Filomat 38:10 (2024), 3313–3328
https://doi.org/10.2298/FIL2410313Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Two-weight inequalities for commutators of fractional integrals

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Abstract. In this article, via the relation between the higher order commutator $I_\alpha^{b,m}$ and its sparse counterparts, we establish two different types of the two-weight boundedness of $I_\alpha^{b,m}$ on Morrey spaces. The main novelties of these results include the predual of the weighted Morrey space $L^{p,\lambda}(\sigma)$, $X_\sigma^{p',\lambda}(\mathbb{R}^n)$, the boundedness of generalized fractional Orlicz maximal operators on Morrey spaces and the boundedness of generalized Orlicz maximal operators on $X_\sigma^{p',\lambda}(\mathbb{R}^n)$.

1. Introduction

Fractional integrals and their commutators, which are important linear operators in harmonic analysis, were widely used in the theory of function spaces and partial differential equations. See, for examples, [1, 7, 8, 17, 30] for the classical weighted case and [6, 9, 16, 27, 28] for the two-weight case. For the Morrey-type spaces and fractional integrals with rough kernel, we refer the reader to [12, 13, 18, 23, 26]. See also [3, 4, 15] for multilinear operators and [11, 19, 20] for other extensions.

Via the equivalence among dyadic fractional integrals and their sparse counterparts, and fractional integrals (cf., for example, [9]), Pan and Sun [20] established two-weight norm inequalities for fractional maximal operators and fractional integrals on two-weight Morrey spaces.

A natural question is whether or not the two-weight norm inequalities remain true for the higher order commutators of fractional integrals on two-weight Morrey spaces. In this article, we give an affirmative answer to this question (see Theorem 1.12 below).

Meanwhile, via establishing the sparse dominations for higher order commutators of fractional integrals $I_\alpha^{b,m}$, Accomazzo et al. [1] obtained qualitative Bloom type estimates for $I_\alpha^{b,m}$. Based on the aforementioned sparse dominations for $I_\alpha^{b,m}$, Wen and Wu [31] obtained the two-weight bounds for $I_\alpha^{b,m}$ under more general bump conditions, the necessity of two-weight bump conditions and the converse of Bloom type estimates for $I_\alpha^{b,m}$.

Motivated by [20, 31], in this article, we show two different types of the two-weight boundedness of $I_\alpha^{b,m}$ on Morrey spaces (see Theorems 1.12 and 1.13 below); see, for instance, [2, 20, 23–26] for more information on the theory of Morrey spaces and its extensions and applications.

To state the main results of this paper, we recall some basic notions and notation. Recall that the space $L^1_{\rm loc}(\mathbb{R}^n)$ is defined to be the *set of all locally integrable functions*.

2020 *Mathematics Subject Classification*. Primary 42B20; Secondary 42B35, 42B25.
Keywords. Two-weight inequality; Commutator; Fractional integral; Morrey space.
Received: 27 July 2023; Accepted: 02 October 2023
Communicated by Dragan S. Djordjević
This project is supported by the National Natural Science Foundation of China (Grant No. 11701160, 11871100).
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Definition 1.1. *Let n, $m \in \mathbb{N}$, $\alpha \in (0, n)$ and $b \in L^m_{\rm loc}(\mathbb{R}^n)$, namely, $|b|^m \in L^1_{\rm loc}(\mathbb{R}^n)$. Then*

(i) the fractional integral I_α *(cf. [29, p. 117]) is defined by setting, for any suitable function f on $\mathbb{R}^n$,*

$$I_\alpha(f)(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad \forall\, x \in \mathbb{R}^n;$$

(ii) the higher order commutator of the fractional integral $I^{b,m}_\alpha$ *(cf., for example, [1, p. 1209]) is defined by setting, for any suitable function f on $\mathbb{R}^n$,*

$$I^{b,m}_\alpha(f)(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} [b(x)-b(y)]^m \frac{f(y)}{|x-y|^{n-\alpha}}\, dy, \quad \forall\, x \in \mathbb{R}^n,$$

where $\gamma(\alpha) := \pi^{n/2} 2^\alpha \Gamma(\frac{\alpha}{2})/\Gamma(\frac{n-\alpha}{2})$ and Γ is the Gamma function.

Recall that a measurable function u on $\mathbb{R}^n$ is called a *weight* if

(i) $0 < u < \infty$ almost everywhere on $\mathbb{R}^n$;

(ii) $u \in L^1_{\rm loc}(\mathbb{R}^n)$.

Definition 1.2. *([20]) Let $p \in (1, \infty)$, $\lambda \in [0, 1)$, σ and u be weights. The* two-weight Morrey Space *is defined to be the set of all functions $f \in L^1_{\rm loc}(\mathbb{R}^n)$ with*

$$\|f\|_{L^{p,\lambda}(\sigma,u)} := \sup_Q \left[\frac{1}{[u(Q)]^\lambda} \int_Q |f(x)|^p \sigma(x)\, dx\right]^{1/p} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. Let

$$L^{p,\lambda}(\sigma) := L^{p,\lambda}(\sigma, 1) \quad \text{and} \quad L^{p,\lambda}(\mathbb{R}^n) := L^{p,\lambda}(1, 1)$$

denote the weighted Morrey Space *and the* classical Morrey space*, respectively.*

Moreover, $L^p_\sigma(\mathbb{R}^n) := L^{p,0}(\sigma)$ is the usual weighted Lebesgue space *on $\mathbb{R}^n$.*

Definition 1.3. *([2]) Given $d \in (0, n]$, the d*-dimensional Hausdorff capacity *of a set $E \subset \mathbb{R}^n$ is defined by*

$$\Lambda^d(E) := \inf \left\{ \sum_{j=1}^\infty r_j^d : E \subset \bigcup_j Q(x_j, r_j), j \in \mathbb{Z} \right\}.$$

Here and thereafter, $Q(x, r)$ represents a cube with the center $x \in \mathbb{R}^n$ *and half of* sidelength $r \in (0, \infty)$.

Definition 1.4. *([22]) Let $\Phi : [0, \infty) \to [0, \infty)$ be a* Young function*, namely, an increasing, convex function with $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t)/t = \infty$. The* localized Luxemburg average *of a measurable function f on a cube Q is defined by*

$$\|f\|_{\Phi,Q} := \inf \left\{ \lambda \in (0, \infty) : \frac{1}{|Q|} \int_Q \Phi\left(\frac{|f(x)|}{\lambda}\right) dx \le 1 \right\}.$$

In particular, when $\Phi(t) = t^p, p \in (1, \infty)$, we simply write $\|f\|_{p,Q} := \|f\|_{t^p,Q}$.

Given a Young function Φ, the *conjugate function* and *left-inverse* (cf. [22]) of Φ are respectively defined by

$$\Phi^*(t) := \sup_{s\in[0,\infty)} \{st - \Phi(s)\}, \quad \forall\, t \in (0, \infty),$$

and

$$\Phi^{-1}(t) := \inf\{s \in [0,\infty) : \ \Phi(s) \geq t\}, \quad \forall\, t \in (0,\infty).$$

Recall from [22, p. 58, Proposition 1] that, the following generalized Hölder inequality holds true for any suitable functions f and g:

$$\frac{1}{|Q|}\int_Q |f(x)g(x)|\, dx \leq 2\|f\|_{\Phi,Q}\|g\|_{\Phi^*,Q}. \tag{1.1}$$

See [14, 22] for more information on the theory of Orlicz spaces.

Now we recall a growth condition on Young functions, which was introduced in [21, Definition 1.6]. Given $p \in (1, n/\alpha)$ and $1/q = 1/p - \alpha/n$, we say that $\Phi \in B_{p,q}$ for some positive constant c, if

$$\int_c^\infty \frac{[\Phi(t)]^{q/p}}{t^q}\frac{dt}{t} < \infty.$$

When $p = q$, we simply write $B_p := B_{p,p}$.

