that $\mathcal{S}$ is $\mathbf{m}-$compact and $\mathbf{m}$ fulfills the Fatou property, then for any $\mathfrak{s}$ in $\mathcal{R}$, $\mathfrak{z} \in \mathcal{S}$ comes into existence such that

$$\mathbf{m}_1(\mathfrak{s}, \mathfrak{z}) \leq \mathcal{H}_{\mathbf{m}}(\mathcal{R}, \mathcal{S}).$$

Lemma 1.9. [1] *Let $(\mathcal{U}, \mathbf{m})$ be an MMS. Assume that $\mathbf{m}$ satisfies the Δ_2-condition. Let $\mathfrak{M}$ be a non-void subset of $\mathcal{U}_{\mathbf{m}}$, $\mathcal{R}_n$ be a sequence of sets $C\mathfrak{B}(\mathfrak{M})$, and suppose $\lim_{n\to\infty} \mathcal{H}_{\mathbf{m}}(\mathcal{R}_n, \mathcal{R}_0) = 0$, where $\mathcal{R}_0 \in C\mathfrak{B}(\mathfrak{M})$. If $\mathfrak{s}_n \in \mathcal{R}_n$ and $\lim_{n\to\infty} \mathfrak{s}_n = \mathfrak{s}_0$, it follows that $\mathfrak{s}_0 \in \mathcal{R}_0$.*

Now, we recollect the concept of $\alpha-$admissibility and some generalizations, as indicated below.

Definition 1.10. *Let $\mathsf{Q} : \mathcal{U} \to \mathcal{U}$ be a self-mapping and $\alpha : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be a function. We contemplate the following circumstances.*

(α_1) $\alpha(\mathfrak{s}, \mathfrak{z}) \geq 1$ implies $\alpha(\mathsf{Q}\mathfrak{s}, \mathsf{Q}\mathfrak{z}) \geq 1$,

(α_2) $\alpha(\mathfrak{s}, \mathsf{Q}\mathfrak{s}) \geq 1$ implies $\alpha\left(\mathsf{Q}\mathfrak{s}, \mathsf{Q}^2\mathfrak{s}\right) \geq 1$,

(α_3) $\alpha(\mathfrak{s}, \mathfrak{z}) \geq 1$ and $\alpha(\mathfrak{z}, \mathsf{Q}\mathfrak{z}) \geq 1$ implies $\alpha(\mathfrak{s}, \mathsf{Q}\mathfrak{z}) \geq 1$.

Taking into account the function (α_i), we put forward that

- *$i = 1$, Q is an $\alpha-$admissible mapping in [31].*
- *$i = 2$, Q is an $\alpha-$orbital admissible mapping [26].*
- *$i = 2, 3$, Q is a triangular $\alpha-$orbital admissible mapping [26].*

The concept of $\alpha-$admissibility is redefined for multivalued mappings, as noted below.

Definition 1.11. [21] *Let $\mathsf{Q} : \mathcal{U} \to C\mathfrak{B}(\mathcal{U})$ be a multivalued mapping and $\alpha : \mathcal{U} \times \mathcal{U} \to \mathbb{R}$ be a function. Q is called α^*-admissible mapping if $\alpha^*(\mathfrak{s}, \mathfrak{z}) \geq 1$* implies *$\alpha^*(\mathsf{Q}\mathfrak{s}, \mathsf{Q}\mathfrak{z}) \geq 1$, where $\alpha^*(A, B) = \inf\{\alpha(\mathfrak{s}, \mathfrak{z}) | \mathfrak{s} \in A, \ \mathfrak{z} \in B\}$.*

Moreover, a mapping $\mathsf{Q} : \mathcal{U} \to C\mathfrak{B}(\mathcal{U})$ is said to be α^*-orbital admissible if

(O) $\alpha^*(\mathfrak{s}, \mathsf{Q}\mathfrak{s}) \geq 1$ implies $\alpha^*\left(\mathsf{Q}\mathfrak{s}, \mathsf{Q}^2\mathfrak{s}\right) \geq 1$, where $\alpha^*(A, B) = \inf\{\alpha(\mathfrak{s}, \mathfrak{z}) | \mathfrak{s} \in A, \ \mathfrak{z} \in B\}$.

In addition to the condition (O), if the following condition ($\mathcal{T}O$) is satisfied, then Q is called triangular α^*-orbital admissible;

$(\mathcal{TO})$ $\alpha(\mathfrak{s},\mathfrak{z}) \geq 1$ and $\alpha^*(\mathfrak{z},\mathsf{Q}\mathfrak{z}) \geq 1$ implies $\alpha^*(\mathfrak{s},\mathsf{Q}\mathfrak{z}) \geq 1$.

Lemma 1.12. [26] *Let $\mathsf{Q}:\mathcal{U}\to\mathcal{U}$ be a triangular $\alpha-$orbital admissible mapping and $\alpha(\mathfrak{s}_0,\mathsf{Q}\mathfrak{s}_0)\geq 1$ with $\mathfrak{s}_0\in\mathcal{U}$. Thereupon, $\alpha\left(\mathfrak{s}_j,\mathfrak{s}_m\right)\geq 1$ for all $j,m\in\mathbb{N}$ with $j<m$, where the sequence $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is defined by $\mathfrak{s}_{j+1}=\mathsf{Q}\mathfrak{s}_j$.*

The following lemma can be presented with respect to Lemma 1.12.

Lemma 1.13. *Let $\mathsf{Q}:\mathcal{U}\to\mathcal{U}$ be a triangular $\alpha-$orbital admissible mapping and $\alpha(\mathfrak{s}_0,\mathsf{Q}\mathfrak{s}_0)\geq 1$ with $\mathfrak{s}_0\in\mathcal{U}$. Thereupon, $\alpha\left(\mathfrak{s}_j,\mathfrak{s}_m\right)\geq 1$ for all $j,m\in\mathbb{N}$ with $j<m$, where the sequence $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is defined by $\mathfrak{s}_{j+1}\in\mathsf{Q}^{n_j}\mathfrak{s}_j$ with $n_j=n\left(\mathfrak{s}_j\right)$.*

Besides, consult [4, 16, 19, 22, 23, 28, 30] for further details about admissible mappings.

With the increasing demand for developing the newly introduced contraction conditions via some auxiliary functions in metric fixed point theory, many authors have made progress, and many new concepts are acquainted with the literature. One of them is Proinov type contraction, which was recently created by Proinov [27] and sums up several contraction mappings in the existing literature.

Definition 1.14. *[27] Let $\mathsf{Q}:\mathcal{U}\to\mathcal{U}$ be a mapping on a metric space $(\mathcal{U},\mathsf{d})$ and $\Phi,\Psi:(0,\infty)\to\mathbb{R}$ be two functions that satisfy the following features:*

- (p_1) *Φ is a non-decreasing function,*
- (p_2) *$\Psi(\mathsf{t})<\Phi(\mathsf{t})$ for all $\mathsf{t}>0$,*
- (p_3) *$\limsup_{\mathsf{t}\to\mathsf{t}_0+}\Psi(\mathsf{t})<\Phi(\mathsf{t}_0+)$ for any $\mathsf{t}_0>0$.*

Thereby, for all $\mathfrak{s},\mathfrak{z}\in\mathcal{U}$, if the inequality

$$\Phi(\mathsf{d}(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z}))\leq\Psi(\mathsf{d}(\mathfrak{s},\mathfrak{z})),$$

is satisfied, the mapping Q is called Proinov type contraction.

