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On the stability of a degenerate vibrating system under fractional derivative controls

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Abstract. In this paper, we consider a coupled system of degenerate wave equations with the fractional feedback acting at the boundary. First, we reformulate each system into an augmented model and using a general criteria of Arendt-Batty, we prove that our models are strongly stable. Next, by using a spectrum method, we establish nonuniform stabilization.

1. Introduction

In this work, we consider a system of coupled wave equations with only one fractional dissipation law. This system defined on $(0,1)\times(0,+\infty)$ takes the following form

$$\begin{cases} u_{tt}(x,t)-(a(x)u_x(x,t))_x+\alpha(u-v)=0 \quad \text{in } (0,1)\times(0,+\infty),\\ v_{tt}(x,t)-(a(x)v_x(x,t))_x+\alpha(v-u)=0 \quad \text{in } (0,1)\times(0,+\infty),\\ (u(x,0),v(x,0),u_t(x,0),v_t(x,0))=(u_0(x),v_0(x),u_1(x),v_1(x)), \end{cases} \tag{1}$$

where $a(x)=\sqrt{1-x^2}$ and α is a strictly positive constants, and the followed boundary conditions

$$\begin{cases} u(0,t)=v(0,t)=0\\ (\sqrt{1-x^2}u_x)(1,t)+\varrho\partial_t^{\tau,\omega}u(1,t)=0, \quad \forall t\in(0,+\infty),\\ (\sqrt{1-x^2}v_x)(1,t)+\tilde{\varrho}\partial_t^{\tau,\omega}v(1,t)=0, \quad \forall t\in(0,+\infty), \end{cases} \tag{2}$$

where $\varrho>0$ and $\tilde{\varrho}>0$, the initial data (u_0,u_1,v_0,v_1) belong to a suitable function space. The notation $\partial_t^{\tau,\omega}$ stands for the generalized Caputo's fractional derivative of order τ, $0<\tau<1$, with respect to the time variable t defined by

$$\partial_t^{\tau,\omega}f=\frac{1}{\Gamma(1-\tau)}\int_0^t(t-s)^{-\tau}e^{-\omega(t-s)}\frac{df}{dt}ds,\ \omega\geq 0, \tag{3}$$

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where Γ denotes the Gamma function. The fractional derivatives are nonlocal and involve singular and non-integrable kernels ($t^{-\tau}, 0 < \tau < 1$). We refer the readers to [21], [22].

Here, the coefficient $a(x) = \sqrt{1-x^2}$ vanishes at the boundary and the problem is weakly degenerate, in the sense that $\frac{1}{a} \in L^1(-1,1)$.

Physically, u and v may represent the displacements of two vibrating objects measured from their equilibrium positions, the coupling terms $\pm\alpha(u-v)$ are the distributed springs linking the two vibrating objects. Indeed, a mathematical model that describes transverse vibration of an elastic string is given by

$$u_{tt}(x,t) - (\frac{T(x)}{\rho(x)}u_x(x,t))_x \pm \text{lower termes} = 0,$$

where T is the tension of a string and ρ is the density of the string.

The exponential stability of the system (1) has been established for $a \equiv 1$ by Najafi et al [19] in the case of linear boundary feedback and by Komornik-Rao [24] in the case of nonlinear boundary feedback.

In [4], Mbodje investigate the asymptotic behavior of solutions with the system

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t)) = 0, & \text{in } (0,1)\times(0,+\infty), \\ u(0,t) = 0, & \text{on } (0,+\infty), \\ u_x(1,t) + \gamma\partial_t^{\alpha,\eta}u(1,t) = 0, & \text{on } (0,+\infty), \eta > 0, \\ u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), & \text{on } (0,1). \end{cases} \tag{4}$$

He proved that the associated semigroup is not exponentially stable, but only strongly asymptotically and the solution of this system will decay, as times goes to infinity, as $1/t$.

This work is a generalization of the work in [1], where the system is described by

$$\begin{cases} u_{tt}(x,t) - u_{xx}(x,t) + \alpha(u-v) = 0 & \text{in } (0,L)\times(0,+\infty), \\ v_{tt}(x,t) - v_{xx}(x,t) + \alpha(v-u) = 0 & \text{in } (0,1)\times(0,+\infty), \\ u(0,t) = v(0,t) = 0 & \\ u_x(1,t) + \varrho\partial_t^{\tau,\omega}u(1,t) = 0 & \text{on } (0,+\infty), \\ u_x(1,t) + \tilde{\varrho}\partial_t^{\tau,\omega}u(1,t) = 0 & \text{on } (0,+\infty), \\ \begin{cases} u(x,0) = u_0(x),\ u_t(x,0) = u_1(x), \\ v(x,0) = v_0(x),\ v_t(x,0) = v_1(x). \end{cases} & \end{cases} \tag{5}$$

The authors considered that coupled wave equations with a two boundary nonlocal control and showed that the energy of the system decays polynomially of type $t^{-2/(1-\tau)}$.

We underline that this is the first paper to consider the stabilization of the system (1)-(2) that couples a degenerate variable coefficient $a(x) = \sqrt{1-x^2}$ in the principal part with a fractional damping acting at a degenerate boundary.

This paper is organized as follows. In section 2, we introduce our notations, functional space, we show that the system (1)-(2) can be replaced by an augmented model by coupling the wave equation with a suitable diffusion equation that can be reformulated into an evolution equation, we deduce the well-posedness property of the problem by the semigroup approach, and using the criteria of Arendt-Batty, we show that the augmented model is strongly stable. In section 3, we show the lack of exponential stability by spectral analysis.

2. Well-posedness and strong stability

This section is concerned with the reformulation of the model (1)-(2) into an augmented system. For that, we need the following claims.

Proposition 2.1. *(see [4]) Let μ be the function:*

$$\mu(\xi) = |\xi|^{(2\tau-1)/2}, \ -\infty < \xi < +\infty, \ 0 < \tau < 1 \tag{6}$$

Then the relationship between the 'input' U and the 'output' O of the system

$$\partial_t \phi(\xi, t) + (\xi^2 + \omega)\phi(\xi, t) - U(t)\mu(\xi) = 0, \ -\infty < \xi < +\infty, \ \omega \geq 0, t \geq 0, \tag{7}$$

$$\phi(\xi, t) = 0 \tag{8}$$

$$O(t) = (\pi)^{-1} \sin(\tau\pi) \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi, t) d\xi \tag{9}$$

is given by

$$O = I^{1-\tau,\omega} U \tag{10}$$

where

$$[I^{\tau,\omega} f](t) = \frac{1}{\Gamma(\tau)} \int_0^t (t-s)^{\tau-1} e^{-\omega(t-s)} f(s) ds. \tag{11}$$

Lemma 2.2. *(see [5]) If $\lambda \in D_\omega = \mathbb{C} \setminus]-\infty, -\omega]$ then*

$$F_n(\lambda) = \int_{-\infty}^{+\infty} \frac{\mu^2(\xi)}{(\lambda + \omega + \xi^2)^n} d\xi = \frac{(1-\tau)(2-\tau)...(n-1-\tau)}{(n-1)!} \frac{\pi}{\sin(\tau\pi)} (\lambda + \omega)^{\tau - n}.$$

2.1. Well-Posedness

We are now in a position to reformulate system(1)–(2). Indeed, by using Propostion (2.1), system (1)–(2) becomes

$$\begin{cases} u_{tt}(x,t) - (\sqrt{1-x^2} u_x)_x(x,t) + \alpha(u - v) = 0 & \text{in } (0,1) \times (0,+\infty), \\ v_{tt}(x,t) - (\sqrt{1-x^2} v_x)_x(x,t) + \alpha(v - u) = 0 & \text{in } (0,1) \times (0,+\infty), \\ \partial_t \phi(\xi,t) + (\xi^2 + \omega)\phi(\xi,t) - \mu(\xi) u_t(1,t) = 0 & \text{in } (-\infty,+\infty) \times (0,+\infty), \\ \partial_t \tilde{\phi}(\xi,t) + (\xi^2 + \omega)\tilde{\phi}(\xi,t) - \mu(\xi) v_t(1,t) = 0 & \text{in } (-\infty,+\infty) \times (0,+\infty), \\ u(-1,t) = v(-1,t) = 0 & \\ (\sqrt{1-x^2} u_x)(1,t) = -\zeta \int_{-\infty}^{+\infty} \mu_(\xi)\phi(\xi,t) d\xi & \text{on } (0,+\infty), \\ (\sqrt{1-x^2} v_x)(1,t) = -\tilde{\zeta} \int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}(\xi,t) d\xi & \text{on } (0,+\infty), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x) & \text{on } (0,1), \\ v(x,0) = v_0(x), \ v_t(x,0) = v_1(x) & \text{on } (0,1), \\ \phi(\xi,0) = \tilde{\phi}(\xi,0) & \text{on } (-\infty,+\infty), \end{cases} \tag{12}$$

where $\zeta = (\pi)^{-1} \sin(\tau\pi)\gamma$, and $\tilde{\zeta} = (\pi)^{-1} \sin(\tau\pi)\tilde{\gamma}$. For a solution $(u, v, \phi, \tilde{\phi})$ of (12), we define the energy associated to the solution of the problem (P') by the following formula:

$$\begin{aligned} E(t) = \tfrac{1}{2} \int_0^1 (|u_t|^2 + |v_t|^2 + a(x)\,|u_x|^2 + a(x)\,|v_x|^2 + \alpha|u - v|^2) dx \\ + \tfrac{1}{2} \int_{-\infty}^{+\infty} (\zeta\,|\phi|^2 + \tilde{\zeta}\,|\tilde{\phi}|^2) d\xi, \end{aligned} \tag{13}$$

where $a(x) = \sqrt{1-x^2}$.

