Filomat 38:10 (2024), 3635–3637
https://doi.org/10.2298/FIL2410635Z

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Random approach to Tribonacci identity

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Abstract. This paper gives an expansion of the work of Benjamin, Arthur T., et al in "random approached to Fibonacci identities" to a random approach to Tribonacci Identity.

1. Introduction

In an original work conducted by Benjamin, Arthur T., et al. in [1] where, adopting a probabilistic perspective, they show that many Fibonacci identities can be explained combinatorially. Building upon their research, we aim to further investigate this topic by presenting an expansion based on Binet's formula for T_n the $n-th$ number of the Tribonacci sequence [2].

The model presented in this study serves to generate random tilings where each tiling of length n depends only on n.

To begin, we establish a combinatorial interpretation of Tribonacci numbers. Let c_n denote the number of series 1's, 2's and 3's that sum to n. We observe that $c_n = T_{n+1}$, where T_n denotes the Tribonacci number. Specifically, we find that $c_1 = 1 = T_2$, $c_2 = 1 = T_3$, $c_3 = 2 = T_4$, and by considering the initial element in the series, $c_n = c_{n-1} + c_{n-2} + c_{n-3}$. Hence, for $n \geq 1$, the Tribonacci number T_n can be defined combinatorially as the number of ways to tile a board of length $n-1$ using squares, dominoes and triominoes.

For the knowledge of the reader, our tiling is taking place on a board of size $1 \times n$, where n is a natural number in $\mathbb{N}$.

2. Bicolored Model

The studied model involves an infinite board containing cells numbered as 1, 2, and so on. In this model, each cell is independently colored either black or white, with a probability of 1/2 for each color. An example of such coloring is illustrated in Figure 1. It is important to note that for any coloring of the first n cells, the probability of such a coloring is $(1/2)^n$.

An infinite tiling is defined as alternating strings having black and white cells of varying lengths.

For example, the tiling in Figure 1 has a black string of length 7 followed by a white string of length 2 followed by a black string of length 5 and so on.

2020 *Mathematics Subject Classification*. Primary 60C05, 11B39, 05B45, 11B50, 60E05.
Keywords. Tribonacci identity, tilings, random approach.
Received: 26 July 2023; Accepted: 25 October 2023
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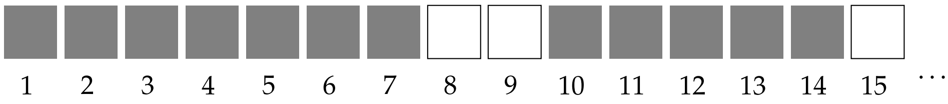

Figure 1: A random black and white board.

Let X be the random variable denoting the location of the end of the first black string of length of the form $3k+2$ where k non negative integer. In our example, $X = 14$. Our objective is to determine the probability of $X = n$ for $n \geq 1$. The theorem presented below provides the answer to this question.

Theorem 2.1. *Let X be the random variable defined previously, for $n \geq 1$ the probability that $X = n$ is*

$$P(X = n) = T_n/2^{n+1}. \tag{1}$$

As consequence we have the following combinatorial identity

$$\sum_{n\geq 1} \frac{T_n}{2^n} = 2. \tag{2}$$

Proof. A tiling is categorized as having A tiling has $X = n$ if and only if the following conditions are satisfied: both cells n and $n-1$ are covered by a black domino, cell $n+1$ is covered by a white square, and cells 1 through $n-2$ can be covered by white or black squares and black triominoes. The number of ways to tile the first $n-2$ cells in this manner is T_n. Therefore, the probability $P(X = n)$ is given by $T_n/2^{n+1}$. Where the sum of all probabilities is equal to 1. □

By employing a similar approach, we can provide an explanation for the following theorem.

Theorem 2.2. *Let X be the random variable defined previously, for $n \geq 1$ the expected value of X is*

$$E(X) = \sum_{n\geq 1} n\frac{T_n}{2^{n+1}} = 13. \tag{3}$$

Proof. To prove that $E(X) = 13$, we express X as the sum of three random variables,

$$X = 2B + 3T + R. \tag{4}$$

B is a geometric random variable with mean 2 where $2B$ denotes the location of the first of the two black squares. T is the number of black triominoes, which refers to pairs of three black squares, that directly succeed the first of the two black squares. The random variable $T+1$ follows a geometric distribution as well, with a mean of $\frac{8}{7}$. Finally, the variable R represents the remaining number of white and black tiles required to reach the end of the first black string with a length in the form of $3k+2$ where k is non negative integer. (For the given tiling illustrated in Figure 1, we have $B = 1$, $T = 1$, and $R = 9$.) The colors of the three squares that cover cells $2B+3T+1$, $2B+3T+2$ and $2B+3T+3$ are randomly assigned with equal probability from the following options: white-white-white, white-white-black, white-black-white, white-black-black, black-white-white, black-white-black, or black-black-white. In the first four cases, where the color combinations are white-white-white, white-white-black, white-black-white, and white-black-black, the value of R is 0. This is because cell $2B+3T$ marks the end of a black string with a length $3k+2$, where k is a non-negative integer. In the fifth and sixth cases, where the color combinations are black-white-white and black-white-black, the cell $2B+3T+1$ marks the end of a black string with a length that is a multiple of 3. The cell $2B+3T+2$ contains a white square, indicating that we return to the starting point or "drawing board". In the last case, when the color combination is black-black-white, the cell $2B+3T+2$ marks the end of a black string with a length in the form of $3k+1$, where k is a non-negative integer. The cell $2B+3T+3$ contains a white square, indicating that we return to the starting point or "drawing board".

Therefore, by applying the principle of linearity of expectation,

$$\begin{aligned} E(X) &= 2E(B) + 3E(T) + E(R) \\ &= 4 + 3(\frac{8}{7} - 1) + \frac{4}{7}(0) + \frac{2}{7}(2 + E(X)) + \frac{2}{7}(2 + E(X)) \end{aligned} \tag{5}$$

Solving the equation using the principle of linearity of expectation yields the value $E(X) = 13$, as desired. □

From an analytical standpoint, it is indeed possible to evaluate the variance and moments of order k for the random variable X. However, the approach presented here may not be sufficient to extend the calculation of the second moment. Additional methods or techniques may be required to calculate the moment of order 2 for X.

This concludes our expansion of the work of Benjamin, Arthur T., et al and confirms again Mach's citation to the Tribonacci.

References

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