Theorem 1.15. [27] *Let $(\mathcal{U},\mathsf{d})$ be a complete metric space and $\mathsf{Q}:\mathcal{U}\to\mathcal{U}$ be a Proinov type contraction. Then Q admits a unique fixed point in $\mathcal{U}$.*

Various fixed point results appear in the literature involving Proinov type contraction. Some examples are in [5, 14, 15, 17, 18, 20, 29].

2. Fixed Point Results for Multivalued Mappings

Throughout this session, we will consider and draw on the following property:

(Ω) Let **m** be a convex regular modular, which obeys the Fatou property and the Δ_2-condition, and $\mathfrak{M}$ be a non-void $\mathbf{m}-$complete subset of $\mathcal{U}_{\mathbf{m}}$.

Definition 2.1. *Let $(\mathcal{U},\mathbf{m})$ be an MMS, $\mathfrak{M}$ be a nonempty bounded subset of $\mathcal{U}_{\mathbf{m}}$, and $\alpha:\mathcal{U}_{\mathbf{m}}\to\mathbb{R}_+$ be a function. A multivalued mapping $\mathsf{Q}:\mathfrak{M}\to\mathcal{CB}(\mathfrak{M})$ is called multivalued Sehgal-Proinov type $(\alpha^*,\Phi,\Psi)-$contraction mapping if there exists $\Phi,\Psi:(0,\infty)\to\mathbb{R}$ such that for each $\mathfrak{s},\mathfrak{z}\in\mathfrak{M}$, there exists a positive integer $n(\mathfrak{s})$ such that:*

$$\alpha(\mathfrak{s},\mathfrak{z})\,\Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)\right)\leq\Psi(\mathcal{R}(\mathfrak{s},\mathfrak{z})),\tag{3}$$

where $\Phi,\Psi:(0,\infty)\to\mathbb{R}$ are two functions satisfying

- (c_1) *Φ is a lower semi-continuous and non-decreasing function;*

(c_2) $\Psi(\mathrm{t}) < \Phi(\mathrm{t})$ *for all* $\mathrm{t} > 0$;

(c_3) $\limsup\limits_{\mathrm{t}\to\mathrm{t}_0+} \Psi(\mathrm{t}) < \Phi(\mathrm{t}_0+)$ *for any* $\mathrm{t}_0 > 0$,

and also,

$$\mathcal{R}(\mathfrak{s},\mathfrak{z}) = \max\left\{ \mathbf{m}_1(\mathfrak{s},\mathfrak{z}), \frac{\delta_2\left(\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)+\delta_2\left(\mathfrak{z},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}\right)}{2}, \frac{\delta_1\left(\mathfrak{z},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)\delta_2\left(\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)+\delta_1\left(\mathfrak{z},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)\delta_2\left(\mathfrak{z},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}\right)}{\delta_2\left(\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)+\delta_1\left(\mathfrak{z},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}\right)+1} \right\}$$

for all $\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right) > 0$.

Theorem 2.2. *Let* $(\mathcal{U},\mathbf{m})$ *be an* $\mathbf{m}-$*complete MMS. Presume that the property* (Ω) *holds and* $\mathsf{Q}:\mathfrak{M}\to\mathcal{K}(\mathfrak{M})$ *is multivalued Sehgal-Proinov type* $(\alpha^*,\Phi,\Psi)-$*contraction mapping. If the circumstances*

- (i) *there is a point* $\mathfrak{s}_0\in\mathfrak{M}$ *that has the property* $\alpha^*(\mathfrak{s}_0,\mathsf{Q}\mathfrak{s}_0)\geq 1$,
- (ii) Q *is a triangular* α^*-*orbital admissible mapping, i.e.,* Q *satisfies the conditions of* $(\mathcal{O})$ *and* $(\mathcal{TO})$,
- (iii) Q *is an* $\mathbf{m}-$*continuous mapping,*
- (iv) *there exist* $\mathfrak{s},\mathfrak{z}\in M_{Fix}(\mathsf{Q}^{n(\mathfrak{s})})$, *which denotes the set of fixed points of multivalued mappings of* $\mathsf{Q}^{n(x)}$, *such that* $\alpha(\mathfrak{s},\mathfrak{z})\geq 1$,

are provided, then, Q *owns a fixed point* $\mathfrak{s}^*$ *in* $\mathfrak{M}\subseteq\mathcal{U}_{\mathbf{m}}$*; where* $\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1)<\infty$ *for some* $\mathfrak{s}_0,\mathfrak{s}_1\in\mathcal{U}_{\mathbf{m}}$.

Proof. Let $\mathfrak{s}_0\in\mathfrak{M}$ be a point in the condition (i) such that $\alpha^*(\mathfrak{s}_0,\mathsf{Q}\mathfrak{s}_0)\geq 1$. Now, since Q is α^*-orbital admissible mapping, we achieve $\alpha^*\left(\mathsf{Q}\mathfrak{s}_0,\mathsf{Q}^2\mathfrak{s}_0\right)\geq 1$, which yields that

$$\alpha^*\left(\mathsf{Q}^{k-1}\mathfrak{s}_0,\mathsf{Q}^k\mathfrak{s}_0\right)\geq 1, \quad \text{for all } k\in\mathbb{N}. \tag{4}$$

Taking (ii) and (4) into account, we acquire that $\alpha^*(\mathfrak{s}_0,\mathsf{Q}^n\mathfrak{s}_0)\geq 1$. So, by choosing $\mathfrak{s}_1\in\mathsf{Q}^{n_0}\mathfrak{s}_0$ with $n_0=n(\mathfrak{s}_0)$, we get $\alpha(\mathfrak{s}_0,\mathfrak{s}_1)\geq 1$ and $\mathfrak{s}_0\neq\mathfrak{s}_1$. As a result of this, from (3) and the fact that (c_1), we have

$$\begin{aligned}\Phi(\delta_1(\mathfrak{s}_1,\mathsf{Q}^{n_1}\mathfrak{s}_1)) \leq \Phi(\mathcal{H}_{\mathbf{m}}(\mathsf{Q}^{n_0}\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1)) &\leq \alpha(\mathfrak{s}_0,\mathfrak{s}_1)\,\Phi(\mathcal{H}_{\mathbf{m}}(\mathsf{Q}^{n_0}\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1))\\ &\leq \Psi(\mathcal{R}(\mathfrak{s}_0,\mathfrak{s}_1)).\end{aligned}$$