Lemma 2.3. *Let $(u, v, \phi, \tilde{\phi})$ be a regular solution of the problem (12). Then, the energy functional defined by* (13) *satisfies*

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \omega) \left|\phi(\xi, t)\right|^2 d\xi - \tilde{\zeta} \int_{-\infty}^{+\infty} (\xi^2 + \omega) \left|\tilde{\phi}(\xi, t)\right|^2 d\xi. \tag{14}$$

Proof. Multiplying the first equation in (12) by $\bar{u}_t$, integrating by parts over $(-1, 1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(|u_t|^2 + a(x)\,|u_x|^2 - \Re\,[(a(x)u_x(1,t)u(1,t))]_{-1}^{1} + \alpha\Re\int_0^1 (u - v)\bar{u}_t dx = 0. \tag{15}$$

Multiplying the second equation in(12) by $\bar{v}_t$, integrating by parts over $(-1, 1)$, we obtain

$$\frac{1}{2}\frac{d}{dt}\int_{0}^{1}(|v_t|^2 + a(x)\,|v_x|^2 - \Re\,[(a(x)v_x(1,t)v(1,t)]_{-1}^{1} + \alpha\Re\int_0^1 (u - v)\bar{v}_t dx = 0. \tag{16}$$

Adding the two equations above, we obtain

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(|u_t|^2 + |v_t|^2 + ax)\,|u_x|^2 + a(x)\,|v_x|^2 + \alpha\,|u - v|^2)dx \\
&- \Re(a(x)u_x(1,t)u(1,t)) - \Re(a(x)v_x(1,t)v(1,t)) \\
&= 0.
\end{aligned} \tag{17}$$

From boundary condition $(12)_6 - (12)_7$, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-1}^{1}(|u_t|^2 + |v_t|^2 + a(x)\,|u_x|^2 + a(x)\,|v_x|^2 + \alpha\,|u - v|^2)dx \\
&+ \zeta\Re\bar{u}_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\phi(\xi,t)d\xi + \zeta\Re\bar{v}_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\tilde{\phi}(\xi,t)d\xi \\
&= 0.
\end{aligned} \tag{18}$$

Multiplying the third and the fourth equations in (12) by $\zeta\bar{\phi}$, $\tilde{\zeta}\bar{\tilde{\phi}}$ and integrating over $(-\infty, +\infty)$, we obtain:

$$\frac{\zeta}{2}\frac{d}{dt}\left\|\phi\right\|_2^2 + \zeta\int_{-\infty}^{+\infty}(\xi^2 + \omega)\left|\phi(\xi,t)\right|^2 d\xi - \zeta\Re u_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)\bar{\phi}(\xi,t)d\xi = 0, \tag{19}$$

$$\frac{\tilde{\zeta}}{2}\frac{d}{dt}\left\|\tilde{\phi}\right\|_2^2 + \tilde{\zeta}\int_{-\infty}^{+\infty}(\xi^2 + \omega)\left|\tilde{\phi}(\xi,t)\right|^2 d\xi - \tilde{\zeta}\Re v_t(1,t)\int_{-\infty}^{+\infty}\mu(\xi)[\bar{\tilde{\phi}}(\xi,t)d\xi = 0. \tag{20}$$

Consequently, it is resulted from (13) , (18) and (19) , (20).

$$E'(t) = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \omega) \left|\phi(\xi, t)\right|^2 d\xi - \tilde{\zeta} \int_{-\infty}^{+\infty} (\xi^2 + \omega) \left|\tilde{\phi}(\xi, t)\right|^2 d\xi. \tag{21}$$

This completes the proof of the lemma. □

2.2. Global Existence

Now, we introduce, as in [25], the following weighted spaces:
For $a(x) = \sqrt{1-x^2}$, we define the Hilbet space $H^1_{1,a}(-1,1)$, as

$$H^1_{1,a}(-1,1) = \left\{u \in L^2(-1,1): \ \sqrt{a(x)}u_x \in L^2(0,1) \ /u(-1) = 0\right\},$$

$$H^1_a(-1,1) = \left\{u \in L^2(-1,1): \ \sqrt{a(x)}u_x \in L^2(0,1)\right\}.$$

We remark that $H^1_a(-1,1)$ is Hilbert space with the scalar product

$$(u,v) = \int_0^1 (u\overline{v} + a(x)u'(x)\overline{v'(x)})dx, \quad \forall u,v \in H^1_a(-1,1).$$

Proposition 2.4. *(see [25]). There is a positive constant $C = C(a)$ such that*

$$\|u\|^2_{L^2(-1,1)} \leq C\,\|u\|^2_{H^1_{1,a}(-1,1)} \quad \forall u \in H^1_a(-1,1).$$

Next, we define

$$H^2_a(-1,1) = \left\{u \in H^1_a(-1,1): \ \sqrt{1-x^2}u'(x) \in H^1(0,1)\right\}.$$

Notice that if $u \in H^2_a(-1,1)$, $1/a \notin L^2(-1,1)$, we have $(a(x)u_x)(\pm 1) \equiv 0$.

In order to study the system (12) we use a reduction order argument. First, we introduce the following Hilbert space (the energy space):

$$\mathcal{H} = H^1_{1,a}(-1,1) \times L^2(-1,1) \times H^1_{1,a}(-1,1) \times L^2(-1,1) \times L^2(\mathbb{R}) \times L^2(\mathbb{R}).$$

For $U = (u,\tilde{u},v,\tilde{v},\phi,\tilde{\phi})^T$, $U_1 = (u_1,\tilde{u}_1,v_1,\tilde{v}_1,\phi_1,\tilde{\phi}_1)^T$, we define the following inner product in $\mathcal{H}$

$$\begin{aligned}\langle U, U_1\rangle = &\int_{-1}^1 (\tilde{u}\overline{\tilde{u}_1} + a(x)u_x\bar{u}_{1x})dx + \int_{-1}^1 (\tilde{v}\overline{\tilde{v}_1} + a(x)v_x\bar{v}_{1x})dx \\ &+ \alpha\int_{-1}^1 (u-v)\overline{(u_1-v_1)}dx + \zeta\int_{-\infty}^{+\infty}\phi\bar{\phi}_1 d\xi + \tilde{\zeta}\int_{-\infty}^{+\infty}\tilde{\phi}\bar{\tilde{\phi}}d\xi.\end{aligned}$$

Let $(u,\tilde{u},v,\tilde{v},\phi,\tilde{\phi})^T$ and rewrite (12) is equivalent to

$$\begin{cases} U' = \mathcal{A}U, \\ U(0) = U_0 = (u_0,u_1,v_0,v_1,0,0), \end{cases} \tag{22}$$

where the operator $\mathcal{A}$ is defined by

$$\mathcal{A}\begin{pmatrix} u \\ \tilde{u} \\ v \\ \tilde{v} \\ \phi \\ \tilde{\phi} \end{pmatrix} = \begin{pmatrix} \tilde{u} \\ (a(x)u_x)_x - \alpha(u-v) \\ \tilde{v} \\ (a(x)v_x)_x - \alpha(v-u) \\ -(\xi^2+\omega)\phi + \mu(\xi)\tilde{u}(1) \\ -(\xi^2+\omega)\tilde{\phi} + \mu(\xi)\tilde{v}(1) \end{pmatrix}. \tag{23}$$

The domain of $\mathcal{A}$ is

$$D(\mathcal{A}) = \left\{ \begin{array}{l} (u, \tilde{u}, v, \tilde{v}, \phi, \tilde{\phi})^T \text{ in } \mathcal{H} : u \in H_a^2(-1,1) \cap H_{1,a}^1(-1,1), \\ \tilde{u} \in H_{1,a}^1(-1,1), v \in H_a^2(-1,1) \cap H_{1,a}^1(-1,1), \tilde{v} \in H_{1,a}^1(-1,1), \\ (a(x)u_x)(1) + \zeta \int_{-\infty}^{+\infty} \mu(\xi)\phi(\xi,)d\xi = 0, \\ (a(x)v_x)(1) + \tilde{\zeta} \int_{-\infty}^{+\infty} \mu(\xi)\tilde{\phi}(\xi,)d\xi = 0, \\ |\xi|\,\phi, |\xi|\,\tilde{\phi} \in L^2(\mathbb{R}). \end{array} \right\} \tag{24}$$

The well-posedness of problem (12) is ensured by the following theorem.