Definition 1.5. *([20, Section 2]) A set of cubes $\mathcal{D}$ in $\mathbb{R}^n$ is said to be* a general dyadic lattice *if*

(i) when $Q \in \mathcal{D}$, its side-length $l(Q) = 2^k$ for some $k \in \mathbb{Z}_+ := \{0\} \cup \mathbb{N}$;

(ii) when $Q, R \in \mathcal{D}$, $Q \cap R = \{Q, R, \emptyset\}$;

(iii) the subset $\mathcal{D}_k = \{Q \in \mathcal{D} : \ l(Q) = 2^k\} \subset \mathcal{D}$ forms a partition of $\mathbb{R}^n$ for any $k \in \mathbb{Z}_+$.

Definition 1.6. *([1, Section 2]) Given a general dyadic lattice $\mathcal{D}$ and $\eta \in (0,1)$, a subset $\mathcal{S} \subset \mathcal{D}$ is said to be* η-sparse, *if*

$$\left|\cup_{R\in\mathcal{S}, R\subsetneq Q} R\right| \leq (1-\eta)|Q|, \quad \forall\, Q \in \mathcal{S}.$$

Remark 1.7. *Let $\mathcal{S} \subset \mathcal{D}$ be η-sparse and, for any $Q \in \mathcal{S}$,*

$$E(Q) := Q \backslash \left(\cup_{R\in\mathcal{S}, R\subsetneq Q} R\right).$$

Then $\{E(Q)\}_{Q\in\mathcal{S}}$ are mutually disjoint and $|E(Q)| \geq \eta|Q|$ for any $Q \in \mathcal{S}$.

Definition 1.8. *([17]) Let $p \in [1,\infty)$. The* class of A_p weights *is defined to be the set of all functions $u \in L^1_{\rm loc}(\mathbb{R}^n)$ such that*

$$[u]_{A_p} := \begin{cases} \sup_Q \left(\frac{1}{|Q|}\int_Q u(x)\, dx\right)\left(\frac{1}{|Q|}\int_Q [u(x)]^{1-p'}\, dx\right)^{p-1} < \infty, & p \in (1,\infty); \\ \sup_Q \left(\frac{1}{|Q|}\int_Q u(x)\, dx\right)\|u^{-1}\|_{L^\infty(Q)} < \infty, & p = 1, \end{cases}$$

where the suprema above are taken over all cubes $Q \subset \mathbb{R}^n$.

The A_∞ weights *is defined by $A_\infty := \cup_{p\in[1,\infty)} A_p$.*

Definition 1.9. *([31]) Let $\alpha \in (0,n), m \in \mathbb{Z}_+, \eta \in (0,1), b \in L^m_{loc}(\mathbb{R}^n)$ and $p, q \in (1,\infty)$ with $p \leq q$. A pair of weight (u,σ) is said to belong to $A^\alpha_{p,q,\Phi,\Psi}$ if*

$$[u,\, \sigma]_{A^\alpha_{p,q,\Phi,\Psi}} := \sup_{Q\in\mathcal{D}} |Q|^{\frac{\alpha}{n}+\frac{1}{q}-\frac{1}{p}}\, \|u\|_{\Phi,Q}\, \|\sigma\|_{\Psi,Q} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$. In particular, when $\Phi(t) = t^q$ and $\Psi(t) = t^{p'}$, we simply write

$$[u,\, \sigma]_{A^\alpha_{p,q}} := [u,\, \sigma]_{A^\alpha_{p,q,t^q,t^{p'}}}\, .$$

Hereafter, for any η-sparse general dyadic lattice $\mathcal{S}$, measurable function f with compact support and $x \in \mathbb{R}^n$,

$$\mathcal{I}_{\mathcal{S},\alpha}^{b,m}(f)(x) := \sup_{Q\in\mathcal{S}} |Q|^{\alpha/n-1} \int_Q |b(y) - b_Q|^m |f(y)|\, dy\, \chi_Q(x)$$

and

$$(\mathcal{I}_{\mathcal{S},\alpha}^{b,m})^*(f)(x) := \sup_{Q\in\mathcal{S}} |Q|^{\alpha/n-1} |b(x) - b_Q|^m \int_Q |f(y)|\, dy\, \chi_Q(x).$$

Definition 1.10. *Let $\sigma \in A_\infty$. A function $b \in L^1_{\rm loc}(\mathbb{R}^n, \sigma(x)\,dx)$ is said to belong to the space* $\mathrm{BMO}_\sigma(\mathbb{R}^n)$ *if*

$$\sup_B \frac{1}{\sigma(B)} \int_B |b(x) - b_{\sigma,B}| \sigma(x)\, dx < \infty,$$

where $b_{\sigma,B} := \frac{1}{\sigma(B)} \int_B b(y)\sigma(y)\, dy$ and the supremum is taken over all balls $B \subset \mathbb{R}^n$. When $\sigma = 1$, we simply write $b_B := b_{1,B}$ and $\mathrm{BMO}(\mathbb{R}^n) := \mathrm{BMO}_1(\mathbb{R}^n)$, which is just the classical space of functions with bounded mean oscillation.

Remark 1.11. *Let $q \in (1,\infty)$ and $\sigma \in A_\infty$. It was shown by the John–Nirenberg inequality on spaces of homogeneous type (see the proof of [5, Theorem B]) that, for any $b \in \mathrm{BMO}_\sigma(\mathbb{R}^n)$,*

$$\|b\|_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \sim \sup_B \left[\frac{1}{\sigma(B)} \int_B |b(x) - b_{\sigma,B}|^q \sigma(x)\, dx\right]^{1/q}, \tag{1.2}$$

where the supremum is taken over all balls $B \subset \mathbb{R}^n$.

Now we are ready to state the two main results of this article.

Theorem 1.12. *Let $\alpha \in (0,n)$, $\lambda \in [0,1)$, $\eta \in (0,1)$, $m \in \mathbb{N}$, $b \in \mathrm{BMO}_\sigma(\mathbb{R}^n) \cap \mathrm{BMO}_u(\mathbb{R}^n)$ and $p,q \in (1,\infty)$ with $p \le q$. If u, $\sigma \in A_\infty$ satisfying*

$$c_{(u,\sigma)} := \left[u^{1/q},\, \sigma^{1/p'}\right]_{A^\alpha_{p,q}} < \infty, \tag{1.3}$$

then there exists a positive constant c such that, for any η-sparse general dyadic lattice $\mathcal{S}$ and $f \in L^{p,\lambda p/q}(\sigma,u)$ with compact support,

$$\begin{aligned}&\left\|\mathcal{I}_{\mathcal{S},\alpha}^{b,m}(f\sigma)\right\|_{L^{q,\lambda}(u,u)} + \left\|(\mathcal{I}_{\mathcal{S},\alpha}^{b,m})^*(f\sigma)\right\|_{L^{q,\lambda}(u,u)}\\ &\le c c_{(u,\sigma)} \left[\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} + \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)}\right] \|f\|_{L^{p,\lambda p/q}(\sigma,u)};\end{aligned}$$

in particular,

$$\left\|I_\alpha^{b,m}(f\sigma)\right\|_{L^{q,\lambda}(u,u)} \le c c_{(u,\sigma)} \left[\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} + \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)}\right] \|f\|_{L^{p,\lambda p/q}(\sigma,u)}.$$