Hence, there exists $\mathfrak{s}_2\in\mathsf{Q}^{n_1}\mathfrak{s}_1$ with $n_1=n(\mathfrak{s}_1)$ such that

$$\Phi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)) \leq \alpha(\mathfrak{s}_0,\mathfrak{s}_1)\,\Phi(\mathcal{H}_{\mathbf{m}}(\mathsf{Q}^{n_0}\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1)) \leq \Psi(\mathcal{R}(\mathfrak{s}_0,\mathfrak{s}_1)), \tag{5}$$

where

$$\mathcal{R}(\mathfrak{s}_0,\mathfrak{s}_1) = \max\left\{ \mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1), \frac{\delta_2(\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1)+\delta_2(\mathfrak{s}_1,\mathsf{Q}^{n_0}\mathfrak{s}_0)}{2}, \frac{\delta_1(\mathfrak{s}_1,\mathsf{Q}^{n_1}\mathfrak{s}_1)\delta_2(\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1)+\delta_1(\mathfrak{s}_0,\mathsf{Q}^{n_0}\mathfrak{s}_0)\delta_2(\mathfrak{s}_1,\mathsf{Q}^{n_0}\mathfrak{s}_0)}{\delta_2(\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1)+\delta_1(\mathfrak{s}_1,\mathsf{Q}^{n_0}\mathfrak{s}_0)+1} \right\}$$

and so forth

- $\delta_2(\mathfrak{s}_0,\mathsf{Q}^{n_1}\mathfrak{s}_1) = \inf\limits_{\mathfrak{s}_2\in\mathsf{Q}^{n_1}\mathfrak{s}_1}\{\mathbf{m}_2(\mathfrak{s}_0,\mathfrak{s}_2)\} \leq \mathbf{m}_2(\mathfrak{s}_0,\mathfrak{s}_2)$,
- $\delta_2(\mathfrak{s}_1,\mathsf{Q}^{n_0}\mathfrak{s}_0) = \inf\limits_{\mathfrak{s}_1\in\mathsf{Q}^{n_0}\mathfrak{s}_0}\{\mathbf{m}_2(\mathfrak{s}_1,\mathfrak{s}_1)\} = 0$,
- $\delta_1(\mathfrak{s}_0,\mathsf{Q}^{n_0}\mathfrak{s}_0) = \inf\limits_{\mathfrak{s}_1\in\mathsf{Q}^{n_0}\mathfrak{s}_0}\{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1)\} \leq \mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1)$,
- $\delta_1(\mathfrak{s}_1,\mathsf{Q}^{n_1}\mathfrak{s}_1) = \inf\limits_{\mathfrak{s}_2\in\mathsf{Q}^{n_1}\mathfrak{s}_1}\{\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)\} \leq \mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)$.

Thereby, we procure that

$$\begin{aligned}\mathcal{R}(\mathfrak{s}_0,\mathfrak{s}_1) &\leq \max\left\{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1), \tfrac{\mathbf{m}_2(\mathfrak{s}_0,\mathfrak{s}_2)}{2}, \tfrac{\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)\mathbf{m}_2(\mathfrak{s}_0,\mathfrak{s}_2)}{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_2)+1}\right\} \\ &\leq \max\left\{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1), \tfrac{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1)+\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)}{2}, \tfrac{\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)\mathbf{m}_2(\mathfrak{s}_0,\mathfrak{s}_2)}{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_2)+1}\right\} \\ &\leq \max\left\{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1), \mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)\right\}.\end{aligned}$$

If $\max\{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1), \mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)\} = \mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)$, then, by considering (5) and (c_2), we get

$$\Phi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)) \leq \Psi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)) < \Phi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)),$$

such that a contradiction arises. Then, we deduce $\max\{\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1), \mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)\} = \mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1)$, which implies $\Phi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)) < \Phi(\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1))$. Likewise, due to the fact that Q is triangular α^*−orbital admissible, we can write $\alpha(\mathfrak{s}_1,\mathfrak{s}_2) \geq 1$ with $\mathfrak{s}_1 \neq \mathfrak{s}_2$ and so, for $\mathfrak{s}_3 \in \mathsf{Q}^{n_2}\mathfrak{s}_2$, we gain

$$\begin{aligned}\Phi(\mathbf{m}_1(\mathfrak{s}_2,\mathfrak{s}_3)) \leq \Phi(\delta_1(\mathfrak{s}_2,\mathsf{Q}^{n_2}\mathfrak{s}_2)) &\leq \alpha(\mathfrak{s}_1,\mathfrak{s}_2)\,\Phi(\mathcal{H}_{\mathbf{m}}(\mathsf{Q}^{n_1}\mathfrak{s}_1,\mathsf{Q}^{n_2}\mathfrak{s}_2)) \\ &\leq \Psi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)) < \Phi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2))\end{aligned}$$

that is,

$$\Phi(\mathbf{m}_1(\mathfrak{s}_2,\mathfrak{s}_3)) < \Phi(\mathbf{m}_1(\mathfrak{s}_1,\mathfrak{s}_2)).$$

Inevitably, by proceeding with this procedure, we contrive a sequence $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ with initial point $\mathfrak{s}_0$ such that $\mathfrak{s}_{j+1} \in \mathsf{Q}^{n_j}\mathfrak{s}_j$ with $n_j = n\left(\mathfrak{s}_j\right)$. On the other hand, considering the assumptions (*ii*) and (4), we get $\alpha^*\left(\mathfrak{s}_j,\mathsf{Q}\mathfrak{s}_j\right) \geq 1$ and $\alpha^*\left(\mathsf{Q}\mathfrak{s}_j,\mathsf{Q}^2\mathfrak{s}_j\right) \geq 1$ such that $\alpha^*\left(\mathfrak{s}_j,\mathsf{Q}^2\mathfrak{s}_j\right) \geq 1$. Thereupon, $\alpha^*\left(\mathfrak{s}_j,\mathsf{Q}^n\mathfrak{s}_j\right) \geq 1$ for all $n \in \mathbb{N}$, which stands for $\alpha\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right) \geq 1$. Then, by (3), we have

$$\begin{aligned}\Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right) \leq \Phi\left(\delta_1\left(\mathfrak{s}_{j+1},\mathsf{Q}^{n_{j+1}}\mathfrak{s}_{j+1}\right)\right) &\leq \alpha\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right)\Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n_j}\mathfrak{s}_j,\mathsf{Q}^{n_{j+1}}\mathfrak{s}_{j+1}\right)\right) \\ &\leq \Psi\left(\mathcal{R}\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right)\right),\end{aligned} \tag{6}$$

where

$$\begin{aligned}\mathcal{R}\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right) &= \max\left\{\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right), \tfrac{\delta_2\left(\mathfrak{s}_j,\mathsf{Q}^{n_{j+1}}\mathfrak{s}_{j+1}\right)+\delta_2\left(\mathfrak{s}_{j+1},\mathsf{Q}^{n_j}\mathfrak{s}_j\right)}{2}, \tfrac{\delta_1\left(\mathfrak{s}_{j+1},\mathsf{Q}^{n_{j+1}}\mathfrak{s}_{j+1}\right)\delta_2\left(\mathfrak{s}_j,\mathsf{Q}^{n_{j+1}}\mathfrak{s}_{j+1}\right)+\delta_1\left(\mathfrak{s}_j,\mathsf{Q}^{n_j}\mathfrak{s}_j\right)\delta_2\left(\mathfrak{s}_{j+1},\mathsf{Q}^{n_j}\mathfrak{s}_j\right)}{\delta_2\left(\mathfrak{s}_j,\mathsf{Q}^{n_{j+1}}\mathfrak{s}_{j+1}\right)+\delta_1\left(\mathfrak{s}_{j+1},\mathsf{Q}^{n_j}\mathfrak{s}_j\right)+1}\right\} \\ &\leq \max\left\{\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right), \tfrac{\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right)+\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)}{2}, \tfrac{\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\mathbf{m}_2\left(\mathfrak{s}_j,\mathfrak{s}_{j+2}\right)}{\mathbf{m}_2\left(\mathfrak{s}_j,\mathfrak{s}_{j+2}\right)+1}\right\} \\ &\leq \max\left\{\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right), \mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right\}.\end{aligned}$$