Theorem 2.5. ***(Existence and uniquenesss).***

(1) If $U_0 \in D(\mathcal{A})$, then system (22)*has a unique trong solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}) \cap C^1(\mathbb{R}_+, \mathcal{H})$$

(2) If $U \in \mathcal{H}$, then system (22) *has a unique weak solution*

$$U \in C^0(\mathbb{R}_+, \mathcal{H}).$$

Proof. We use the semigroup approach. In what follows, we prove that $\mathcal{A}$ is monotone. For any $U_0 \in D(\mathcal{A})$, and using (21) and the fact that

$$\Re \langle \mathcal{A}U, U \rangle = -\zeta \int_{-\infty}^{+\infty} (\xi^2 + \omega) \left|\phi(\xi)\right|^2 d\xi - \tilde{\zeta} \int_{-\infty}^{+\infty} (\xi^2 + \omega) \left|\tilde{\phi}(\xi)\right|^2 d\xi. \tag{25}$$

Hence, A is monotone. Next, we prove that the operator $\lambda I - \mathcal{A}$ is surjective for $\lambda > 0$. Given $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$, we prove that there exists $U = (u, \tilde{u}, v, \tilde{v}, \phi, \tilde{\phi}) \in D(\mathcal{A})$ satisfying

$$\lambda U - \mathcal{A}U = F \tag{26}$$

$$\left\{ \begin{array}{l} \lambda u - \tilde{u} = f_1, \\ \lambda \tilde{u} - (\sqrt{1-x^2}u_x)_x + \alpha(u - v) = f_2, \\ \lambda v - \tilde{v} = f_3, \\ \lambda \tilde{v} - (\sqrt{1-x^2}v_x)_x + \alpha(u - v) = f_4, \\ \lambda \phi + (\xi^2 + \omega)\phi - \mu(\xi)\tilde{u}(1) = f_5, \\ \lambda \tilde{\phi} + (\xi^2 + \omega)\tilde{\phi} - \mu(\xi)\tilde{v}(1) = f_6. \end{array} \right. \tag{27}$$

Suppose u and v is found with the appropriate regularity. Then, $(27)_1$ and $(27)_3$ yield

$$\left\{ \begin{array}{l} \lambda u - f_1 = \tilde{u} \in H_{1,a}^1(-1,1) \\ \lambda v - f_3 = \tilde{v} \in H_{1,a}^1(-1,1). \end{array} \right. \tag{28}$$

Furthemore, by $(27)_5$and $(27)_6$, we can find ϕ and $\tilde{\phi}$ as

$$\begin{aligned} \phi &= \frac{f_5(\xi) + \mu(\xi)\tilde{u}(1)}{\xi^2 + \omega + \lambda}, \\ \tilde{\phi} &= \frac{f_6(\xi) + \mu(\xi)\tilde{v}(1)}{\xi^2 + \omega + \lambda}. \end{aligned} \tag{29}$$

By using(27)and (28) it can easily be shown that u and v satisfies

$$\begin{cases} \lambda^2 u - (\sqrt{1-x^2}u_x)_x + \alpha(u-v) = f_2 + \lambda f_1 \\ \lambda^2 v - (\sqrt{1-x^2}v_x)_x + \alpha(v-u) = f_4 + \lambda f_3. \end{cases} \tag{30}$$

Solving system (30) is equivalent to finding $u \in H_a^2(-1,1) \cap H_{1,a}^1(-1,1)$ and $v \in H_a^2(-1,1) \cap H_{1,a}^1(-1,1)$ such that

$$\int_{-1}^{1} (\lambda^2 u\overline{w_1} - ((a(x)u_x)_x\overline{w_1}) + \alpha(u-v)\overline{w_1})dx = \int_{-1}^{1} (f_2 + \lambda f_1)\overline{w_1}dx, \tag{31}$$
$$\int_{-1}^{1} (\lambda^2 v\overline{w_2} - ((a(x)v_x)_x\overline{w_2}) + \alpha(u-v)\overline{w_2})dx = \int_{-1}^{1} (f_4 + \lambda f_3)\overline{w_2}dx,$$

for all $w_1, w_2 \in H_{1,a}^1(-1,1)$. By using (31), the boundary conditions (24) , (28) and (29) , the functions u and v satisfy the following system

$$\begin{cases} \lambda^2 \int_{-1}^{1} (u\overline{w_1} + v\overline{w_2})dx + \int_{-1}^{1} (a(x)u_x)_x\overline{w_1} + (a(x)v_x)_x\overline{w_2})dx + \lambda(\lambda+\omega)^{\tau-1}(\gamma u(1)\overline{w_1}(1) \\ +\tilde{\gamma}v(1)\overline{w_2}(1)) + \alpha\int_{-1}^{1} (u-v)(\overline{w_1} - \overline{w_2})dx = \int_{-1}^{1} ((f_2+\lambda f_1)\overline{w_1} + (f_4+\lambda f_3)\overline{w_2})dx \\ -\zeta\overline{w_1}(1)\int_{-\infty}^{+\infty} \frac{f_5(\xi)\mu(\xi)}{\xi^2+\omega+\lambda}d\xi - \tilde{\zeta}\overline{w_2}(1)\int_{-\infty}^{+\infty} \frac{f_6(\xi)\mu(\xi)}{\xi^2+\omega+\lambda}d\xi + (\lambda+\omega)^{\tau-1}(\gamma f_1(1)\overline{w_1}(1) \\ +\tilde{\gamma}f_3(1)\overline{w_2}(1)), \end{cases} \tag{32}$$

where we have used the fact that $\int_{-\infty}^{+\infty} \mu^2(\xi)/(\lambda+\omega+\xi^2)d\xi = \pi/\sin(\tau\pi)(\lambda+\omega)^{\tau-1}$. Consequently, problem (32) is of the form

$$\mathcal{B}((u,v),(w_1,w_{2)}) = \mathcal{L}(w_1,w_2), \tag{33}$$

where the sesquilinear form $\mathcal{B} : \left[H_{1,a}^1(-1,1) \times H_{1,a}^1(-1,1)\right]^2 \to \mathbb{C}$, and the antilinear form $\mathcal{L} : H_{1,a}^1(-1,1) \times H_{1,a}^1(-1,1) \to \mathbb{C}$ are defined by

$$\begin{aligned} \mathcal{B}((u,v),(w_1,w_{2)}) &= \lambda^2 \int_0^1 (u\overline{w_1} + v\overline{w_2})dx + \int_0^1 (a(x)u_x)_x\overline{w_1} + (a(x)v_x)_x\overline{w_2})dx \\ &\quad + \lambda(\lambda+\omega)^{\tau-1}(\gamma u(1)\overline{w_1}(1) + \tilde{\gamma}v(1)\overline{w_2}(1)) \\ &\quad + \alpha\int_0^1 (u-v)(\overline{w_1} - \overline{w_2})dx, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}(w_1,w_2) &= \int_{-1}^{1} ((f_2+\lambda f_1)\overline{w_1} + (f_4+\lambda f_3)\overline{w_2})dx - \zeta\overline{w_1}(1)\int_{-\infty}^{+\infty} \frac{f_5(\xi)\mu(\xi)}{\xi^2+\omega+\lambda}d\xi \\ &\quad - \tilde{\zeta}\overline{w_2}(1)\int_{-\infty}^{+\infty} \frac{f_6(\xi)\mu(\xi)}{\xi^2+\omega+\lambda}d\xi + (\lambda+\omega)^{\tau-1}(\gamma f_1(1)\overline{w_1}(1) + \tilde{\gamma}f_3(1)\overline{w_2}(1)). \end{aligned}$$

It is easy to verify that $\mathcal{B}$ is continuous and coercive, and $\mathcal{L}$ is continuous. Applying the Lax-Milgram Theorem, we infer that for all $(w_1,w_2) \in H_{1,a}^1(-1,1) \times H_{1,a}^1(-1,1)$ problem (33) has a unique solution $H_a^2(-1,1) \times H_a^2(-1,1)$. Applying the classical elliptic regularity arguments (see [9]), it follows from (32) that $(u,v) \in H_a^2(-1,1) \times H_a^2(-1,1)$.Therefore, the operator $\lambda I - \mathcal{A}$ is surjective for any $\lambda > 0$. At last, the result of Theorem 2.5 follows from the Hille-Yosida theorem.

□

2.3. Strong stability of the system

In this part, we use a general criteria of Arendt-Batty and Lyubich-Vu.

Theorem 2.6. *([10]) Let $\mathcal{A}$ be the generator of uniformly bounded C_0-semigroup of contractions $\left\{e^{t\mathcal{A}}\right\}_{t\geq 0}$ in Hilbert space X. If*

(i) $\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.