Theorem 1.13. *Let $\alpha \in (0,n)$, $\lambda \in [0,1)$, $\eta \in (0,1)$, $m \in \mathbb{N}$, $b \in L^m_{\rm loc}(\mathbb{R}^n)$ and $p,q \in (1,\infty)$ with $p \le q$. Assume that A, B, C, D are Young functions which satisfy $A^*, C^* \in B_{q'}$ and $B^*, D^* \in B_{p,q}$. Then the following statements hold true:*

(i) *If a pair of weights (u,σ) satisfies*

$$\widetilde{C}_{(u,\sigma,b)} := \left[u^{1/q},\, (b - b_{(\cdot)})^m \sigma^{-1/p}\right]_{A^\alpha_{p,q,A,B}} + \left[(b - b_{(\cdot)})^m u^{1/q},\, \sigma^{-1/p}\right]_{A^\alpha_{p,q,C,D}} < \infty,$$

then exists a positive constant $\widetilde{C}$ such that, for any η-sparse general dyadic lattice $\mathcal{S}$ and $f \in L^{p,\lambda}(\sigma)$ with compact support,

$$\left\|\mathcal{I}_{\mathcal{S},\alpha}^{b,m}(f)\right\|_{L^{q,\lambda}(u)} + \left\|(\mathcal{I}_{\mathcal{S},\alpha}^{b,m})^*(f)\right\|_{L^{q,\lambda}(u)} \le \widetilde{C}\widetilde{C}_{(u,\sigma,b)} \|f\|_{L^{p,\lambda}(\sigma)}; \tag{1.4}$$

in particular,

$$\left\|I_\alpha^{b,m}(f)\right\|_{L^{q,\lambda}(u)} \lesssim \widetilde{C}_{(u,\sigma,b)} \|f\|_{L^{p,\lambda}(\sigma)}.$$

(ii) *If* (1.4) *holds true with* $\lambda = 0$, *then*

$$\left[u^{1/q},\, (b - b_{(\cdot)})^m \sigma^{-1/p}\right]_{A^{\alpha}_{p,q}} + \left[(b - b_{(\cdot)})^m u^{1/q},\, \sigma^{-1/p}\right]_{A^{\alpha}_{p,q}} < \infty.$$

Remark 1.14. *When* $\lambda = 0$, *Theorem 1.13 returns to [31, Theorems 1.1 and 1.2].*

This paper is organized as follows.

In Section 2, we recall some preliminary results which are necessary to the proofs of Theorems 1.12 and 1.13.

Section 3 is devoted to proofs of Theorems 1.12 and 1.13. To prove Theorem 1.12, by Lemma 2.1 below, we first reduce the estimates to those of some sparse counterparts $\{\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f)\}_{j=1}^{3^n}$ and $\{(\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f)\}_{j=1}^{3^n}$, which follows by the duality between $L^q_u(\mathbb{R}^n)$ and $L^{q'}_u(\mathbb{R}^n)$, the Hölder inequality (1.1), equivalent norms of functions in $\mathrm{BMO}_\sigma(\mathbb{R}^n)$ [see (1.2)], boundedness of weighted maximal operators on weighted Lebesgue spaces, η-sparsity of $\mathcal{S}$ and some ideas from the proof of [20, Theorem 1.2].

Theorem 1.13 follows by some similar arguments as those used in the proof of [31, Theorem 1.1]. The main novelties of this proof include the applications of duality between $X^{p',\lambda}_\sigma(\mathbb{R}^n)$ and $L^{p,\lambda}(\sigma)$ (see Theorem 2.3 below), the boundedness of generalized fractional Orlicz maximal operators on Morrey spaces (see Lemma 2.5 below) and the boundedness of generalized Orlicz maximal operators on the predual of $L^{p,\lambda}(\sigma)$, $X^{p',\lambda}_\sigma(\mathbb{R}^n)$ (see Corollary 2.8 below).

Finally, we list some conventions on notation. Let $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{Z}_+ := \{0\} \bigcup \mathbb{N}$ and $\mathbb{Z} := \{0, \pm1, \pm2, \ldots\}$. Let $\widetilde{C}$ denote a *positive constant* which is independent of the main parameters, but it may change from line to line. For two real functions f and g, we write $f \lesssim g$ if $f \le Cg$; $f \sim g$ if $f \lesssim g \lesssim f$. For any $p \in [1, \infty]$, let $p' := p/(p-1)$. Moreover, denote by E^* the dual space of a Banach spaces E.

2. Preliminaries

In Section 2, we recall and establish some preliminary results which are important to the proofs of Theorems 1.12 and 1.13. First, by the proof of [1, Theorem 2.1] and [31, Lemmas 2.3 and 2.4], we have the following conclusion.

Lemma 2.1. *Let* $\alpha \in (0, n)$, $\eta \in (0, 1)$, $b \in L^m_{\mathrm{loc}}(\mathbb{R}^n)$ *and* $m \in \mathbb{N}$. *Then there exist a family* $\{\mathcal{D}_j\}_{j=1}^{3^n}$ *of general dyadic lattices and a family* $\{\mathcal{S}_j\}_{j=1}^{3^n}$ *of* η*-sparse general dyadic lattices with* $\mathcal{S}_j \subset \mathcal{D}_j$, *for any* $j \in \{1, \ldots, 2^n\}$, *such that, for any measurable function* f *with compact support,*

$$\left|I^{b,m}_\alpha(f)(x)\right| \lesssim \sum_{j=1}^{3^n} \left[\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f)(x) + (\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f)(x)\right], \quad \forall\, x \in \mathbb{R}^n.$$

The following conclusion is adapted from [2, Theorem 5.2]. Recall that $L^1(\Lambda^d)$ denotes the set of all Λ^d*-quasi-continuous* functions f (namely, those functions f such that, for any $\epsilon \in (0, \infty)$, there exists a subset $E \subset \mathbb{R}^n$ such that $\Lambda^d(E) < \epsilon$ and f restricted to $\mathbb{R}^n \setminus E$ is continuous) for which

$$\|f\|_{L^1(\Lambda^d)} := \int_{\mathbb{R}^n} |f(x)|\, d\Lambda^d(x) = \int_0^\infty \Lambda^d(\{x \in \mathbb{R}^n : |f(x)| \ge t\})\, dt < \infty.$$

Lemma 2.2. *Let* $p \in (1, \infty)$, $\gamma \in (0, n)$ *and* $\lambda n = n - \gamma$. *Then*

$$\|f\|_{L^{p,\lambda}(\sigma)} = \sup_w \left[\int_{\mathbb{R}^n} |f(x)|^p w(x)\sigma(x)\, dx\right]^{1/p},$$

where the supremum is taken over all non-negative measurable functions w *with* $\|w\|_{L^1(\Lambda^d)} \le 1$ *and* $d = \lambda n$.