If we consider $\max\left\{\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right), \mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right\} = \mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)$, then, we achieve that $\mathcal{R}\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right) \leq \mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)$. Hence, by using (c_2), the inequality (6) becomes

$$\Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right) \leq \Psi\left(\Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right)\right) < \Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right),$$

but this causes a contradiction. Then, we conclude that $\mathcal{R}\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right) \leq \mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right)$ and so, we get $\Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right)\right) < \Phi\left(\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right)\right)$. Moreover, since Φ is non-decreasing, we obtain

$$\mathbf{m}_1\left(\mathfrak{s}_{j+1},\mathfrak{s}_{j+2}\right) < \mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}_{j+1}\right).$$

Therefore, we deduce that the sequence $\left\{\mathbf{m}_1\left(\mathfrak{s}_j, \mathfrak{s}_{j+1}\right)\right\}_{j\in\mathbb{N}}$ is non-increasing. Hence, a number $\mathcal{L} \geq 0$ exists such that $\lim_{j\to\infty} \mathbf{m}_1\left(\mathfrak{s}_j, \mathfrak{s}_{j+1}\right) = \mathcal{L}$. The main aim of this step is to signify that $\mathcal{L} = 0$. Contrarily, let $\mathcal{L} > 0$. Then, from (6), we attain

$$\Phi(\mathcal{L}) = \lim_{j\to\infty} \Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j+1}, \mathfrak{s}_{j+2}\right)\right) \leq \limsup_{j\to\infty} \Psi\left(\mathbf{m}_1\left(\mathfrak{s}_j, \mathfrak{s}_{j+1}\right)\right) < \limsup_{s\to\mathcal{L}} \Phi(s)$$

such that this contradicts with the supposition (c_3). We perceive that our supposition is incorrect; that is,

$$\lim_{j\to\infty} \mathbf{m}_1\left(\mathfrak{s}_j, \mathfrak{s}_{j+1}\right) = 0. \tag{7}$$

It is required to indicate $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is an $\mathbf{m}-$Cauchy sequence. Rather, presume that $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is not an $\mathbf{m}-$Cauchy sequence. Then, we write the following;

$$\mathbf{m}_1\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)}\right) \geq \varepsilon. \tag{8}$$

In this case, for at least a $\varepsilon > 0$ and $m(k) > j(k) \geq k$ whenever $k \in \mathbb{N} \cup \{0\}$, the following expression is provided. Let $m(k)$ be the smallest index satisfying (8). We have $\mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)-1}\right) < \varepsilon$.

Hence, by using $(\mathbf{m}_3)$ and (7), we procure

$$\varepsilon \leq \mathbf{m}_1\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)}\right) \leq \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{j(k)+1}\right) + \mathbf{m}_{\frac{1}{4}}\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{m(k)+1}\right) + \mathbf{m}_{\frac{1}{4}}\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{m(k)+1}\right)$$

such that

$$\limsup_{k\to\infty} \mathbf{m}_{\frac{1}{4}}\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{m(k)+1}\right) \geq \varepsilon. \tag{9}$$

Likewise, considering (7), we infer that

$$\mathbf{m}_1\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)}\right) \leq \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)-1}\right) + \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{m(k)-1}, \mathfrak{s}_{m(k)}\right)$$

such that

$$\limsup_{k\to\infty} \mathbf{m}_1\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)}\right) \leq \varepsilon. \tag{10}$$

Again, one can reason out

$$\mathbf{m}_2\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)+1}\right) \leq \mathbf{m}_1\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)-1}\right) + \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{m(k)-1}, \mathfrak{s}_{m(k)}\right) + \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{m(k)}, \mathfrak{s}_{m(k)+1}\right),$$

and

$$\mathbf{m}_2\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{m(k)}\right) \leq \mathbf{m}_1\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{j(k)}\right) + \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)-1}\right) + \mathbf{m}_{\frac{1}{2}}\left(\mathfrak{s}_{m(k)-1}, \mathfrak{s}_{m(k)}\right)$$

which means that

$$\limsup_{k\to\infty} \mathbf{m}_2\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)+1}\right) = \limsup_{k\to\infty} \mathbf{m}_2\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{m(k)}\right) \leq \varepsilon. \tag{11}$$

On the other hand, from Lemma 1.13, we gain $\alpha\left(\mathfrak{s}_j, \mathfrak{s}_m\right) \geq 1$ for all $j, m \in \mathbb{N}$ with $j < m$. Thereby, by considering (3) with $\mathfrak{s}_{j(k)+1} \in \mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}$ and $\mathfrak{s}_{m(k)+1} \in \mathsf{Q}^{n_{m(k)}}\mathfrak{s}_{m(k)}$, we attain

$$\begin{aligned}\Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j(k)+1}, \mathfrak{s}_{m(k)+1}\right)\right) &\leq \alpha\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)}\right)\Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}, \mathsf{Q}^{n_{m(k)}}\mathfrak{s}_{m(k)}\right)\right) \\ &\leq \Psi\left(\mathcal{R}\left(\mathfrak{s}_{j(k)}, \mathfrak{s}_{m(k)}\right)\right),\end{aligned} \tag{12}$$

where

$$\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)}\right)=\max\left\{\begin{array}{c}\mathbf{m}_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)}\right),\frac{\delta_1\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n_{m(k)}}\mathfrak{s}_{m(k)}\right)+\delta_1\left(\mathfrak{s}_{m(k)},\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)}{2},\\ \frac{\delta_1\left(\mathfrak{s}_{m(k)},\mathsf{Q}^{n_{m(k)}}\mathfrak{s}_{m(k)}\right)\delta_1\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n_{m(k)}}\mathfrak{s}_{m(k)}\right)+\delta_1\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)\delta_1\left(\mathfrak{s}_{m(k)},\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)}{\delta_1\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n_{m(k)}}\mathfrak{s}_{m(k)}\right)+\delta_1\left(\mathfrak{s}_{m(k)},\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)+1}\end{array}\right\}$$

$$\leq\max\left\{\begin{array}{c}\mathbf{m}_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)}\right),\frac{\delta_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)+1}\right)+\delta_1\left(\mathfrak{s}_{m(k)},\mathfrak{s}_{j(k)+1}\right)}{2},\\ \frac{\delta_1\left(\mathfrak{s}_{m(k)},\mathfrak{s}_{m(k)+1}\right)\delta_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)+1}\right)+\delta_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{j(k)+1}\right)\delta_1\left(\mathfrak{s}_{m(k)},\mathfrak{s}_{j(k)+1}\right)}{\delta_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)+1}\right)+\delta_1\left(\mathfrak{s}_{m(k)},\mathfrak{s}_{j(k)+1}\right)+1}\end{array}\right\}. \tag{13}$$