(ii) The intersection of spectrum $\sigma(\mathcal{A})$ with $i\mathbb{R}$ is at most a countable set, then the semigroup $\left\{e^{t\mathcal{A}}\right\}_{t\geq 0}$ is asymptotically stable, i.e, $\left\|e^{t\mathcal{A}}z\right\|_X \to 0$ as $t \to +\infty$, for any $z \in X$.

Theorem 2.7. *The C_0-semigroup of contractions $e^{t\mathcal{A}}$ is strongly stable in $\mathcal{H}$; i.e, for all $U_0 \in \mathcal{H}$, the solution of* (22) *satisies*

$$\lim_{t\to\infty}\left\|e^{t\mathcal{A}}U_0\right\| = 0.$$

For the proof of Theorem(2.7), we need the following two lemmas.

Lemma 2.8. *$\mathcal{A}$ does not have eigenvalues on $i\mathbb{R}$.*

Proof. We make a distinction between $i\lambda = 0$, and $i\lambda \neq 0$.

Step 1. Solving for $\mathcal{A}U = 0$ lesds to the following system

$$\begin{cases} \tilde{u} = 0 \\ (\sqrt{1-x^2}u_x)_x - \alpha(u-v) = 0 \\ \tilde{v} = 0 \\ (\sqrt{1-x^2}v_x)_x - \alpha(v-u) = 0 \\ -(\xi^2+\omega)\phi + \mu(\xi)\tilde{u}(1) = 0 \\ -(\xi^2+\omega)\tilde{\phi} + \mu(\xi)\tilde{v}(1) = 0 \end{cases} \tag{34}$$

Then, from (24), $(34)_1$ and $(34)_3$ we have

$$\phi = \tilde{\phi} = 0, \ \tilde{u} = \tilde{v} = 0. \tag{35}$$

Than, from (35) and (24)

$$(\sqrt{1-x^2}v_x)(1) = (\sqrt{1-x^2}u_x)(1) = 0 \tag{36}$$

Multiplyng equation $(34)_2$ by $\overline{u}$ and $(34)_4$ by $\overline{v}$, using Green formila, we get

$$\begin{aligned} &\int_{-1}^{1}\sqrt{1-x^2}\,|u_x|^2\,dx + \alpha\int_{-1}^{1}(u-v)dx = 0 \\ &\int_{-1}^{1}\sqrt{1-x^2}\,|v_x|^2\,dx + \alpha\int_{-1}^{1}(v-u)dx = 0. \end{aligned} \tag{37}$$

we have

$$\int_{-1}^{1}\sqrt{1-x^2}\left(|u_x|^2+|v_x|^2\right)dx = 0. \tag{38}$$

Than

$$\left(\sqrt{1-x^2}\,|u_x|^2\right)(x) = \left(\sqrt{1-x^2}\,|v_x|^2\right)(x) = 0 \ \ \forall x \in (-1,1). \tag{39}$$

From boundary conditions (36), we obtain

$$(\sqrt{1-x^2}u_x) = (\sqrt{1-x^2}v_x) = 0. \tag{40}$$

Moreover, from (40), we have

$$u_x(x) = 0,\ v_x(x) = 0 \quad \text{on } (-1,1). \tag{41}$$

Hence u and v are constants in $(-1,1)$. As $u(-1) = v(-1) = 0$, then $u = v = 0$, and consequently, we obtain $U \equiv 0$. Hence, $i\lambda = 0$ is not an eigenvalue of $\mathcal{A}$.

Step 2. We will argue by contradiction. let us suppose that there $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and $\|U\| \neq 0$,such that $\mathcal{A}U = i\lambda U$. Then, we get

$$\begin{cases} i\lambda u - \tilde{u} = 0 \\ i\lambda\tilde{u} - (\sqrt{1-x^2}u_x)_x + \alpha(u-v) = 0 \\ i\lambda v - \tilde{v} = 0 \\ i\lambda\tilde{v} - (\sqrt{1-x^2}v_x)_x + \alpha(u-v) = 0 \\ i\lambda\phi + (\xi^2+\omega)\phi - \mu(\xi)\tilde{u}(1) = 0 \\ i\lambda\tilde{\phi} + (\xi^2+\omega)\tilde{\phi} - \mu(\xi)\tilde{v}(1) = 0. \end{cases} \tag{42}$$

Then, from boundary conditions (24). Using (14) and (42), we find

$$\phi = 0 \text{ and } \tilde{\phi} = 0. \tag{43}$$

From $(42)_5$ and $(42)_6$, we have

$$\tilde{u}(1) = 0 \text{ and } \tilde{v}(1) = 0. \tag{44}$$

Hence, from (42)and (24), we obtain

$$\begin{cases} \lambda^2 u - (\sqrt{1-x^2}u_x)_x + \alpha(u-v) = 0 \\ \lambda^2 v - (\sqrt{1-x^2}v_x)_x + \alpha(v-u) = 0 \\ u(-1) = u(1) = 0 \\ v(-1) = v(1) = 0 \\ u(\sqrt{1-x^2}u_x)(1) = 0, \\ u(\sqrt{1-x^2}v_x)(1) = 0, \end{cases} \tag{45}$$

Thenξ

$$\begin{cases} \varphi = u + v \\ \psi = u - v \end{cases} \tag{46}$$

Hence

$$\begin{cases} \lambda^2\varphi + (\sqrt{1-x^2}\varphi_x)_x = 0, \\ \left(\lambda^2 - 2\alpha\right)\Psi + (\sqrt{1-x^2}\psi_x)_x = 0, \\ \varphi(-1) = \varphi(1) = (\sqrt{1-x^2}\varphi_x)(1) = 0, \\ \psi(-1) = \psi(1) = (\sqrt{1-x^2}\psi_x)(1) = 0. \end{cases} \tag{47}$$

From $(47)_1$ for such λ, we find

$$(x^2-1)\varphi_{xx} + x\varphi_x + \lambda^2\sqrt{1-x^2}\varphi = 0 \tag{48}$$

we take $\varphi(x) = \theta(\xi)$, $x = \cos\xi$, $k\pi < \xi < (k+1)\pi$, $x \in (-1, 1)$, we find

$$\theta_{\xi\xi} - (\lambda^2 \sin\xi)\theta = 0 \tag{49}$$

see [26], and equation (49) can write by

$$\theta_{tt} - (4\lambda^2 \cos 2t)\theta = 0 \tag{50}$$

where $\xi = (2t + \frac{\pi}{2})$. Then we have

$$\theta(2t + \frac{\pi}{2}) = c_1 J_0(2\lambda e^{i\frac{\pi}{4}}) + c_2 Y_0(2\lambda e^{i\frac{\pi}{4}}), \tag{51}$$

(see ,[14], p.665).Then it becomes

$$\varphi(x) = c_1 J_0(2\lambda e^{i\frac{\pi}{4}}) + c_2 Y_0(2\lambda e^{i\frac{\pi}{4}}) \tag{52}$$

where J_n and Y_n are defined by:

$$J_n(y) = \sum_{m=0}^{\infty} \frac{(-1)^n}{(m!)^2} \left(\frac{y}{2}\right)^{2m+n}. \tag{53}$$

If $n \neq 0$

$$\begin{aligned} Y_n = \frac{2}{\pi} \ln(\frac{x}{2}) J_n(x) - \frac{1}{\pi} \sum_{k=0}^{n-1} \frac{(n-k-1)!}{k!} \left(\frac{x}{2}\right)^{2k-n} \\ - \frac{1}{\pi} \sum_{k=0}^{\infty} \left[\frac{\Gamma'(k+1)}{\Gamma(k+1)} + \frac{\Gamma'(k+n+1)}{\Gamma(k+n+1)}\right] \frac{(-1)^n \left(\frac{x}{2}\right)^{n+2k}}{k!(n+k)!} \end{aligned} \tag{54}$$

and

$$Y_0 = \left(\frac{2}{\pi} \ln(\frac{x}{2}) + \gamma\right) J_0(x) + \frac{2}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k+1} H_k}{(k!)^2} \left(\frac{x}{2}\right)^{2k}, \; H_k = \sum_{j=1}^{k} \frac{1}{j} \tag{55}$$

which γ is the Euler-Mascheroni constant,

wehre J_n and Y_n are Bessel functions of the first kind and second kind of order n and are linearly independent and therefore the pair (J_n, Y_n) (classical result) forms a fundamental system of solutions of (48).