Proof. *From [2, Theorem 5.1], we deduce that*

$$\int_{\mathbb{R}^n} |f(x)|^p w(x)\sigma(x)\,dx = \int_{\mathbb{R}^n} \left(|f(x)|[\sigma(x)]^{\frac{1}{p}}\right)^p w(x)\,dx$$
$$\leq \|w\|_{L^1(\Lambda^d)} \left\|f\sigma^{\frac{1}{p}}\right\|_{L^{p,\lambda}(\mathbb{R}^n)} = \|w\|_{L^1(\Lambda^d)}\|f\|_{L^{p,\lambda}(\sigma)}.$$

On the other hand, if $w_0 := \chi_{Q(x_0,r_0)} r_0{}^{-\lambda n}$, *then* $\|w_0\|_{L^1(\Lambda^d)} = \int_{Q(x_0,r_0)} r_0{}^{-\lambda n}\,d\Lambda^d = 1$,

$$\|f\|_{L^{p,\lambda}(\sigma)} = \sup_{(x_0,r_0)\in(\mathbb{R}^n\times\infty)} \left[\int_{\mathbb{R}^n} |f(x)|^p r_0{}^{-\lambda n}\chi_{Q(x_0,r_0)}(x)\sigma(x)\,dx\right]^{\frac{1}{p}}$$
$$= \sup_{(x_0,r_0)\in(\mathbb{R}^n\times\infty)} \left[\int_{\mathbb{R}^n} |f(x)|^p w_0(x)\sigma(x)\,dx\right]^{\frac{1}{p}} \leq \sup_{w}\left[\int_{\mathbb{R}^n} |f(x)|^p w(x)\sigma(x)\,dx\right]^{\frac{1}{p}},$$

where the supremum is taken over all non-negative functions w *on* $\mathbb{R}^n$ *with* $\|w\|_{L^1(\Lambda^d)} \leq 1, d = \lambda n$. *This completes the proof of Lemma 2.2.* □

We also need to introduce the notion of the predual of the weighted Morrey space $X_\sigma^{p,\lambda}(\mathbb{R}^n)$. Given $p \in (1,\infty)$ and $\lambda \in (0,1)$, we say $f \in X_\sigma^{p,\lambda}(\mathbb{R}^n)$ if

$$\|f\|_{X_\sigma^{p,\lambda}(\mathbb{R}^n)} = \inf_{w}\left(\int_{\mathbb{R}^n} |f(x)|^p [w(x)]^{1-p}\sigma(x)\,dx\right)^{1/p} < \infty, \tag{2.1}$$

where the supremum is over all non-negative functions w with $\|w\|_{L^1(\Lambda^d)} \leq 1, d = \lambda n$. When $\sigma(\cdot) := |\cdot|$ is the Lebesgue measure on $\mathbb{R}^n$, let $X^{p,\lambda}(\mathbb{R}^n) := X_{|\cdot|}^{p,\lambda}(\mathbb{R}^n)$.

By borrowing some ideas from the proof [2, Theorem 5.3], we obtain the following conclusion.

Theorem 2.3. *Let* $p \in (1,\infty), \lambda \in (0,1)$. *Then the predual of* $L^{p,\lambda}(\sigma)$ *is* $X_\sigma^{p',\lambda}(\mathbb{R}^n)$ *in the sense that*

(i) if $f \in L^{p,\lambda}(\sigma)$, *then the linear functional* L_f *defined by*

$$L_f(g) := \langle f, g\rangle = \int_{\mathbb{R}^n} f(x)g(x)\sigma(x)\,dx, \quad \forall\, g \in X_\sigma^{p',\lambda}(\mathbb{R}^n) \tag{2.2}$$

is a bounded linear functional on $X_\sigma^{p',\lambda}(\mathbb{R}^n)$ *as in* (2.2) *with* $\|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*} \leq 2\|f\|_{L^{p,\lambda}(\sigma)}$;

(ii) if $L \in (X_\sigma^{p',\lambda}(\mathbb{R}^n))^*$, *then there exists a function* $f \in L^{p,\lambda}(\sigma)$ *such that* $L = L_f$ *with*

$$\|f\|_{L^{p,\lambda}(\sigma)} \leq \|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*}.$$

Proof. (i) For any $g \in X_\sigma^{p',\lambda}(\mathbb{R}^n)$, by (2.1), we may choose a non-negative functions w with $\|w\|_{L^1(\Lambda^d)} \leq 1$ and $d = \lambda n$ such that

$$\int_{\mathbb{R}^n} |f(x)|^p [w(x)]^{1-p}\sigma(x)\,dx \leq 2\|f\|_{L^{p,\lambda}(\sigma)},$$

which, combined with the Hölder inequality and Lemma 2.2, implies that

$$|L_f(g)| \leq \int_{\mathbb{R}^n} |f(x)|\,|g(x)|[w(x)]^{\frac{1}{p}}[w(x)]^{-\frac{1}{p}}[\sigma(x)]^{\frac{1}{p}}[\sigma(x)]^{\frac{1}{p'}}\,dx$$

$$\leq \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\sigma(x)\, dx \right)^{\frac{1}{p}} \left(\int_{\mathbb{R}^n} |g(x)|^{p'} [w(x)]^{1-p'} \sigma(x)\, dx \right)^{\frac{1}{p'}}$$
$$\leq 2 \left\| f \right\|_{L^{p,\lambda}(\sigma)} \left\| g \right\|_{X_\sigma^{p',\lambda}(\mathbb{R}^n)},$$

which shows (i).

(ii) Let L be a bounded linear functional on $(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*$. Then we claim that, for any cube $Q := Q(x_1, r_1)$ with $x_1 \in \mathbb{R}^n$ and $r_1 \in (0, \infty)$, L induces a bounded linear functional on $L_\sigma^{p'}(Q)$. Indeed, choose $w = r_1^{\gamma-n} \chi_{Q(x_1,r_1)}$ in (2.1). Then we have

$$\|g\|_{X_\sigma^{p',\lambda}(\mathbb{R}^n)} \leq r_1^{\frac{n-\gamma}{p}} \left(\int_{Q(x_1,r_1)} |g(x)|^{p'} \sigma(x)\, dx \right)^{\frac{1}{p'}} = r_1^{\frac{n-\gamma}{p}} \|g\|_{L_\sigma^{p'}(Q(x_1,r_1))},$$

which implies that

$$|L(g)| \leq \|L\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*} \left\|g\right\|_{X_\sigma^{p',\lambda}(\mathbb{R}^n)} \leq r_1^{\frac{n-\gamma}{p}} \|L\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*} \left\|g\right\|_{L_u^{p'}(Q(x_0,r_0))},$$

as claimed.

Fix a cube $Q_0 := Q(x_0, r_0)$ with $x_0 \in \mathbb{R}^n$ and $r_0 \in (0, \infty)$. Since $\left(L_\sigma^{p'}(Q_0)\right)^* = L_\sigma^p(Q_0)$, we know that there exists $f^{Q_0} \in L_\sigma^p(Q_0)$, such that

$$L(g) = \int_{\mathbb{R}^n} f^{Q_0}(x)\chi_{Q_0}(x)g(x)\sigma(x)\, dx, \quad \forall\, g \in L_\sigma^{p'}(Q_0).$$

Let $Q_1 := Q(0, 1) \cap Q(x_0, r_0)$. Then there exists $f^{Q_1} \in L_\sigma^p(Q_1)$ such that

$$L(g) = \int_{\mathbb{R}^n} f^{Q_1}(x)\chi_{Q_1}(x)g(x)\sigma(x)\, dx, \quad \forall\, g \in L_\sigma^{p'}(Q_1).$$

Let $Q_2 := Q(0, 2) \cap Q(x_0, r_0)$. Then there exists $f^{Q_2} \in L_\sigma^p(Q_2)$ such that

$$L(g) = \int_{\mathbb{R}^n} f^{Q_2}(x)\chi_{Q_2}(x)g(x)\sigma(x)\, dx, \quad \forall\, g \in L_\sigma^{p'}(Q_2).$$

Furthermore

$$\int_{\mathbb{R}^n} f^{Q_1}(x)\chi_{Q_1}(x)g(x)\sigma(x)\, dx = \int_{\mathbb{R}^n} f^{Q_2}(x)\chi_{Q_2}(x)g(x)\sigma(x)\, dx, \quad \forall\, g \in L_\sigma^{p'}(Q_2) \subset L_\sigma^{p'}(Q_1),$$

which proves $f^{Q_1}\chi_{Q_1} = f^{Q_2}\chi_{Q_1}$.