If we take the limit superior in (13) and use (7), (10), and (11), we achieve

$$\limsup_{k\to\infty}\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)}\right)\leq\max\left\{\varepsilon,\frac{\varepsilon+\varepsilon}{2},0\right\}=\varepsilon.$$

Hereupon, taking (7), (9), and(c_2) into account and from (12), we induce

$$\begin{aligned}\Phi(\varepsilon)\leq\limsup_{k\to\infty}\Phi\left(\mathbf{m}_1\left(\mathfrak{s}_{j(k)+1},\mathfrak{s}_{m(k)+1}\right)\right)&\leq\limsup_{k\to\infty}\Psi\left(\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)}\right)\right)\\ &<\Phi\left(\limsup_{k\to\infty}\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}_{m(k)}\right)\right)\\ &<\Phi(\varepsilon),\end{aligned}$$

but this results in a contradiction, that is, we get $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is an $\mathbf{m}$−Cauchy sequence in $(\mathcal{U},\mathbf{m})$, which implies there exists a point $\mathfrak{s}^*\in\mathfrak{M}$ such that

$$\lim_{j\to\infty}\mathbf{m}_1\left(\mathfrak{s}_j,\mathfrak{s}^*\right)=0. \tag{14}$$

By the condition (*iii*), let Q be an $\mathbf{m}$−continuous multivalued mapping and $\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}_j$ be a sequence in $\mathcal{CB}(\mathfrak{M})$. Then, from (14), we deduce that $\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}_j\to\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*$, and so $\lim_{j\to\infty}\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}_j,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)=0$, where $\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\in\mathcal{CB}(\mathfrak{M})$. If $\mathfrak{s}_{j+1}\in\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}_j$ and $\lim_{j\to\infty}\mathfrak{s}_{j+1}=\mathfrak{s}^*$, then, considering Lemma 1.9, we acquire that $\mathfrak{s}^*\in\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*$, which implies that $\mathfrak{s}^*$ is the fixed point of $\mathsf{Q}^{n(\mathfrak{s}^*)}$. The next step is to indicate the uniqueness of the fixed point of $\mathsf{Q}^{n(\mathfrak{s}^*)}$. For this purpose, we assume that a point z^* resides in $\mathfrak{M}$ with $\mathfrak{s}^*\neq\mathsf{z}^*$ such that $\mathsf{z}^*\in\mathsf{Q}^{n(\mathfrak{s}^*)}\mathsf{z}^*$. From the hypothesis, we have $\alpha(\mathfrak{s}^*,\mathsf{z}^*)\geq1$. Thereupon, by using (3), we obtain

$$\Phi\left(\mathbf{m}_1\left(\mathfrak{s}^*,\mathsf{z}^*\right)\right)\leq\alpha\left(\mathfrak{s}^*,\mathsf{z}^*\right)\Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathsf{z}^*\right)\right)\leq\Psi\left(\mathcal{R}\left(\mathfrak{s}^*,\mathsf{z}^*\right)\right), \tag{15}$$

where

$$\mathcal{R}(\mathfrak{s}^*,\mathsf{z}^*)=\max\left\{\mathbf{m}_1(\mathfrak{s}^*,\mathsf{z}^*),\frac{\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathsf{z}^*\right)+\delta_2\left(\mathsf{z}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)}{2},\frac{\delta_1\left(\mathsf{z}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathsf{z}^*\right)\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathsf{z}^*\right)+\delta_1\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\delta_2\left(\mathsf{z}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)}{\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathsf{z}^*\right)+\delta_1\left(\mathsf{z}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)+1}\right\}$$

$$\leq\max\left\{\mathbf{m}_1(\mathfrak{s}^*,\mathsf{z}^*),\mathbf{m}_2(\mathfrak{s}^*,\mathsf{z}^*),0\right\}=\mathbf{m}_1(\mathfrak{s}^*,\mathsf{z}^*).$$

So, using (c_2), the inequality (15) turns into

$$\Phi\left(\mathbf{m}_1\left(\mathfrak{s}^*,\mathsf{z}^*\right)\right)\leq\Psi\left(\mathbf{m}_1\left(\mathfrak{s}^*,\mathsf{z}^*\right)\right)<\Phi\left(\mathbf{m}_1\left(\mathfrak{s}^*,\mathsf{z}^*\right)\right)$$

such that this contradiction proves that $\mathsf{Q}^{n(\mathfrak{s}^*)}$ owns a unique fixed point.

Eventually, we assert that $\mathfrak{s}^* \in \mathsf{Q}\mathfrak{s}^*$. However, to demonstrate this, let us theorize the opposite, that is, $\mathfrak{s}^* \notin \mathsf{Q}\mathfrak{s}^*$. Hence, by regarding the uniqueness of the fixed point, from (3), we attain

$$\begin{aligned}\Phi\left(\delta_1\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right)\right) &\leq \Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*, \mathsf{Q}\left(\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\right)\right)\\ &\leq \alpha\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right)\Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\left(\mathsf{Q}\mathfrak{s}^*\right)\right)\right) \leq \Psi\left(\mathcal{R}\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right)\right),\end{aligned} \tag{16}$$

where

$$\begin{aligned}\mathcal{R}\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right) = \max\Bigg\{ &\delta_1\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right), \frac{\delta_2\left(\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}(\mathsf{Q}\mathfrak{s}^*)\right)+\delta_2\left(\mathsf{Q}\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)}{2}, \\ &\frac{\delta_1\left(\mathsf{Q}\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}(\mathsf{Q}\mathfrak{s}^*)\right)\delta_2\left(\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}(\mathsf{Q}\mathfrak{s}^*)\right)+\delta_1\left(\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\delta_2\left(\mathsf{Q}\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)}{\delta_2\left(\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}(\mathsf{Q}\mathfrak{s}^*)\right)+\delta_1\left(\mathsf{Q}\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)+1} \Bigg\}\\ &\leq \delta_1\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right).\end{aligned}$$

Consequently, considering (c_2), the expression (16) becomes

$$\Phi\left(\delta_1\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right)\right) \leq \Psi\left(\mathcal{R}\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right)\right) < \Phi\left(\delta_1\left(\mathfrak{s}^*, \mathsf{Q}\mathfrak{s}^*\right)\right).$$

However, this is not possible due to (c_1). Then, we achieve $\mathfrak{s}^* \in \mathsf{Q}\mathfrak{s}^*$. □

The above theorem can be demonstrated without the continuity of the mapping by replacing it with a suitable property, as indicated below.