The solution of the system (47) is given by

$$\begin{cases} \varphi(x) = c_1 \Phi_+(x) + c_2 \Phi_-(x), \\ \psi(x) = c_3 \Phi_{++}(x) + c_4 \Phi_{--}(x), \end{cases}$$

where

$$\begin{cases} \Phi_+(x) = J_0(2i\lambda e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}), \\ \Phi_-(x) = Y_0(2i\lambda e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}), \\ \Phi_{++}(x) = J_0(2i\sqrt{\lambda^2 - 2\alpha} e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}), \\ \Phi_{--}(x) = Y_0(2i\sqrt{\lambda^2 - 2\alpha} e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}), \end{cases} \tag{56}$$

from boundary condition $(47)_3$,we deduce that

$$\begin{cases} c_1 J_0(2\lambda e^{i\frac{\pi}{4}}) + c_2 Y_0(2\lambda e^{i\frac{\pi}{4}}) = 0, \\ c_1 J_1(2\lambda e^{i\frac{\pi}{4}}) + c_2 Y_1(2\lambda e^{i\frac{\pi}{4}}) = 0, \\ c_1 J_0(2i\lambda e^{i\frac{\pi}{4}}) + c_2 Y_0(2i\lambda e^{i\frac{\pi}{4}}) = 0. \end{cases} \tag{57}$$

Using the fact (the Wronskian w)

$$J_0(2\lambda e^{i(\frac{\pi}{4})})Y_1(2\lambda e^{i\frac{\pi}{4}}) - J_1(2\lambda e^{i\frac{\pi}{4}})Y_0(2\lambda e^{i\frac{\pi}{4}}) = -\frac{1}{\pi\lambda e^{i\frac{\pi}{4}}} \neq 0. \tag{58}$$

We deduce that $c_1 = c_2 = 0$, and similarly, we get $c_3 = c_4 = 0$. Hence

$$U \equiv 0$$

Therfore $U = 0$, which contradicts $\|U\| \neq 0$. Consequently, $\mathcal{A}$ does not have purely imagunary eigenvalues. $\square$

To prove scond condition of theorem (2.6), we need the following generalization of the Lax-Milgram Lemma.

Lemma 2.9. *(**Lax-Milgram-Fredholm**, see [11])*
Let V and H be Hilbert space such that the embedding $V \hookrightarrow H$ is compact and dense. Suppose that $a_V : V \times V \to \mathbb{C}$ and $a_H : H \times H \to \mathbb{C}$
are two bounded sesquilinar forms such that a_V is V-coercive and $G : v \to \mathbb{C}$ is continuous conjugate linear form. The equation

$$a_H(u, v) + a_V(u, v) = G(v), \quad \forall v \in V$$

has either a unique solution $u \in V$ for all $G \in V'$ or has a nontrivial solution for $G = 0$.

Lemma 2.10. *If $\lambda \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.*
If $\lambda = 0$ and $\omega \neq 0$, the operator $i\lambda I - \mathcal{A}$ is surjective.

Proof. **Case 1:** $\lambda \neq 0$. Let $F = (f_1, f_2, f_3, f_4, f_5, f_6)^T \in \mathcal{H}$ be given, and let $X = (u, \tilde{u}, v, \tilde{v}, \phi, \tilde{\phi}) \in D(\mathcal{A})$ be such that

$$\begin{cases} i\lambda u - \tilde{u} = f_1 \\ i\lambda\tilde{u} - (a(x)u_x)_x + \alpha(u - v) = f_2 \\ i\lambda v - \tilde{v} = f_3 \\ i\lambda\tilde{v} - (a(x)v_x)_x + \alpha(v - u) = f_4 \\ i\lambda\phi + (\xi^2 + \omega)\phi - \mu(\xi)\tilde{u}(1) = f_5 \\ i\lambda\tilde{\phi} + (\xi^2 + \omega)\tilde{\phi} - \mu(\xi)\tilde{v}(1) = f_6. \end{cases} \tag{59}$$

We divide the proof into three steps, as follows:
Step 1. Inserting $(59)_1, (59))_3$ into $(59)_2, (59))_4$, we get

$$\begin{cases} \lambda^2 u - (\sqrt{1 - x^2}u_x)_x + \alpha(u - v) = (f_2 + i\lambda f_1) \\ \lambda^2 v - (\sqrt{1 - x^2}v_x)_x + \alpha(v - u) = (f_4 + i\lambda f_3). \end{cases} \tag{60}$$

Solving system (59) is equivalent to finding $(u, v) \in \left[H_a^2(-1, 1) \cap H_{1,a}^1(-1, 1)\right]^2$ such that

$$\begin{cases} \displaystyle\int_0^1 (-\lambda^2 u\overline{w_1} - (\sqrt{1 - x^2}u_x)_x\overline{w_1}) + \alpha(u - v)\overline{w_1})dx = \int_0^1 (f_2 + i\lambda f_1)\overline{w_1}dx \\ \displaystyle\int_0^1 (-\lambda^2 v\overline{w_2} - (\sqrt{1 - x^2}v_x)_x\overline{w_2}) + \alpha(u - v)\overline{w_2})dx = \int_0^1 (f_4 + i\lambda f_3)\overline{w_2}dx \end{cases} \tag{61}$$

for all $w_1, w_2 \in H_{1,a}^1(-1, 1)$. By using (60) the functions u and v satisfying the following system

$$\begin{cases} \displaystyle\int_{-1}^{1}(-\lambda^2 u\overline{w_1} - \sqrt{1-x^2}u_x\overline{w_1})dx + \int_{-1}^{1}(-\lambda^2 v\overline{w_2} - \sqrt{1-x^2}v_x\overline{w_2})dx \\ \displaystyle +\alpha\int_{0}^{1}(u-v)(\overline{w_1}-\overline{w_2})dx + i\varrho\lambda(i\lambda+\omega)^{\tau-1}u(1)\overline{w_1}(1) + i\tilde{\varrho}\lambda(i\lambda+\omega)^{\tau-1}v(1)\overline{w_2}(1) \\ \displaystyle = \int_{-1}^{1}((f_2+i\lambda f_1)\overline{w_1} + \int_{-1}^{1}(f_4+i\lambda f_3)\overline{w_2})dx - \zeta\overline{w_1}(1)\int_{-\infty}^{+\infty}\tfrac{f_5(\xi)\mu(\xi)}{\xi^2+\omega+i\lambda}d\xi \\ \displaystyle -\tilde{\zeta}\overline{w_2}(1)\int_{-\infty}^{+\infty}\tfrac{f_6(\xi)\mu(\xi)}{\xi^2+\omega+i\lambda}d\xi + (i\lambda+\omega)^{\tau-1}(\varrho f_1(1)\overline{w_1}(1) + \tilde{\varrho}f_3(1)\overline{w_2}(1)). \end{cases} \tag{62}$$

We can rewrite (62)as

$$L_\lambda(U,V) + a_{(H^1_{1,a}(-1,1))^2}(U,V) = l(V), \tag{63}$$

where the sesquilinear froms $L_\lambda : \left[L^2(0,1)\times L^2(0,1)\right]^2 \to \mathbb{C}$,
$a_{(H^1_*(0,1))^2} : \left[H^1_{1,a}(-1,1)\times H^1_{1,a}(-1,1)\right]^2 \to \mathbb{C}$ and the antilinear form
$l : H^1_{1,a}(-1,1)\times H^1_{1,a}(-1,1) \to \mathbb{C}$ are defined by

$$L_\lambda(U,V) = -\int_{-1}^{1}\lambda^2 u\overline{w_1}dx + \int_{-1}^{1}\lambda^2 v\overline{w_2}dx \tag{64}$$

$$\begin{aligned} a_{(H^1_{1,a}(-1,1))^2}(U,V) = &\int_{-1}^{1}a(x)u_x\overline{w_1})dx + \int_{-1}^{1}a(x)v_x\overline{w_2}dx + \alpha\int_{-1}^{1}(u-v)(\overline{w_1}-\overline{w_2})dx \\ &+ (i\lambda+\omega)^{\tau-1}(\gamma f_1(1)\overline{w_1}(1) + \tilde{\gamma}f_3(1)\overline{w_2}(1)) \end{aligned} \tag{65}$$

and

$$\begin{aligned} l(V) = &\int_{-1}^{1}((f_2+i\lambda f_1)\overline{w_1} + \int_{-1}^{1}(f_4+i\lambda f_3)\overline{w_2})dx \\ &- \zeta\overline{w_1}(1)\int_{-\infty}^{+\infty}\frac{f_5(\xi)\mu(\xi)}{\xi^2+\omega+i\lambda}d\xi - \tilde{\zeta}\overline{w_2}(1)\int_{-\infty}^{+\infty}\frac{f_6(\xi)\mu(\xi)}{\xi^2+\omega+i\lambda}d\xi \\ &+ (i\lambda+\omega)^{\tau-1}(\gamma f_1(1)\overline{w_1}(1) + \tilde{\gamma}f_3(1)\overline{w_2}(1)). \end{aligned} \tag{66}$$