Repeat the process countable infinite times, there exists $\{f^{Q_j}\}_{j=1}^\infty$ satisfying that, for any $j \in \mathbb{N}, Q_j := Q(0, j) \cap Q(x_0, r_0), f^{Q^j} \in L_\sigma^p(Q_j)$,

$$f^{Q_j}\chi_{Q_j} = f^{Q_{j+1}}\chi_{Q_j}$$

and

$$L(g) = \int_{\mathbb{R}^n} f^{Q_j}(x)\chi_{Q_j}(x)g(x)\sigma(x)\, dx, \quad \forall\, g \in L_\sigma^{p'}(Q_j).$$

Thus, we construct $f \in L_\sigma^p(\mathbb{R}^n)$ such that $f = f^{Q_j}$ on Q_j for any $j \in \mathbb{N}$ and hence

$$L(g) = \int_{\mathbb{R}^n} f(x)g(x)\sigma(x)\, dx = L_f(g)$$

with $g \in X_\sigma^{p',\gamma}(\mathbb{R}^n)$ supporting on some cube of $\{Q_j\}_{j=0}^\infty$.

Thus, if we choose $g = \chi_{Q_0}|f|^p f^{-1}, w_0 = \chi_{Q_0} r_0{}^{\gamma-n}$, then

$$\begin{aligned}\int_{Q_0} |f(x)|^p \sigma(x)\,dx &= L_f(g) \le \|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*} \|g\|_{X_\sigma^{p',\lambda}(\mathbb{R}^n)} \\ &\le \|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*} \left(r_0{}^{(\gamma-n)(1-p')} \int_{Q_0} |g(x)|^{p'} \sigma(x)\,dx \right)^{\frac{1}{p'}} \\ &= \|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*} \left(r_0{}^{(\gamma-n)(1-p')} \int_{Q_0} |f(x)|^{(p-1)p'} \sigma(x)\,dx \right)^{\frac{1}{p'}} \\ &\le \|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*}\ r_0{}^{\frac{n-\gamma}{p}} \left(\int_{Q_0} |f(x)|^p \sigma(x)\,dx \right)^{\frac{1}{p'}},\end{aligned}$$

which shows that $\|f\|_{L^{p,\lambda}(\sigma)} \le \|L_f\|_{(X_\sigma^{p',\lambda}(\mathbb{R}^n))^*}$. This completes the proof of (ii) and hence of Theorem 2.3. □

Let $\alpha \in [0, n)$ and Φ be a Young function. Recall from [7] that the *generalized fractional Orlicz maximal operator*, $M_{\alpha,\Phi}$, is defined by

$$M_{\alpha,\Phi}(f)(x) := \sup_{Q \ni x} |Q|^{\alpha/n} \|f\|_{Q,\Phi}, \quad \forall\, x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes Q in $\mathbb{R}^n$ containing x. Let $M_\Phi := M_{0,\Phi}$.

We now recall the following conclusion from [7, Lemma 5.28].

Lemma 2.4. *Let Φ be a Young function. Then, for any cube Q, one has*

$$M_\Phi(f\chi_Q)(x) = \sup_{x \in R \subset Q} \left\|f\right\|_{\Phi,R}, \quad \forall\, x \in Q,$$

where the supremum is taken over all cubes R containing x with $R \subset Q$.

Adapting the proof of [8, Theorem 3.3], we further have the following conclusion.

Lemma 2.5. *Assume that $\alpha \in (0, n)$, $p \in (1, \alpha)$, $\lambda \in [0, 1)$, $1/q = 1/p - \alpha/n$ and $\Phi \in B_{p,q}$. Then there exists a positive constant $\widetilde{C}$ such that, for any $f \in L^{p,\lambda}(\mathbb{R}^n)$,*

$$\left\|M_{\alpha,\Phi}(f)\right\|_{L^{q,\lambda}(\mathbb{R}^n)} \le \widetilde{C} \left(\int_0^\infty \frac{[\Phi(t)]^{q/p}}{t^q} \frac{dt}{t} \right)^{\frac{1}{q}} \left\|f\right\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Proof. For any $s \in (0, \infty)$, let $\Phi_p(s) := \Phi(s^{1/p})$. By a result in [8, p.443], we know that

$$\||f|^p\|_{Q,\Phi_p} = \|f\|_{Q,\Phi}^p$$

and hence

$$[M_{\alpha,\Phi} f]^q = [M_{p\alpha,\Phi_p}(|f|^p)]^{q/p}.$$

From [7, Lemma 5.49], it follows that

$$\left|\{x \in Q : M_{\alpha p,\Phi_p}(|f|^p)(x) > s\}\right|^{\frac{n-p\alpha}{n}} \lesssim \int_{\{x \in Q:\ |f(x)|^p > s/c)\}} \Phi_p\left(\frac{|f(x)|^p}{s} \right) dx,$$

which, combined with the Minkowski inequality and the change of variables $t = \frac{|f(x)|}{s^{1/p}}$,

$$\left(\frac{1}{|Q|^\lambda} \int_Q [M_{\alpha,\Phi} f(x)]^q\,dx \right)^{\frac{1}{q}} = \left(\frac{1}{|Q|^\lambda} \int_Q [M_{p\alpha,\Phi_p}(|f|^p)(x)]^{q/p}\,dx \right)^{\frac{1}{q}}$$

$$= \left(\frac{1}{|Q|^{\lambda}} \int_0^{\infty} s^{q/p} \left| \{x \in Q : M_{p\alpha,\Phi_p}(|f|^p)(x) > s\} \right| \frac{ds}{s} \right)^{\frac{1}{q}}$$
$$\lesssim \left\{ \frac{1}{|Q|^{\lambda}} \int_0^{\infty} s^{q/p} \left[\int_{\{x\in Q:\ |f(x)|^p > s/c\}} \Phi_p \left(\frac{|f(x)|^p}{s} \right) dx \right]^{q/p} \frac{ds}{s} \right\}^{\frac{1}{q}}$$
$$\lesssim \left\{ \frac{1}{|Q|^{\lambda}} \int_Q \left[\int_0^{c|f(x)|^p} \left\{ \Phi \left(\frac{|f(x)|}{s^{1/p}} \right) \right\}^{q/p} s^{q/p} \frac{ds}{s} \right]^{p/q} dx \right\}^{\frac{1}{p}}$$
$$\lesssim \left\{ \frac{1}{|Q|^{\lambda}} \int_Q \left[\int_0^{\infty} [\Phi(t)]^{q/p} \left(\frac{|f(x)|}{t} \right)^q \frac{dt}{t} \right]^{p/q} dx \right\}^{\frac{1}{p}}$$
$$\sim \left(\int_0^{\infty} \frac{[\Phi(t)]^{q/p}}{t^q} \frac{dt}{t} \right)^{\frac{1}{q}} \left(\frac{1}{|Q|^{\lambda}} \int_Q |f(x)|^p \right)^{\frac{1}{p}}.$$

Take the supremum over all cubes Q in $\mathbb{R}^n$ on both sides of the above inequality. We finish the proof of Lemma 2.5. □

Remark 2.6. *When $\lambda = 0$, Lemma 2.5 returns to [8, Theorem 3.3].*

The following conclusion is taken from [10, Theorem 3.1].

Lemma 2.7. *Assume that $p \in (1, \infty)$, A, B and C are Young functions which satisfy*

$$B^{-1}(t)C^{-1}(t) \le A^{-1}(t), \quad \forall\, t \in (0, \infty), \quad \text{and} \quad C \in B_p \text{ is doubling}. \tag{2.3}$$

If the weight u satisfies

$$\sup_{Q \text{ cube}} \left(\frac{1}{|Q|} \int_Q u(x)\, dx \right)^{\frac{1}{p}} \|u^{-\frac{1}{p}}\|_{B,Q} < \infty,$$

then, there is a positive constant $\widetilde{C}$ such that

$$\left\| M_A(f) \right\|_{L^p_u(\mathbb{R}^n)} \le \widetilde{C} \left\| f \right\|_{L^p_u(\mathbb{R}^n)}, \quad \forall\, f \in L^p_u(\mathbb{R}^n). \tag{2.4}$$

In particular, M_A is bounded on $L^p(\mathbb{R}^n)$.