Theorem 2.3. *Let $(\mathcal{U}, \mathbf{m})$ be an $\mathbf{m}-$complete MMS. Presume that the property (Ω) holds and $\mathsf{Q} : \mathfrak{M} \to \mathcal{K}(\mathfrak{M})$ be a multivalued Sehgal-Proinov type $(\alpha^*, \Phi, \Psi)-$contraction mapping. If the following conditions hold:*

- (i) *there is a point $\mathfrak{s}_0 \in \mathfrak{M}$ that has the property $\alpha^*(\mathfrak{s}_0, \mathsf{Q}\mathfrak{s}_0) \geq 1$,*
- (ii) *Q is a triangular α^*-orbital admissible mapping,*
- (iii) *if $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is a sequence satisfying*
  - (a) *$\alpha\left(\mathfrak{s}_j, \mathfrak{s}_{j+1}\right) \geq 1$ for all j,*
  - (b) *$\mathfrak{s}_j \to \mathfrak{s}^* \in \mathfrak{M}$ as $j \to \infty$,*

  then a subsequence $\left\{\mathfrak{s}_{j(k)}\right\}$ of $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ exists such that $\alpha\left(\mathfrak{s}_{j(k)}, \mathfrak{s}^*\right) \geq 1$,
- (iv) *there exist $\mathfrak{s}, \mathfrak{z} \in M_{Fix}(\mathsf{Q}^{n(\mathfrak{s})})$, which denotes the set of fixed points of multivalued mappings of $\mathsf{Q}^{n(\mathfrak{s})}$ such that $\alpha(\mathfrak{s}, \mathfrak{z}) \geq 1$.*

Then, Q owns a fixed point $\mathfrak{s}^*$ in $\mathfrak{M} \subseteq \mathcal{U}_{\mathbf{m}}$; where $\mathbf{m}_1(\mathfrak{s}_0, \mathfrak{s}_1) < \infty$ for some $\mathfrak{s}_0, \mathfrak{s}_1 \in \mathcal{U}_{\mathbf{m}}$.

Proof. Similar to the proof of Theorem 2.2, we can construct a sequence $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ in $\mathfrak{M}$ by $\mathfrak{s}_{j+1} \in \mathsf{Q}\mathfrak{s}_j$ and conclude that $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ is an $\mathbf{m}-$Cauchy sequence, which converges to $\mathfrak{s}^* \in \mathfrak{M}$.

From the hypothesis, there exists a subsequence $\left\{\mathfrak{s}_{j(k)}\right\}$ of $\left\{\mathfrak{s}_j\right\}_{j\in\mathbb{N}}$ such that $\alpha\left(\mathfrak{s}_{j(k)}, \mathfrak{s}^*\right) \geq 1$. Because $K(\mathfrak{M})$ is compact, an element $\mathfrak{s}^* \in K(\mathfrak{M}) \subseteq \mathcal{U}_{\mathbf{m}}$ exists such that $\mathfrak{s}_j \to \mathfrak{s}^*$. Then, from the Fatou property, we procure

$$\begin{aligned}\delta_1\left(\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right) \leq \liminf_{k\to\infty}\mathbf{m}_1\left(\mathfrak{s}_{j(k)+1}, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right) &= \liminf_{k\to\infty}\mathbf{m}_1\left(\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\\ &\leq \mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right),\end{aligned}$$

and as Φ is a non-decreasing map, by (3) and (c_2), we achieve

$$\Phi\left(\delta_1\left(\mathfrak{s}^*, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\right) \leq \alpha\left(\mathfrak{s}_{j(k)}, \mathfrak{s}^*\right)\Phi\left(\mathcal{H}_{\mathbf{m}}\left(\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}, \mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\right) \leq \Psi\left(\mathcal{R}\left(\mathfrak{s}_{j(k)}, \mathfrak{s}^*\right)\right), \tag{17}$$

where

$$\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}^*\right)=\max\left\{\begin{array}{l}\mathbf{m}_1\left(\mathfrak{s}_{j(k)},\mathfrak{s}^*\right),\frac{\delta_2\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)+\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)}{2},\\ \frac{\delta_1\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\delta_2\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)+\delta_1\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)}{\delta_2\left(\mathfrak{s}_{j(k)},\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)+\delta_1\left(\mathfrak{s}^*,\mathsf{Q}^{n_{j(k)}}\mathfrak{s}_{j(k)}\right)+1}\end{array}\right\}. \tag{18}$$

If we assume $\mathfrak{s}^*\notin\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*$ and take the limit as $k\to\infty$ in (17) and (18), we attain

$$\begin{aligned}\Phi\left(\delta_1\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\right)&\leq\lim_{k\to\infty}\Psi\left(\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}^*\right)\right)\\&<\Phi\left(\lim_{k\to\infty}\mathcal{R}\left(\mathfrak{s}_{j(k)},\mathfrak{s}^*\right)\right)\\&\leq\Phi\left(\max\left\{0,\frac{\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)}{2},\frac{\delta_1\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)}{\delta_2\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)+1}\right\}\right)\\&\leq\Phi\left(\delta_1\left(\mathfrak{s}^*,\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*\right)\right),\end{aligned}$$

which is a contradiction. Hence, we gain $\mathfrak{s}^*\in\mathsf{Q}^{n(\mathfrak{s}^*)}\mathfrak{s}^*$. Consequently, an analogous method as in Theorem 2.2 can be followed for the rest of the proof and induce that $\mathfrak{s}^*\in\mathsf{Q}\mathfrak{s}^*$. □

If we accept $n(\mathfrak{s})=1$ in Theorem 2.2 (also Theorem 2.3), then the following results can be obtained.

Corollary 2.4. *Let $(\mathcal{U},\mathbf{m})$ be an $\mathbf{m}-$complete MMS, and the property (Ω) holds. Let $\mathsf{Q}:\mathfrak{M}\to\mathcal{K}(\mathfrak{M})$ be multivalued self-mapping and $\alpha:\mathcal{U}_{\mathbf{m}}\to\mathbb{R}_+$ be a function. Presume that the following conditions hold:*

(*i*) *there exist $\Phi,\Psi:(0,\infty)\to\mathbb{R}$ such that for each $\mathfrak{s},\mathfrak{z}\in\mathfrak{M}$;*

$$\alpha(\mathfrak{s},\mathfrak{z})\,\Phi\left(\mathcal{H}_{\mathbf{m}}(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z})\right)\leq\Psi\left(\mathcal{R}(\mathfrak{s},\mathfrak{z})\right),$$

where Φ,Ψ are two functions satisfying the conditions $(c_1)-(c_3)$ and

$$\mathcal{R}(\mathfrak{s},\mathfrak{z})=\max\left\{\mathbf{m}_1(\mathfrak{s},\mathfrak{z}),\frac{\delta_2(\mathfrak{s},\mathsf{Q}\mathfrak{z})+\delta_2(\mathfrak{z},\mathsf{Q}\mathfrak{s})}{2},\frac{\delta_1(\mathfrak{z},\mathsf{Q}\mathfrak{z})\delta_2(\mathfrak{s},\mathsf{Q}\mathfrak{z})+\delta_1(\mathfrak{z},\mathsf{Q}\mathfrak{z})\delta_2(\mathfrak{z},\mathsf{Q}\mathfrak{s})}{\delta_2(\mathfrak{s},\mathsf{Q}\mathfrak{z})+\delta_1(\mathfrak{z},\mathsf{Q}\mathfrak{s})+1}\right\}$$

for all $\mathcal{H}_{\mathbf{m}}(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z})>0$.