It is easy to verify that $a_{(H^1_{1,a}(-1,1))^2}$ is continuous and coercive and L_λ is bounded. Furthermore

$$\begin{aligned} \Re a_{(H^1_{1,a}(-1,1))^2}(U,V) &= \left\|(1-x^2)u_x\right\|_2^2 + \left\|(1-x^2)v_x\right\|_2^2 + \alpha\|u-v\|_2^2 \\ &\quad + \lambda\Re(i(i\lambda+\omega)^{\tau-1})(\varrho|u(1)|^2 + \tilde{\varrho}|v(1)|^2) \\ &\geq \left\|(1-x^2)u_x\right\|_2^2 + \left\|(1-x^2)v_x\right\|_2^2 \end{aligned} \tag{67}$$

where we have used the fact that

$$\Re(i(i\lambda+\omega)^{\tau-1}) = \lambda^2\int_{-\infty}^{+\infty}\frac{\mu^2(\xi)}{\lambda+(\omega+\xi^2)^2}d\xi > 0. \tag{68}$$

Now, following Fredholm alternative, we still need to prove that the operator l is injective to obtain that the operator l is an isomorphism. Let $(u,v)\in\ker(l)$, then

$$L_\lambda(U,V) + a_{(H^1_{1,a}(-1,1))^2}(U,V) = 0\ \forall(U,V)\in H^1_{1,a}(-1,1). \tag{69}$$

In particular for $U = V$, it follows that

$$\begin{aligned}&\lambda^2(\|u\|_2^2 + \|v\|_2^2) - i\varrho\lambda(i\lambda + \omega)^{\tau-1}|u(1)|^2\\&- i\tilde{\varrho}\lambda(i\lambda + \omega)^{\tau-1} - |v(1)|^2\\&= \left\|(1-x^2)u_x\right\|_2^2 + \left\|(1-x^2)v_x\right\|_2^2 + \alpha\|u-v\|_2^2.\end{aligned} \tag{70}$$

Hence, we have

$$u(1) = v(1) = 0. \tag{71}$$

From (69) , we obtain

$$(\sqrt{1-x^2}u_x)(1) = (\sqrt{1-x^2}v_x)(1) = 0 \tag{72}$$

and then

$$\begin{cases}\lambda^2 u - (\sqrt{1-x^2}u_x)_x + \alpha(u-v) = 0\\ \lambda^2 v - (\sqrt{1-x^2}v_x)_x + \alpha(v-u) = 0\\ u(-1) = v(-1) = 0\\ u(1) = (\sqrt{1-x^2}u_x)(1) = 0\\ v(1) = (\sqrt{1-x^2}v_x)(1) = 0.\end{cases} \tag{73}$$

We deduce that $U = 0$. Hence $i\lambda I - \mathcal{A}$ is surjective for all $\lambda \in \mathbb{R}^*$.
Case 1: $\lambda = 0$ and $\omega \neq 0$. Using Lax-Milgram Lemma, we obtain the result. □

3. Spectral analysis and lack of uniform stability

This section will be devoted to the study of the lack of exponential decay of solutions associated with the system (22). To do this, we shall usethe following well-known result from semigroup theory.

Theorem 3.1. *([12]-[13]) Let $S(t)$ be a C_0-semigroup of contractions on Hilbert space $\mathcal{H}$ with generator $\mathcal{A}$. Then $S(t)$ is exponentially stable if and only if*

$$i\mathbb{R} \equiv \{i\beta : \beta \in \mathbb{R}\} \subset \rho(\mathcal{A}) \tag{74}$$

$$\overline{\lim_{|\beta|\to\infty}} \left\|(i\beta I - \mathcal{A}.)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{75}$$

Our main result is

Theorem 3.2. *The semigroup generated by the operator A is not exponentially stable in the energy space $\mathcal{H}$.*

Proof. We will examine two cases.
Case 1 $\omega = 0$.
We shall show that $i\lambda = 0$ is not in the resolvent set of the operator $\mathcal{A}$. Indeed, noting that $(\sin(x+1), 0, \sin(x+1), 0, 0, 0) \in \mathcal{H}$, and denoting by $(u, \tilde{u}, v, \tilde{v}, \phi, \tilde{\phi})^T$ the image of $(\sin x, 0 \sin x, 0, 0, 0)^T$ by $\mathcal{A}^{-1}$, we see that $\phi(\xi) = |\xi|^{\frac{2\tau-5}{2}}\sin 2$. But $\phi \notin L^2(\mathbb{R})$, since $\tau \in (0,1)$ and $(u, \tilde{u}, v, \tilde{v}, \phi, \tilde{\phi})^T \notin D(\mathcal{A})$.
Case 2 $\omega \neq 0$.
We aim to show that an infinite number of eigenvalues of $\mathcal{A}$ approach the imaginary axis which prevents the system (1)–(2) from being exponentially stable. Indeed, we first compute the characteristic equation

that gives the eigenvalues of $\mathcal{A}$. Let be an eigenvalue of $\mathcal{A}$ with associated eigenvector $U = (u, \tilde{u}, v, \tilde{v}, \phi, \tilde{\phi})^T$. Then $\mathcal{A}U = \lambda U$ is equivalent to

$$\begin{cases} i\lambda u - \tilde{u} = 0, \\ i\lambda\tilde{u} - (\sqrt{1-x^2}u_x)_x + \alpha(u - v) = 0, \\ i\lambda v - \tilde{v} = 0, \\ i\lambda\tilde{u} - (\sqrt{1-x^2}u_x)_x + \alpha(v - u) = 0, \\ i\lambda\phi + (\xi^2 + \omega)\phi - \mu(\xi)\tilde{u}(1) = 0, \\ i\lambda\tilde{\phi} + (\xi^2 + \omega)\tilde{\phi} - \mu(\xi)\tilde{v}(1) = 0. \end{cases} \tag{76}$$

Inserting $(76)_1, (76)_3$ into $(76)_2, (76)_4$, we get

$$\begin{cases} \lambda^2 - (\sqrt{1-x^2}u_x)_x + \alpha(u - v) = 0 \\ \lambda^2 v - (\sqrt{1-x^2}v_x)_x + \alpha(v - u) = 0 \\ i\lambda\phi + (\xi^2 + \omega)\phi - \mu(\xi)\tilde{u}(1) = 0, \\ i\lambda\tilde{\phi} + (\xi^2 + \omega)\tilde{\phi} - \mu(\xi)\tilde{v}(1) = 0. \end{cases} \tag{77}$$

With boundary conditions

$$\begin{cases} (\sqrt{1-x^2}u_x)(1) - \varrho\lambda(\lambda + \omega)^{\tau-1} + \mu(\xi)u(1) = 0, \\ (\sqrt{1-x^2}v_x)(1) - \tilde{\varrho}\lambda(\lambda + \omega)^{\tau-1} + \mu(\xi)v(1) = 0, \\ u(-1) = v(-1). \end{cases} \tag{78}$$

Finally, we get the follwing system

$$\begin{cases} \lambda^2 u - (\sqrt{1-x^2}u_x)_x + \alpha(u - v) = 0, \\ \lambda^2 v - (\sqrt{1-x^2}v_x)_x + \alpha(v - u) = 0, \\ u(-1) = v(-1) = 0, \\ \sqrt{1-x^2}u_x)(1) - \varrho\lambda(\lambda + \omega)^{\tau-1} + \mu(\xi)u(1) = 0, \\ (\sqrt{1-x^2}v_x)(1) - \tilde{\varrho}\lambda(\lambda + \omega)^{\tau-1} + \mu(\xi)v(1) = 0. \end{cases} \tag{79}$$

Let us set

$$\begin{cases} \varphi = u + v, \\ \tilde{\varphi} = u - v. \end{cases} \tag{80}$$

Then, we obtain

$$\begin{cases} \lambda^2\varphi - (\sqrt{1-x^2}\varphi_x)_x = 0, \\ (\lambda^2 + 2\alpha)\tilde{\varphi} - (\sqrt{1-x^2}\tilde{\varphi}_x)_x = 0. \end{cases} \tag{81}$$

It is well-known that Bessel functions play an important role in this type of problem. Assume that φ is a solution of (81) associated to eigenvalue $-\lambda^{2,}$ then one easily cheks that the function.