As an easy consequence of Lemma 2.7, we have the following conclusion.

Corollary 2.8. *Let $p \in (1, \infty)$,$\lambda \in (0, 1)$ and A be a Young function. Then there exists a positive constant $\widetilde{C}$ such that, for any $f \in X^{p,\lambda}(\mathbb{R}^n)$,*

$$\left\| M_A(f) \right\|_{X^{p,\lambda}(\mathbb{R}^n)} \le \widetilde{C} \left\| f \right\|_{X^{p,\lambda}(\mathbb{R}^n)}.$$

Proof. By [2, Theorem 5.5], we know that $X^{p,\lambda}(\mathbb{R}^n)$ is equivalent to

$$H^{p,\lambda}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n), \|f\|_{H^{p,\lambda}(\mathbb{R}^n)} := \inf_w \left[\int_{\mathbb{R}^n} |f(x)|^p [w(x)]^{1-p}\, dy \right]^{\frac{1}{p}} < \infty \right\},$$

where the infimum is taken over all weights $w \in A_1(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} w(x)\, d\Lambda^{\lambda n}(x) = \int_0^{\infty} \Lambda^{\lambda n} \left(\{x \in \mathbb{R}^n : w(x) > t\} \right) dt \le 1.$$

From this, (2.4) with $u = w^{1-p}$, $B(t) = t^{p'}$ for any $t \in (0, \infty)$, and $C(t)$ satisfying (2.3), we deduce that

$$\left\| M_A(f) \right\|_{X^{p,\lambda}(\mathbb{R}^n)} \sim \left\| M_A(f) \right\|_{H^{p,\lambda}(\mathbb{R}^n)} \lesssim \left\| f \right\|_{H^{p,\lambda}(\mathbb{R}^n)} \sim \left\| f \right\|_{X^{p,\lambda}(\mathbb{R}^n)},$$

as desired. □

3. Proofs of Theorems 1.12 and 1.13

In this section, we aim to prove Theorems 1.12 and 1.13. First, we show Theorem 1.12.

Proof. [Proof of Theorem 1.12] (i) By Lemma 2.1, it suffices to show that, for any $j \in \{1, \ldots, 3^n\}$,

$$\left\| \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma) \right\|_{L^{q,\lambda}(u,u)} + \left\| (\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f\sigma) \right\|_{L^{q,\lambda}(u,u)} \lesssim \|f\|_{L^{p,\lambda p/q}(\sigma,u)}.$$

We first show that, for any general dyadic lattice $\mathcal{D}$ and $j \in \{1, \ldots, 3^n\}$,

$$\sup_{R\in\mathcal{D}} \frac{1}{[u(R)]^{\lambda/q}} \left\| \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)\chi_R \right\|_{L^q_u(\mathbb{R}^n)} \lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)}. \tag{3.1}$$

Fix an $R \in \mathcal{D}$. By duality $(L^{q'}_u(\mathbb{R}^n))^* = L^q_u(\mathbb{R}^n)$, we have

$$\begin{aligned}
&\frac{1}{[u(R)]^{\lambda/q}} \left\| \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)\chi_R \right\|_{L^q_u(\mathbb{R}^n)} \qquad (3.2)\\
&= \sup_{\|g\|_{L^{q'}_u(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \left| \int_{\mathbb{R}^n} \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)(x) g(x) u(x)\, \chi_R(x)\, dx \right| \\
&= \sup_{\|g\|_{L^{q'}_u(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \int_{\mathbb{R}^n} \sup_{Q\in\mathcal{S}_j} |Q|^{\alpha/n-1} \int_Q |b(y)-b_Q|^m |f(y)|\sigma(y)\, dy \\
&\quad \times \chi_Q(x)\chi_R(x)|g(x)|u(x)\, dx \\
&\le \sup_{\|g\|_{L^{q'}_u(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \int_{\mathbb{R}^n} \sum_{Q\in\mathcal{S}_j} |Q|^{\alpha/n-1} \int_Q |b(y)-b_Q|^m |f(y)|\sigma(y)\, dy \\
&\quad \times \chi_Q(x)\chi_R(x)|g(x)|u(x)\, dx \\
&\le \sup_{\|g\|_{L^{q'}_u(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} |Q|^{\alpha/n-1} \int_Q |b(y)-b_Q|^m |f(y)|\sigma(y)\, dy \int_{Q\cap R} |g(x)|u(x)\, dx \\
&\quad + \sup_{\|g\|_{L^{q'}_u(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} \cdots \\
&=: \mathrm{I}_1 + \mathrm{I}_2.
\end{aligned}$$

Choose $r \in (1, p)$. From the Hölder inequality, the facts that, for any measurable function g on $\mathbb{R}^n$ and $u \in A_\infty$,

$$\frac{1}{u(Q)} \int_Q |g(y)|u(y)\, dy \le \inf_{Q\ni x} M_u(g)(x),$$

where the weighted maximal function $M_u(g)$ for any suitable function g is defined by

$$M_u(g)(x) := \sup_{Q\ni x} \frac{1}{u(Q)} \int_Q |g(y)|u(y)\, dy$$

with the supremum taking over all cubes $Q \subset \mathbb{R}^n$,

$$u(Q) \lesssim u(E(Q))$$

and (1.2), it follows that

$$\mathrm{I}_1 \le \sup_{\|g\|_{L^{q'}_u(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} |Q|^{\alpha/n-1} \left[\int_Q |b(y)-b_Q|^{mr'} \sigma(y)\, dy \right]^{1/r'}$$

$$\times\left[\int_Q |f(y)|^r\chi_R(y)\sigma(y)\,dy\right]^{1/r} u(Q)\inf_{Q\ni x} M_u(g)(x)$$
$$\lesssim \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\sum_{Q\in\mathcal{S}_j,Q\subset R}|Q|^{\alpha/n-1}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}[\sigma(Q)]^{1/r'}[\sigma(Q)]^{1/r}\left[\inf_{Q\ni x}M_\sigma\left(|f|^r\chi_R\right)(x)\right]^{1/r}$$
$$\times\, u(Q)\inf_{Q\ni x}M_u(g)(x)$$
$$\lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\sum_{Q\in\mathcal{S}_j,Q\subset R}[\sigma(Q)]^{1/p}\left\{\inf_{Q\ni x}\left[M_\sigma\left(|f|^r\chi_R\right)(x)\right]^{p/r}\right\}^{1/p}$$
$$\times\,[u(Q)]^{1/q'}\inf_{Q\ni x}M_u(g)(x)$$
$$\lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\sum_{Q\in\mathcal{S}_j,Q\subset R}[\sigma(E(Q))]^{1/p}\left\{\inf_{E(Q)\ni x}\left[M_\sigma\left(|f|^r\chi_R\right)(x)\right]^{p/r}\right\}^{1/p}$$
$$\times\,[u(E(Q))]^{1/q'}\inf_{E(Q)\ni x}M_u(g)(x),$$

which, combined with $p' > q'$ and the boundedness of M_u on $L_u^p(\mathbb{R}^n)$ (see, for example, [5, (3.6)]) for any $u \in A_\infty$ and $p \in (1,\infty)$, further implies that