(*ii*) *there is a point $\mathfrak{s}_0\in\mathfrak{M}$ which has the property $\alpha^*(\mathfrak{s}_0,\mathsf{Q}\mathfrak{s}_0)\geq1$,*

(*iii*) *Q is a triangular α^*-orbital admissible mapping,*

(*iv*) *Q is an $\mathbf{m}-$continuous mapping,*

(*v*) *there exist $\mathfrak{s},\mathfrak{z}\in M_{Fix}(\mathsf{Q})$, $M_{Fix}(\mathsf{Q})$ indicates the set of fixed points of Q such that $\alpha(\mathfrak{s},\mathfrak{z})\geq1$.*

Thereby, Q owns a fixed point $\mathfrak{s}^*$ in $\mathfrak{M}\subseteq\mathcal{U}_{\mathbf{m}}$, where $\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1)<\infty$ for some $\mathfrak{s}_0,\mathfrak{s}_1\in\mathcal{U}_{\mathbf{m}}$.

The following one can be given for a single-valued mapping considering Theorem 2.2 (also Theorem 2.3).

Corollary 2.5. *Let $\mathcal{U}^*_{\mathbf{m}}$ be an $\mathbf{m}-$complete MMS, $\mathsf{Q}:\mathcal{U}^*_{\mathbf{m}}\to\mathcal{U}^*_{\mathbf{m}}$ be a self-mapping and $\alpha:\mathcal{U}_{\mathbf{m}}\to\mathbb{R}_+$ be a function. Presume that*

(i) *there exist* $\Phi, \Psi : (0,\infty) \to \mathbb{R}$ *and a positive integer* $n(\mathfrak{s})$ *such that:*

$$\alpha(\mathfrak{s},\mathfrak{z})\,\Phi\left(\mathbf{m}_1\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)\right) \le \Psi(\mathcal{R}(\mathfrak{s},\mathfrak{z})),$$

where Φ, Ψ *are two functions satisfying the statements* $(c_1)-(c_3)$ *and*

$$\mathcal{R}(\mathfrak{s},\mathfrak{z}) = \max\left\{ \mathbf{m}_1(\mathfrak{s},\mathfrak{z}), \frac{\mathbf{m}_2(\mathfrak{s},\mathsf{Q}\mathfrak{z})+\mathbf{m}_2(\mathfrak{z},\mathsf{Q}\mathfrak{s})}{2}, \frac{\mathbf{m}_1(\mathfrak{z},\mathsf{Q}\mathfrak{z})\mathbf{m}_2(\mathfrak{s},\mathsf{Q}\mathfrak{z})+\mathbf{m}_1(\mathfrak{z},\mathsf{Q}\mathfrak{z})\mathbf{m}_2(\mathfrak{z},\mathsf{Q}\mathfrak{s})}{\mathbf{m}_2(\mathfrak{s},\mathsf{Q}\mathfrak{z})+\mathbf{m}_1(\mathfrak{z},\mathsf{Q}\mathfrak{s})+1} \right\} \tag{19}$$

for all $\mathfrak{s},\mathfrak{z} \in \mathcal{U}^*_{\mathbf{m}}$ *and* $\mathbf{m}_1(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z}) > 0$,

(ii) *there is a point* $\mathfrak{s}_0 \in \mathcal{U}^*_{\mathbf{m}}$ *such that* $\alpha(\mathfrak{s}_0,\mathsf{Q}\mathfrak{s}_0) \ge 1$,

(iii) Q *is a triangular* $\alpha-$*orbital admissible mapping,*

(iv) Q *is an* $\mathbf{m}-$*continuous mapping,*

(v) *there exist* $\mathfrak{s},\mathfrak{z} \in Fix(\mathsf{Q})$, *Fix*($\mathsf{Q}$) *indicates the set of fixed points of* Q *such that* $\alpha(\mathfrak{s},\mathfrak{z}) \ge 1$.

Thereby, Q *owns a fixed point* $\mathfrak{s}^*$ *in* $\mathcal{U}^*_{\mathbf{m}}$, *where* $\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1) < \infty$ *for some* $\mathfrak{s}_0,\mathfrak{s}_1 \in \mathcal{U}^*_{\mathbf{m}}$.

Corollary 2.6. *Let* $\mathcal{U}^*_{\mathbf{m}}$ *be an* $\mathbf{m}-$*complete MMS and* $\mathsf{Q} : \mathcal{U}^*_{\mathbf{m}} \to \mathcal{U}^*_{\mathbf{m}}$ *be a self-mapping.* $\Phi, \Psi : (0,\infty) \to \mathbb{R}$ *and a positive integer exist* $n(\mathfrak{s})$ *such that:*

$$\Phi\left(\mathbf{m}_1\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s},\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}\right)\right) \le \Psi(\mathcal{R}(\mathfrak{s},\mathfrak{z})),$$

where $\mathcal{R}(\mathfrak{s},\mathfrak{z})$ *is defined by* (19) *for all* $\mathbf{m}_1(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z}) > 0$ *and for each* $\mathfrak{s},\mathfrak{z} \in \mathcal{U}^*_{\mathbf{m}}$.

Thereby, Q *owns a fixed point* $\mathfrak{s}^*$ *in* $\mathcal{U}^*_{\mathbf{m}}$; *where* $\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1) < \infty$ *for some* $\mathfrak{s}_0,\mathfrak{s}_1 \in \mathcal{U}^*_{\mathbf{m}}$.

Proof. The proof is obtained by taking $\alpha(\mathfrak{s},\mathfrak{z}) = 1$ in Corollary 2.5, . □

Corollary 2.7. *Let* $\mathcal{U}^*_{\mathbf{m}}$ *be an* $\mathbf{m}-$*complete MMS and* $\mathsf{Q} : \mathcal{U}^*_{\mathbf{m}} \to \mathcal{U}^*_{\mathbf{m}}$ *be a self-mapping. There exists* $\Phi : (0,\infty) \to \mathbb{R}$, *which is left-continuous and non-decreasing such that*

$$\Phi(\mathbf{m}_1(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z})) \le \kappa\Phi(\mathcal{R}(\mathfrak{s},\mathfrak{z})),$$

where $\mathcal{R}(\mathfrak{s},\mathfrak{z})$ *is defined by* (19) *and* $\kappa \in [0,1)$ *for all* $\mathbf{m}_1(\mathsf{Q}\mathfrak{s},\mathsf{Q}\mathfrak{z}) > 0$ *and for each* $\mathfrak{s},\mathfrak{z} \in \mathcal{U}^*_{\mathbf{m}}$.

Thereby, Q *owns a fixed point* $\mathfrak{s}^*$ *in* $\mathcal{U}^*_{\mathbf{m}}$, *where* $\mathbf{m}_1(\mathfrak{s}_0,\mathfrak{s}_1) < \infty$ *for some* $\mathfrak{s}_0,\mathfrak{s}_1 \in \mathcal{U}^*_{\mathbf{m}}$.