The general solution of system (81) is given by

$$\begin{aligned} \varphi(x) &= C_1 J_0(2\lambda e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}) + C_2 Y_0(2\lambda e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}), \\ \tilde{\varphi}(x) &= C_3 J_0(2i\lambda_\alpha e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}) + C_4 Y_0(2i\lambda_\alpha e^{i(\frac{ar\cos x}{2} - \frac{\pi}{4})}). \end{aligned} \tag{82}$$

Where $\lambda_\alpha = \sqrt{\lambda^2 + 2\alpha}$. Thus the boundary conditions may be written as the following system

$$M(\lambda)\begin{pmatrix} C_1 \\ C_2 \\ C_3 \\ C_4 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \tag{83}$$

where

$$M(\lambda)=\begin{pmatrix} g_{1,\lambda} & g_{2,\lambda} & g_{3,\lambda_\alpha} & g_{4,\lambda_\alpha} \\ g_{1,\lambda} & g_{2,\lambda} & -g_{3,\lambda_\alpha} & -g_{4,\lambda_\alpha} \\ J_0(2\lambda e^{i\frac{\pi}{4}}) & Y_0(2\lambda e^{i\frac{\pi}{4}}) & J_0(2\lambda_\alpha e^{i\frac{\pi}{4}}) & Y_0(2\lambda_\alpha e^{i\frac{\pi}{4}}) \\ J_0(2\lambda e^{i\frac{\pi}{4}}) & Y_0(2\lambda e^{i\frac{\pi}{4}}) & -J_0(2\lambda_\alpha e^{i\frac{\pi}{4}}) & -Y_0(2\lambda_\alpha e^{i\frac{\pi}{4}}) \end{pmatrix}, \tag{84}$$

and

$$\begin{cases} g_{1,\lambda}=i\lambda e^{-i\frac{\pi}{4}}J_1(2\lambda e^{-i\frac{\pi}{4}})+\varrho\lambda(\lambda+\omega)^{\tau-1}J_0(2\lambda e^{-i\frac{\pi}{4}}), \\ g_{2,\lambda}=i\lambda e^{-i\frac{\pi}{4}}Y_1(2\lambda e^{-i\frac{\pi}{4}})+\varrho\lambda(\lambda+\omega)^{\tau-1}Y_0(2\lambda e^{-i\frac{\pi}{4}}), \\ g_{3,\lambda_\alpha}=i\lambda e^{-i\frac{\pi}{4}}J_1(2\lambda_\alpha e^{-i\frac{\pi}{4}})+\tilde{\varrho}\lambda(\lambda+\omega)^{\tau-1}J_0(2\lambda_\alpha e^{-i\frac{\pi}{4}}), \\ g_{4,\lambda_\alpha}=i\lambda e^{-i\frac{\pi}{4}}Y_1(2\lambda_\alpha e^{-i\frac{\pi}{4}})+\tilde{\varrho}\lambda(\lambda+\omega)^{\tau-1}Y_0(2\lambda_\alpha e^{-i\frac{\pi}{4}}). \end{cases} \tag{85}$$

Hence a non-trivial solution exist if only if the determinant of $M(\lambda)$ vanishes.

Set $f(\lambda)=\det M(\lambda)$, thus characteristic equation is $f(\lambda)=0$. Our purpose in the sequel is to prove, thanks to Rouch's theorem, that there is a sequence of eigenvalues for which their real part tends to 0. since $\mathcal{A}$ is dissipative, we study the asymptotic behavior of the large eigenvalues λ of $\mathcal{A}$ in the strip $-\alpha_0\le\Re(\lambda)\le 0$, for some $\alpha_0>0$ large enough.and for such λ, we remark that J_0, Y_0 remains bounded. □

Lemma 3.3. *The large eigenvalues of the dissipative operator are simple and can be slit into two families $(\lambda_k^j)_{k\in\mathbb{Z}}$, $j=1,2$, ($k_0\in\mathbb{N}$, chosen large enough).*

Moreover, the followine asymptotic expansions for the eigenvalues hold:

$$\begin{aligned} &\lambda_k^1=i\frac{(2k+1)}{4\sqrt{2}}+\frac{\tilde{\alpha}_1}{k^{1-\tau}}+\frac{\tilde{\beta}_1}{k^{1-\tau}}+o\left(\frac{1}{K^{1-\tau}}\right),\ \tilde{\alpha}_1\in i\mathbb{R},\ \tilde{\beta}_1<0,\ \ k\ge k_0, \\ &\lambda_k^1=\overline{\lambda_{-k}^1},\ k\le -k_0, \\ &\lambda_k^2=i\frac{(2k+1)}{4\sqrt{2}}+\frac{\tilde{\alpha}_2}{k^{1-\tau}}+\frac{\tilde{\beta}_2}{k^{1-\tau}}+o\left(\frac{1}{K^{1-\tau}}\right),\ \tilde{\alpha}_2\in i\mathbb{R},\ \tilde{\beta}_2<0,\ \ k\ge k_0, \\ &\lambda_k^2=\overline{\lambda_{-k}^2},\ k\le -k_0. \end{aligned} \tag{86}$$

Proof. **Step1.** We will use thefollowing classical asymptotic development (see [15]) for all $\delta>0$, thefollowing development holds when $|\arg z|\le\pi-\delta$:

$$J_v(z)=\left(\frac{2}{\pi z}\right)^{1/2}\cos(z-v\frac{\pi}{2}-\frac{\pi}{4})\left(1+O(\frac{1}{|z|^2})\right)-\left(\frac{2}{\pi z}\right)^{1/2}\sin(z-v\frac{\pi}{2}-\frac{\pi}{4})O(\frac{1}{|z|^2}) \tag{87}$$

$$Y_v(z)=\left(\frac{2}{\pi z}\right)^{1/2}\sin(z-v\frac{\pi}{2}-\frac{\pi}{4})\left(1+O(\frac{1}{|z|^2})\right)-\left(\frac{2}{\pi z}\right)^{1/2}\cos(z-v\frac{\pi}{2}-\frac{\pi}{4})O(\frac{1}{|z|^2}). \tag{88}$$

The determinant of $M(\lambda)$ is given by

$$\begin{aligned} f(\lambda)&=4\left(Y_0(2\lambda e^{i\frac{\pi}{4}}g_{1,\lambda}-J_0(2\lambda e^{i\frac{\pi}{4}})g_{2,\lambda}\right)\left(J_0(2\lambda_\alpha e^{i\frac{\pi}{4}})g_{4,\lambda_\alpha}-Y_0(2\lambda_\alpha e^{i\frac{\pi}{4}})g_{3,\lambda_\alpha}\right). \\ &=4\,(A\times B) \end{aligned} \tag{89}$$

where

$$\begin{cases} A=Y_0(2\lambda e^{i\frac{\pi}{4}}g_{1,\lambda}-J_0(2\lambda e^{i\frac{\pi}{4}})g_{2,\lambda}, \\ B=J_0(2\lambda_\alpha e^{i\frac{\pi}{4}})g_{4,\lambda_\alpha}-Y_0(2\lambda_\alpha e^{i\frac{\pi}{4}})g_{3,\lambda_\alpha}. \end{cases} \tag{90}$$

Using (87)–(88), we get

$$\left\{\begin{array}{l} A = \frac{ie^{-i\frac{\pi}{4}}}{\pi}\left[\cos(2\sqrt{2}\lambda i) + \frac{\varrho e^{i\frac{\pi}{4}}}{i}(\lambda+\omega)^{\tau-1}\sin(2\sqrt{2}\lambda i) + O\left(\frac{1}{\lambda}\right)\right] \\ B = \frac{ie^{-i\frac{\pi}{4}}}{\pi}\left[\cos(2\sqrt{2}\lambda_\alpha i) + \frac{\tilde{\varrho} e^{i\frac{\pi}{4}}}{i}(\lambda+\omega)^{\tau-1}\cos(2\sqrt{2}\lambda_\alpha i) + O\left(\frac{1}{\lambda}\right)\right] \end{array}\right. \tag{91}$$

we have

$$\left\{\begin{array}{l} A = \frac{ie^{-i\frac{\pi}{4}}}{\pi}e^{-2\sqrt{2}\lambda}\left[(e^{4\sqrt{2}\lambda}+1) + \frac{\varrho e^{i\frac{\pi}{4}}}{\lambda^{1-\tau}}(e^{4\sqrt{2}\lambda}-1) + O\left(\frac{1}{\lambda^{1-\tau}}\right)\right] \\ B = \frac{ie^{-i\frac{\pi}{4}}}{\pi}e^{-2\sqrt{2}\lambda_\alpha}\left[(e^{4\sqrt{2}\lambda_\alpha}+1) + \frac{\tilde{\varrho} e^{i\frac{\pi}{4}}}{\lambda^{1-\tau}}(e^{4\sqrt{2}\lambda_\alpha}-1) + O\left(\frac{1}{\lambda^{1-\tau}}\right)\right]. \end{array}\right. \tag{92}$$

We start by the expansion of λ_α and $e^{4\sqrt{2}\lambda_\alpha}$

$$\lambda_\alpha = \lambda + \frac{\alpha}{\lambda} + O\left(\frac{1}{\lambda^2}\right), \tag{93}$$

$$e^{4\sqrt{2}\lambda_\alpha} = e^{4\sqrt{2}\lambda}\left(1 + \frac{\alpha}{\lambda} + \frac{\alpha^2}{2\lambda^2} + O\left(\frac{1}{\lambda^3}\right)\right). \tag{94}$$

Then

$$(e^{4\sqrt{2}\lambda_\alpha}+1) + \frac{\tilde{\varrho} e^{i\frac{\pi}{4}}}{\lambda^{1-\tau}}(e^{4\sqrt{2}\lambda_\alpha}-1) = (e^{4\sqrt{2}\lambda}+1) + \frac{\tilde{\varrho} e^{i\frac{\pi}{4}}}{\lambda^{1-\tau}}(e^{4\sqrt{2}\lambda}-1) + O\left(\frac{1}{\lambda}\right). \tag{95}$$