$$\mathrm{I}_1 \lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\left\{\sum_{Q\in\mathcal{S}_j,Q\subset R}\sigma(E(Q))\inf_{E(Q)\ni x}\left[M_\sigma\left(|f|^r\chi_R\right)(x)\right]^{p/r}\right\}^{1/p}$$
$$\times\left\{\sum_{Q\in\mathcal{S}_j,Q\subset R}[u(E(Q))]^{p'/q'}\inf_{E(Q)\ni x}\left[M_u(g)(x)\right]^{p'}\right\}^{1/p'}$$
$$\lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\left\{\sum_{Q\in\mathcal{S}_j,Q\subset R}\int_{E(Q)}\left[M_\sigma\left(|f|^r\chi_R\right)(x)\right]^{p/r}\sigma(x)\,dx\right\}^{1/p}$$
$$\times\left\{\sum_{Q\in\mathcal{S}_j,Q\subset R}\int_{E(Q)}\left[M_u(g)(x)\right]^{q'}u(x)\,dx\right\}^{1/q'}$$
$$\lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\left\|M_\sigma\left(|f|^r\chi_R\right)\right\|^{1/r}_{L_\sigma^{p/r}(\mathbb{R}^n)}\left\|M_u(g)\right\|_{L_u^{q'}(\mathbb{R}^n)}$$
$$\lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1}\frac{1}{[u(R)]^{\lambda/q}}\left\|f\chi_R\right\|_{L_\sigma^{p}(\mathbb{R}^n)}\|g\|_{L_u^{q'}(\mathbb{R}^n)}$$
$$\lesssim c_{(u,\sigma)}\|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)}\|f\|_{L^{p,\lambda p/q}(\sigma,u)}.$$

As for I_2, by (1.3), we know that

$$|Q|^{\frac{\alpha}{n}-1} \lesssim c_{(u,\sigma)}[u(Q)]^{-\frac{1}{q}}[\sigma(Q)]^{-\frac{1}{p'}}, \quad \forall\, Q\in\mathcal{D},$$

which, together with the Hölder inequality, (1.2) and a result used in the estimation of J_2 in [20, Theorem 1.2]

$$\sum_{Q\in\mathcal{S}_j,R\subset Q}\left[\frac{u(R)}{u(Q)}\right]^{(1-\lambda)/q} < \infty,$$

implies that

$$
\begin{aligned}
\mathrm{I}_2 \leq & \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} |Q|^{\alpha/n-1} \left[\int_Q |b(y)-b_Q|^{mp'}\sigma(y)\,dy\right]^{1/p'} \\
& \times \left[\int_Q |f(y)|^p \sigma(y)\,dy\right]^{1/p} \left[\int_R |g(x)|^{q'} u(x)\,dx\right]^{1/q'} [u(R)]^{1/q} \\
\lesssim & \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} |Q|^{\alpha/n-1} [\sigma(Q)]^{1/p'} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} [u(Q)]^{\lambda/q} \|f\|_{L^{p,\lambda p/q}(\sigma,u)} [u(R)]^{1/q} \\
\lesssim & c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)} \sum_{Q\in\mathcal{S}_j, R\subset Q} \left[\frac{u(R)}{u(Q)}\right]^{(1-\lambda)/q} \\
\lesssim & c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)}.
\end{aligned}
$$

This, combined with (3.2) and the estimate of I_2, implies (3.1).

Moreover, by the proof of [20, Theorem 1.1], we know that, for any cube R in $\mathbb{R}^n$, there exist $\{R_k\}_{k=1}^{2^n} \subset \mathcal{D}$ such that $R \subset \bigcup_{k=1}^{2^n} R_k$ and $l(R) \leq l(R_k) \leq 2l(R)$ for any $k \in \{1, \ldots, 2^n\}$. Then $|R| \sim |R_k|$ for any $k \in \{1, \ldots, 2^n\}$. From $u \in A_\infty$ and , it follows that $u(R) \sim u(R_k), \forall k \in \{1, ..., 2^n\}$, which shows that

$$
\frac{1}{[u(R)]^{\lambda/q}} \left\| \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)\chi_Q \right\|_{L_u^q(\mathbb{R}^n)} \lesssim \sum_{k=1}^{2^n} \frac{1}{[u(R_k)]^{\lambda/q}} \left\| \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)\chi_{R_k} \right\|_{L_u^q(\mathbb{R}^n)}.
$$

Since $\{R_k\}_{k=1}^{2^n} \subset \mathcal{D}$, by (3.1), we have

$$
\frac{1}{[u(R)]^{\lambda/q}} \left\| \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)\chi_R \right\|_{L_u^q(\mathbb{R}^n)} \lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)}.
$$

Taking the supremum over all cubes R in $\mathbb{R}^n$ in the above equation, we get

$$
\|\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f\sigma)\|_{L^{q,\lambda}(u,u)} \lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)},
$$

which proves (3.1).

Now we prove that, for any general dyadic lattice $\mathcal{D}$ and $j \in \{1, \ldots, 3^n\}$,

$$
\sup_{R\in\mathcal{D}} \frac{1}{[u(R)]^{\lambda/q}} \left\| (\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f\sigma)\chi_R \right\|_{L_u^q(\mathbb{R}^n)} \lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)}. \tag{3.3}
$$

Similar to (3.2), we obtain

$$
\begin{aligned}
& \frac{1}{[u(R)]^{\lambda/q}} \left\| (\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f\sigma)\chi_R \right\|_{L_u^q(\mathbb{R}^n)} \\
&= \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \int_{\mathbb{R}^n} (\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f\sigma)(x) g(x) u(x) \chi_R(x)\,dx \\
&= \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \int_{\mathbb{R}^n} \sup_{Q\in\mathcal{S}_j} |Q|^{\alpha/n-1} |b(x)-b_Q|^m \int_Q |f(y)\sigma(y)|\,dy\, \chi_Q(x)\chi_R(x) g(x) u(x)\,dx \\
&\leq \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} |Q|^{\alpha/n-1} \int_Q |f(y)\sigma(y)|\,dy \int_Q |b(x)-b_Q|^m g(x) u(x) \chi_R(x)\,dx \\
&\quad + \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} \cdots
\end{aligned}
$$

$=: \mathrm{I}_3 + \mathrm{I}_4.$

Choose $r \in (q, \infty)$. By some arguments similar to those used in the estimation of I_1, we find that

$$\begin{aligned}
\mathrm{I}_3 &\leq \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} |Q|^{\alpha/n-1} \left[\int_Q |b(y)-b_Q|^{mr} u(y)\,dy\right]^{1/r} \\
&\quad \times \left[\int_Q |g(y)|^{r'} \chi_R(y) u(y)\,dy\right]^{1/r'} \sigma(Q) \inf_{Q\ni x} M_\sigma(f\chi_R)(x) \\
&\lesssim \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} |Q|^{\alpha/n-1} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} [u(Q)]^{1/r} [u(Q)]^{1/r'} \left[\inf_{Q\ni x} M_\sigma\left(|g|^{r'}\right)(x)\right]^{1/r'} \\
&\quad \times \sigma(Q) \inf_{Q\ni x} M_\sigma(f\chi_R)(x) \\
&\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} [u(Q)]^{1/q'} \left\{\inf_{Q\ni x}\left[M_\sigma\left(|g|^{r'}\right)(x)\right]^{q'/r'}\right\}^{1/q'} \\
&\quad \times [\sigma(Q)]^{1/p} \inf_{Q\ni x} M_\sigma(f\chi_R)(x) \\
&\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, Q\subset R} [u(E(Q))]^{1/q'} \left\{\inf_{E(Q)\ni x}\left[M_u\left(|g|^{r'}\right)(x)\right]^{q'/r'}\right\}^{1/q'} \\
&\quad \times [\sigma(E(Q))]^{1/p} \inf_{E(Q)\ni x} M_\sigma(f\chi_R)(x)
\end{aligned}$$