Proof. The proof is achieved if $\Psi(s) = \kappa\Phi(s)$ in Corollary 2.6. □

3. An Application to Nonlinear Integral Equation

We demonstrate an existence theorem for solving the following nonlinear integral equation.

$$\mathfrak{s}(\iota) = \int_{a_1}^{a_2} \mathcal{Y}(\iota,\varrho,\mathfrak{s}(\varrho))\,d\varrho, \tag{20}$$

where $a_1, a_2 \in \mathbb{R}$ by $a_1 < a_2$, $\mathfrak{s} \in C[a_1,a_2]$ (the set of all continuous functions from $[a_1,a_2]$ into $\mathbb{R}$) and $\mathcal{Y} : [a_1,a_2] \times [a_1,a_2] \times \mathbb{R} \to \mathbb{R}$ is a specified mapping. We endow $\mathcal{U}^*_{\mathbf{m}} = C[a_1,a_2]$ with the following function

$$\mathbf{m}_1(\mathbf{f},\mathbf{g}) = \left|\mathbf{f}(\iota) - \mathbf{g}(\iota)\right| e^{-\flat}, \quad \flat > 0,$$

for all $\mathbf{f},\mathbf{g} \in \mathcal{U}^*_{\mathbf{m}}$. Evidently, $(\mathcal{U}^*_{\mathbf{m}},\mathbf{m})$ is an $\mathbf{m}-$complete MMS.

Furthermore, let $\mathsf{Q} : \mathcal{U}^*_{\mathbf{m}} \to \mathcal{M}^*_{\mathbf{m}}$ be defined by

$$\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}(\iota) = \int_{a_1}^{a_2} \mathcal{Y}(\iota,\varrho,\mathfrak{s}(\varrho))\,d\varrho$$

for all $\mathfrak{s} \in \mathcal{U}^*_{\mathbf{m}}$ and $\iota \in [a_1,a_2]$. Accordingly, the existence of a solution to (20) is equivalent to the existence of a fixed point of Q.

Theorem 3.1. *Contemplate the nonlinear integral equation (20). Presume that the following statements are met:*

i. $\mathcal{Y} : [a_1, a_2] \times [a_1, a_2] \times \mathbb{R} \to \mathbb{R}$ *is continuous and non-decreasing in the third variable,*

ii. there exists $p > 1$ *satisfying the following condition: for each* $\iota, \varrho \in [a_1, a_2]$ *and* $\mathfrak{s}, \mathfrak{z} \in \mathcal{U}^*_{\mathbf{m}}$ *with* $\mathfrak{s}(r) \leq \mathfrak{z}(r)$ *for all* $r \in [a_1, a_2]$*, we acquire*

$$\left|\mathcal{Y}(\iota, \varrho, \mathfrak{s}(\varrho)) - \mathcal{Y}(\iota, \varrho, \mathfrak{z}(\varrho))\right| \leq \sigma(\iota, \varrho)\mathcal{R}(\mathfrak{s}, \mathfrak{z}), \tag{21}$$

where $\sigma : [a_1, a_2] \times [a_1, a_2] \to [0, \infty)$ *is a continuous function identified by*

$$\sup_{\iota \in [a_1, a_2]} \left(\int_{a_1}^{a_2} \sigma(\iota, \varrho) d\varrho \right) \leq \frac{1}{e^{\mathfrak{b}}}, \tag{22}$$

and

$$\mathcal{R}(\mathfrak{s}, \mathfrak{z}) = \max\left\{ \mathbf{m}_1(\mathfrak{s}, \mathfrak{z}), \frac{\mathbf{m}_2(\mathfrak{s}, \mathsf{Q}\mathfrak{z}) + \mathbf{m}_2(\mathfrak{z}, \mathsf{Q}\mathfrak{s})}{2}, \frac{\mathbf{m}_1(\mathfrak{z}, \mathsf{Q}\mathfrak{z})\mathbf{m}_2(\mathfrak{s}, \mathsf{Q}\mathfrak{z}) + \mathbf{m}_1(\mathfrak{z}, \mathsf{Q}\mathfrak{z})\mathbf{m}_2(\mathfrak{z}, \mathsf{Q}\mathfrak{s})}{\mathbf{m}_2(\mathfrak{s}, \mathsf{Q}\mathfrak{z}) + \mathbf{m}_1(\mathfrak{z}, \mathsf{Q}\mathfrak{s}) + 1} \right\}.$$

Then, the nonlinear integral equation (20) owns a solution.

Proof. From (21) and (22), for all $\iota \in [a_1, a_2]$, we attain

$$\begin{aligned}
\Phi\left(\mathbf{m}_1\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}(\iota), \mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}(\iota)\right)\right) &= e^{2\mathfrak{b}}\mathbf{m}_1\left(\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}(\iota), \mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}(\iota)\right) = e^{2\mathfrak{b}} \frac{\left|\mathsf{Q}^{n(\mathfrak{s})}\mathfrak{s}(\iota) - \mathsf{Q}^{n(\mathfrak{s})}\mathfrak{z}(\iota)\right|}{e^{\mathfrak{b}}} \\
&= e^{\mathfrak{b}}\left| \int_{a_1}^{a_2} \mathcal{Y}(\iota, \varrho, \mathfrak{s}(\varrho)) d\varrho - \int_{a_1}^{a_2} \mathcal{Y}(\iota, \varrho, \mathfrak{z}(\varrho)) d\varrho \right| \\
&\leq e^{\mathfrak{b}}\left| \int_{a_1}^{a_2} [\mathcal{Y}(\iota, \varrho, \mathfrak{s}(\varrho)) - \mathcal{Y}(\iota, \varrho, \mathfrak{z}(\varrho))] d\varrho \right| \\
&\leq e^{\mathfrak{b}}\left(\int_{a_1}^{a_2} \left|\mathcal{Y}(\iota, \varrho, \mathfrak{s}(\varrho)) - \mathcal{Y}(\iota, \varrho, \mathfrak{z}(\varrho))\right| d\varrho \right) \\
&\leq e^{\mathfrak{b}}\left(\int_{a_1}^{a_2} \sigma(\iota, \varrho)\mathcal{R}(\mathfrak{s}, \mathfrak{z}) d\varrho \right) \\
&\leq e^{\mathfrak{b}}\left(\int_{a_1}^{a_2} \sigma(\iota, \varrho)^p d\varrho \right)\mathcal{R}(\mathfrak{s}, \mathfrak{z}) \\
&\leq \Psi(\mathcal{R}(\mathfrak{s}, \mathfrak{z})).
\end{aligned}$$

Considering $\Phi(\mathbf{f}) = e^{2\mathfrak{b}}\mathbf{f}$ and $\Psi(\mathbf{f}) = \mathbf{f}$ for all $\mathbf{f} > 0$, we deduce that all the conditions of Corollary 2.6 are held. Consequently, a unique $\mathfrak{s} \in \mathcal{U}^*_{\mathbf{m}}$ exists, such that $\mathfrak{s} \in Fix(\mathsf{Q})$ indicates that $\mathfrak{s}$ is the unique solution for the integral equation (20). □

4. Conclusion

In conclusion, in the modular metric space setting, we bring forward the outcomes of Proinov [27] and Sehgal [33] by combining these notions through both multivalued mappings and α−admissible functions. Incidentally, we demonstrate that the outcomes which we have achieved can be applied to a nonlinear integral equation.

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