We get

$$\begin{aligned} f(\lambda) = & \frac{4}{\pi^2}e^{-2\sqrt{2}\lambda}e^{-2\sqrt{2}\lambda_\alpha}((e^{4\sqrt{2}\lambda}+1)^2 \\ & + \frac{e^{i\frac{\pi}{4}}(\varrho+\tilde{\varrho})}{\lambda^{1-\tau}}(e^{4\sqrt{2}\lambda}+1)(e^{4\sqrt{2}\lambda}-1) \\ & + \frac{\varrho\tilde{\varrho} i}{\lambda^{2-2\tau}}(e^{4\sqrt{2}\lambda}-1)^2) + O\left(\frac{1}{\lambda^{2-2\tau}}\right). \end{aligned} \tag{96}$$

We set

$$\tilde{f}(\lambda) = f_0(\lambda) + \frac{f_1(\lambda)}{\lambda^{1-\tau}} + \frac{f_2(\lambda)}{\lambda^{2-2\tau}} + O\left(\frac{1}{\lambda^{2-2\tau}}\right). \tag{97}$$

Where

$$\begin{aligned} & f_0(\lambda) = (e^{4\sqrt{2}\lambda}+1)^2, \\ & f_1(\lambda) = e^{i\frac{\pi}{4}}(\varrho+\tilde{\varrho})(e^{4\sqrt{2}\lambda}+1)(e^{4\sqrt{2}\lambda}-1), \\ & f_1(\lambda) = \varrho\tilde{\varrho} i(e^{4\sqrt{2}\lambda}-1)^2. \end{aligned} \tag{98}$$

Step2.
We look at the roots of f_0. From (97) has one family of roots that we denote λ_k^0.

$$f_0(\lambda) = 0 \Leftrightarrow (e^{4\sqrt{2}\lambda}+1)^2 = 0. \tag{99}$$

Hence

$$\lambda_k^0 = i\frac{(k+1/2)}{2\sqrt{2}}\pi, \qquad k \in \mathbb{Z}, \tag{100}$$

Now with the help of Rouch's Theorem, we will show that the roots of $\tilde{f}$ are close to those of f_0. Let us start with the first family. Changing in (97) the unknown λ by $u = 4\sqrt{2}\lambda$ then (97) becomes

$$\tilde{f}(u) = (e^u + 1)^2 + O\left(\frac{1}{u^{1-\tau}}\right). \tag{101}$$

The roots of f_0 are $u_k = i\frac{(k+1/2)}{2\sqrt{2}}\pi$, $k \in \mathbb{Z}$, and setting $u = u_k + re^{it}$, $t \in [0, 2\pi]$, , we can easily check that there exists a constant $C > 0$ independent of k such that $|e^u + 1| \geq Cr$ for r small enough. This allows to apply Rouch's Theorem. Consequently, there exists a subsequence of roots of which tends to the roots $\tilde{f}$ which tends to the u_k of f_0. Equivalently, it means that there exists N and a subsequence $(\lambda_k)_{|k|\geq N}$ of the roots of $f(\lambda)$, such that $\lambda_k = \lambda_k^0 + o(1)$, which tends to the roots $i\frac{(2k+1)}{4\sqrt{2}}\pi$, $k \in \mathbb{Z}$. Finally for $|k| \geq N$, λ_k is simple since λ_k^0 is.

Step3. We can write

$$\lambda_k = \lambda_k^0 + \varepsilon_k. \tag{102}$$

Using (102), we get

$$e^{4\sqrt{2}\lambda_k} = -1 - 4\sqrt{2}\varepsilon_k + O(\varepsilon_k^2) \tag{103}$$

Substituting (103) into (94), using that $\tilde{f}(\lambda_k) = 0$, we get:

$$\tilde{f}(\lambda_k) = 32\varepsilon_k^2 + e^{i\frac{\pi}{4}}(\varrho + \tilde{\varrho})\frac{8\sqrt{2}\varepsilon_k}{\left(\frac{ik\pi}{2\sqrt{2}}\right)^{1-\tau}} + \varrho\tilde{\varrho}i\frac{4}{\left(\frac{ik\pi}{2\sqrt{2}}\right)^{2-2\tau}} + o(\varepsilon_k) = 0. \tag{104}$$

Multipying (104) by $k^{2-2\tau}$ leads to:

$$32(k^{1-\tau}\varepsilon_k)^2 + e^{i\frac{\pi}{4}}(\varrho + \tilde{\varrho})\frac{8\sqrt{2}(k^{1-\tau}\varepsilon_k)}{\left(\frac{i\pi}{2\sqrt{2}}\right)^{1-\tau}} + \varrho\tilde{\varrho}i\frac{4}{\left(\frac{i\pi}{2\sqrt{2}}\right)^{2-2\tau}} + o(1) + o(k^{1-\tau}\varepsilon_k) = 0. \tag{105}$$

Thus, $k^{1-\tau}\varepsilon_k$ is bounded and

$$8(k^{1-\tau}\varepsilon_k)^2 + e^{i\frac{\pi}{4}}(\varrho + \tilde{\varrho})\frac{2\sqrt{2}(k^{1-\tau}\varepsilon_k)}{\left(\frac{i\pi}{2\sqrt{2}}\right)^{1-\tau}} + \varrho\tilde{\varrho}i\frac{1}{\left(\frac{i\pi}{2\sqrt{2}}\right)^{2-2\tau}} + o(1) = 0. \tag{106}$$

The previous equation has two solutions

$$k^{1-\tau}\varepsilon_k = \frac{2\sqrt{2}e^{i\frac{\pi}{4}}\left(-(\varrho + \tilde{\varrho}) \pm \left|\varrho - \tilde{\varrho}\right|\right)}{\left(\frac{i\pi}{2\sqrt{2}}\right)^{1-\tau}} + o(1). \tag{107}$$

It holds:

$$\varepsilon_k = \frac{-(\varrho + \tilde{\varrho}) \pm \left|\varrho - \tilde{\varrho}\right|}{\pi^{1-\tau}\left(2\sqrt{2}\right)^{\tau-2} k^{1-\tau}}\left(\cos(1-2\tau)\frac{\pi}{4} - i\sin(1-2\tau)\frac{\pi}{4}\right) + o\left(\frac{1}{\lambda^{\tau-1}}\right). \tag{108}$$

From (108), we have in the case $|k|^{1-\tau}\,\Re\lambda_k^j \sim \beta_j$, $j = 1, 2$, with

$$\begin{cases} \beta_1 = \dfrac{-(\varrho + \tilde{\varrho})\ + \left|\varrho - \tilde{\varrho}\right|}{\pi^{1-\tau}\left(2\sqrt{2}\right)^{\tau-2}}\cos(1-2\tau)\frac{\pi}{4}, \\ \beta_1 = \dfrac{-(\varrho + \tilde{\varrho})\ - \left|\varrho - \tilde{\varrho}\right|}{\pi^{1-\tau}\left(2\sqrt{2}\right)^{\tau-2}}\cos(1-2\tau)\frac{\pi}{4}. \end{cases} \tag{109}$$

The operator $\mathcal{A}$ has two branches of eigenvalues with eigenvalues admitting real partstending to zero. Hence, the energy corresponding to the first and second branch of eigenvalues has no exponential decaying. Now, setting $\tilde{U}_k^j = (\lambda_k^0 I - \mathcal{A})U_k^j$, $j1,2$, where U_k^j is a normalized eigenfunction associated to λ_k^j. We then have

$$\begin{aligned}
\left\|(\lambda_k^0 I - \mathcal{A}.)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} &= \sup_{U\in\mathcal{H},U\neq 0} \frac{\left\|(\lambda_k^0 I - \mathcal{A}.)^{-1}U\right\|_{\mathcal{H}}}{\|U\|_{\mathcal{H}}} \\
&\geq \frac{\left\|(\lambda_k^0 I - \mathcal{A}.)^{-1}\tilde{U}_k^j\right\|_{\mathcal{H}}}{\left\|\tilde{U}_k^j\right\|_{\mathcal{H}}} \\
&\geq \frac{\left\|U_k^j\right\|_{\mathcal{H}}}{\left\|(\lambda_k^0 I - \mathcal{A}.)^{-1}U_k^j\right\|_{\mathcal{H}}}.
\end{aligned}$$

Hence, by Lemme 3.3, we deduce that

$$\left\|(\lambda_k^0 I - \mathcal{A}.)^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \geq c\,|k|^{1-\tau}. \tag{110}$$

Thus, Theorem is not satisfied. So that, the semigroup is not exponentially stable. □

Remark 3.4. *In this work, the polynomial stability was not studied because of some difficulties on the estimation of the integral of the Bessel functions, for this we conjuncture that the polynomial energy decay rate of the system (1)–(2) which depends on the order of the fractional derivative and the type* $t^{-2/(1-\tau)}$.

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