and hence

$$\begin{aligned}
\mathrm{I}_3 &\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \left\{\sum_{Q\in\mathcal{S}_j, Q\subset R} [u(E(Q))]^{p'/q'} \inf_{E(Q)\ni x}\left[M_u\left(|g|^{r'}\right)(x)\right]^{p'/r'}\right\}^{1/p'} \\
&\quad \times \left\{\sum_{Q\in\mathcal{S}_j, Q\subset R} \sigma(E(Q)) \inf_{E(Q)\ni x} [M_\sigma(f\chi_R)(x)]^p\right\}^{1/p} \\
&\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \left\{\sum_{Q\in\mathcal{S}_j, Q\subset R} \int_{E(Q)} \left[M_u\left(|g|^{r'}\right)(x)\right]^{q'/r'} u(x)\,dx\right\}^{1/q'} \\
&\quad \times \left\{\sum_{Q\in\mathcal{S}_j, Q\subset R} \int_{E(Q)} [M_\sigma(f\chi_R)(x)]^p \sigma(x)\,dx\right\}^{1/p} \\
&\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \left\|M_u\left(|g|^{r'}\right)\right\|^{1/r'}_{L_u^{q'/r'}(\mathbb{R}^n)} \left\|M_\sigma(f\chi_R)\right\|_{L_\sigma^p(\mathbb{R}^n)} \\
&\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \left\|f\chi_R\right\|_{L_\sigma^p(\mathbb{R}^n)} \|g\|_{L_u^{q'}(\mathbb{R}^n)} \\
&\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)}.
\end{aligned}$$

Moreover, from the same arguments as those used in the estimation of I_2, it follows that

$$\mathrm{I}_4 = \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} |Q|^{\alpha/n-1} \int_Q |f(y)|\sigma(y)\,dy \int_R |b(x)-b_Q|^m g(x) u(x)\,dx$$

$$\leq \sup_{\|g\|_{L_u^{q'}(\mathbb{R}^n)}=1} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} |Q|^{\alpha/n-1} \int_Q |f(y)|^p \sigma(y)\,dx [\sigma(Q)]^{1/p'}$$

$$\times \left[\int_R |b(x)-b_Q|^{mq} u(x)\,dx\right]^{1/q} \left[\int_R |g(x)|^{q'} u(x)\,dx\right]^{1/q'}$$

$$\lesssim c_{(u,\sigma)} \frac{1}{[u(R)]^{\lambda/q}} \sum_{Q\in\mathcal{S}_j, R\subset Q} [u(Q)]^{-1/q}[u(Q)]^{\lambda/q} \|f\|_{L^{p,\lambda p/q}(\sigma,u)} \|b\|^m_{\mathrm{BMO}_u(\mathbb{R}^n)} [u(R)]^{1/q}$$

$$\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)} \sum_{Q\in\mathcal{S}_j, R\subset Q} \left[\frac{u(R)}{u(Q)}\right]^{(1-\lambda)/q}$$

$$\lesssim c_{(u,\sigma)} \|b\|^m_{\mathrm{BMO}_\sigma(\mathbb{R}^n)} \|f\|_{L^{p,\lambda p/q}(\sigma,u)},$$

which, combined with the estimate of I_4 and the final arguments used in the proof of (3.1), completes the proof of (3.3) and hence of Theorem 1.12. □

Then we begin to prove Theorem 1.13.

Proof. [Proof of Theorem 1.13] The conclusion (ii) is essentially proved in [31, Theorem 1.1(2)], we omit the details here.

To prove (i), by Lemma 2.1, it suffices to show that, for any $j \in \{1,\ldots,3^n\}$,

$$\left\|\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f)\right\|_{L^{q,\lambda}(u)} + \left\|(\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f)\right\|_{L^{q,\lambda}(u)} \lesssim \|f\|_{L^{p,\lambda}(\sigma)}.$$

We now prove that, for any $j \in \{1,\ldots,3^n\}$,

$$\left\|\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f)\right\|_{L^{q,\lambda}(u)} \lesssim \|f\|_{L^{p,\lambda}(\sigma)}. \tag{3.4}$$

Let $\beta \in [0,n)$ satisfy $1/q = 1/p - \beta/n$. By Theorem 2.3, (1.1), Lemma 2.5 and Corollary 2.8, we conclude that

$$\left\|\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f)\right\|_{L^{q,\lambda}(u)}$$

$$= \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\leq 1} \left|\int_{\mathbb{R}^n} \mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha} f(x) g(x) u(x)\,dx\right|$$

$$\leq \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\leq 1} \sum_{Q\in\mathcal{S}_j} |Q|^{\alpha/n-1} \int_Q |b(y)-b_Q|^m |f(y)|\,dy \int_Q g(x)u(x)\,dx$$

$$\lesssim \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\leq 1} \sum_{Q\in\mathcal{S}_j} |Q|^{1+\frac{\alpha}{n}} \left\|f\sigma^{1/p}\right\|_{B^*,Q} \left\|(b-b_Q)^m \sigma^{-1/p}\right\|_{B,Q} \left\|u^{1/q}\right\|_{A,Q} \|gu^{1/q'}\|_{A^*,Q}$$

$$\lesssim \widetilde{C}_{(u,\sigma,b)} \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\leq 1} \sum_{Q\in\mathcal{S}_j} |Q|^{1+\frac{\beta}{n}} \left\|f\sigma^{1/p}\right\|_{B^*,Q} \|gu^{1/q'}\|_{A^*,Q},$$

which, together with Lemma 2.5 and Corollary 2.8, implies that

$$\left\|\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha}(f)\right\|_{L^{q,\lambda}(u)}$$

$$\lesssim \widetilde{C}_{(u,\sigma,b)} \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\leq 1} \sum_{Q\in\mathcal{S}_j} |E(Q)||Q|^{\frac{\beta}{n}} \left\|f\sigma^{1/p}\right\|_{B^*,Q} \|gu^{1/q'}\|_{A^*,Q}$$

$$\lesssim \widetilde{C}_{(u,\sigma,b)} \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\leq 1} \int_{\mathbb{R}^n} M_{\beta,B^*}(f\sigma^{1/p})(x) M_{A^*}(gu^{1/q'})(x)\,dx$$

$$
\begin{aligned}
&\lesssim \widetilde{C}_{(u,\sigma,b)} \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\le 1} \left\|M_{\beta,B^*}(f\sigma^{1/p})\right\|_{L^{q,\lambda}(\mathbb{R}^n)} \left\|M_{A^*}(gu^{1/q'})\right\|_{X^{q',\lambda}(\mathbb{R}^n)}\\
&\lesssim \widetilde{C}_{(u,\sigma,b)} \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\le 1} \left\|f\sigma^{1/p}\right\|_{L^{p,\lambda}(\mathbb{R}^n)} \left\|gu^{1/q'}\right\|_{X^{q',\lambda}(\mathbb{R}^n)}\\
&\sim \widetilde{C}_{(u,\sigma,b)} \sup_{\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)}\le 1} \|f\|_{L^{p,\lambda}(\sigma)}\|g\|_{X_u^{q',\lambda}(\mathbb{R}^n)} \lesssim \widetilde{C}_{(u,\sigma,b)} \left\|f\right\|_{L^{p,\lambda}(\sigma)}.
\end{aligned}
$$

From some arguments similar to those used in the estimation of (3.4), we deduce that

$$
\|(\mathcal{I}^{b,m}_{\mathcal{S}_j,\alpha})^*(f)\|_{L^{q,\lambda}(u)} \lesssim \|f\|_{L^{p,\lambda}(\sigma)},
$$

which shows (i) and the desired result. □

Acknowledgements. The authors would like to thank the anonymous referee for his / her several enlightening comments on this article, which do improve the presentation of this article.

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