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On adjacent eccentric distance sum index

Mingqiang An[a]

[a]*College of Science, Tianjin University of Science and Technology, Tianjin, 300457, P.R. China*

Abstract. For a connected graph H, the adjacent eccentric distance sum index (AEDSI) is defined as

$$\xi^{sv}(H) = \sum_{v_x \in V(H)} \frac{\varepsilon_H(v_x) \cdot D_H(v_x)}{\deg_H(v_x)},$$

where $\varepsilon_H(v_x)$ denotes the eccentricity of the vertex v_x, $\deg_H(v_x)$ is the degree of v_x and $D_H(v_x) = \Sigma_{v_y \in V(H)} d(v_x, v_y)$ is the sum of all distances from v_x in H. AEDSI is proven to be very helpful on predicting anti-HIV activity. In this paper, we give a best possible upper bound on the AEDSI of H with given radius that guarantees H is ω-connected, β-deficient, ω-Hamiltonian, ω-path-coverable and ω-edge-Hamiltonian, respectively. This supplies a continuation of the results presented by Feng et al. (2017).

1. Introduction

We study simple, undirected, connected and finite graphs throughout this paper. Let H be a graph with vertex set $V(H) = \{v_1, v_2, \ldots, v_p\}$, i.e., $p = |V(H)|$. For a vertex $v_s \in V(H)$, the degree $\deg_H(v_s)\,(= d_s)$ of v_s is the number of edges incident with v_s in H and denote by $(d_1, d_2, \ldots, d_p)$ the *degree sequence* of H with $d_1 \le d_2 \le \cdots \le d_p$. A degree sequence $\pi = (d_1, d_2, \ldots, d_p)$ is called *graphical* if there exists a graph H having π as its vertex degree sequence. Let $d_H(v_s, v_t)$ be the distance between two vertices v_s and v_t in H, and $D_H(v_s) = \Sigma_{v_t \in V(H)} d(v_s, v_t)$ the sum of all distances from the vertex v_s in H. The *eccentricity* $\varepsilon_H(v_s)$ of v_s is the maximum distance from v_s to any other vertex in H. The *radius* $r(H)$ of H is the minimum eccentricity among the vertices of H. We write $\alpha(H)$ for the independence number of H. In the following context, we usually delete the footnote H from the symbols if there is no ambiguity, and refer the reader to [10] for undefined notation and terminologies.

We shall use $H_1 + H_2$, $H_1 \nabla H_2$ to denote the union, the join of two vertex-disjoint graphs H_1 and H_2, respectively. We use K_p to denote the complete graph of order p, and by $\overline{H}$ we denote the complement of H.

A connected graph H with more than ω vertices, remains connected whenever fewer than ω vertices are removed, then H is said to be *ω-connected*. The *deficiency* of H is the number of vertices that are not matched under a maximum matching in H, and is denoted by def(H). Particularly, H has a 1-factor if and only if def(H)= 0. We say H, β-deficient if def$(H)\le \beta$. For a connected graph H and $X \subseteq V(H)$ with $|X| \le \omega$, $\omega \in \mathbb{Z}^+$,

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Email address: anmq@tust.edu.cn (Mingqiang An)

if the subgraph induced by $V(H) \setminus X$ is Hamiltonian, then H is called to be *ω-Hamiltonian*. Thus the notions "0-Hamiltonian" and "Hamiltonian" are equivalent.

A path (cycle) with $|V(H)|$ vertices is called a *Hamiltonian path (cycle)* of H. If H contains a Hamiltonian path, then H is *traceable*. We say a graph H, *ω-path-coverable*, if the vertex set of H can be covered by ω or fewer vertex-disjoint paths. In particular, the notions "1-path-coverable" and "traceable" are equivalent. A graph H is *ω-edge-Hamiltonian* if any collection of pairwise vertex-disjoint paths altogether with at most ω edges belong to a Hamiltonian cycle in H.

Molecular structure descriptors, also regarded as topological indices, are used in theoretical chemistry to depict the characteristics of chemical compounds [17]. Up to now, a large number of topological indices have been found to have a wide range of practical applications [12], such as the connective eccentricity index (CEI) [15], eccentric connectivity index (ECI) [31] and eccentric distance sum (EDS) [16].

The *adjacent eccentric distance sum index* (AEDSI), which was first introduced in [29], is defined as

$$\xi^{sv}(H) = \sum_{v_x \in V(H)} \frac{\varepsilon_H(v_x) \cdot D_H(v_x)}{\deg_H(v_x)}.$$

On the above eccentricity-based topological indices one can refer to [6, 7, 14, 19, 28, 30] for more details.

According to the definition of $D_H(v_x)$, we get

$$D_H(v_x) \geq \deg(v_x) + 2\Big(n - 1 - \deg(v_x)\Big). \tag{1}$$

A popular research topic in graph theory is the study of whether a given graph has some important property (such as Hamiltonicity or traceability). It shows that [20] determining whether a graph has a Hamiltonian cycle is NP-complete. Although there are some literatures [5, 13, 18, 23–27, 33] using the bounds of topological indices to confirm the structure of graphs, there are still few results related to them. Recently, based on the first Zagreb index or reciprocal degree distance, the κ-connectivity, β-deficiency [2, 4], Hamiltonian-connectedness [1] and $\hbar$-Hamiltonicity, $\hbar$-path-coverability and $\hbar$-edge-Hamiltonicity [3] of graphs have been discussed. By employing the Wiener index, some vulnerability parameters (such as integrity, toughness, tenacity and binding number) of graphs have been studied [34]. However, it is still unknown to judge a graph with the above graph properties by means of eccentricity-based topological indices.

In this paper, we have partially solved the problems above, that is to say, we give a best possible upper bound on the AEDSI of a graph H with given radius that guarantees H is ω-connected, β-deficient, ω-Hamiltonian, ω-path-coverable and ω-edge-Hamiltonian, respectively.

In Section 2, we give some necessary lemmas. The major results and their proofs will be given in Section 3.

2. Preliminaries

In this section, some useful lemmas will be given.

Lemma 2.1. [9] *Let $(t_1, \ldots, t_p)$ be a graphical sequence with $1 \leq \omega \leq p - 1$ and $p \geq 2$. If*

$$t_j \leq \omega + j - 2 \Rightarrow t_{p+1-\omega} \geq p - j \text{ for } j \in \left[1, \frac{p - \omega + 1}{2}\right],$$

then any graph with this degree sequence is ω-connected.

Lemma 2.2. [32] *Let $(t_1, \ldots, t_p)$ be a graphical sequence, $0 \leq \beta \leq p$ and $p \equiv \beta \,(mod\ 2)$. If*

$$t_{j+1} \leq j - \beta \Rightarrow t_{p+\beta-j} \geq p - j - 1 \text{ for } j \in \left[1, \frac{p + \beta - 2}{2}\right],$$

then any graph with this degree sequence is β-deficient.

Lemma 2.3. *Let $(t_1, \dots, t_p)$ be a graphical sequence, $\omega \in [0, p-3]$.*
(i) [11] *If*

$$t_j \leq \omega + j \Rightarrow t_{p-j-\omega} \geq p - j \text{ for } j \in \left[1, \frac{p-\omega}{2}\right),$$

then any graph with this degree sequence is ω-hamiltonian.
(ii) [21] *If*

$$t_{j-\omega} \leq j \Rightarrow t_{p-j} \geq p + \omega - j \text{ for } j \in \left[\omega + 1, \frac{\omega + p}{2}\right),$$

then any graph with this degree sequence is ω-edge-hamiltonian.

Lemma 2.4. [8, 22] *Let $(t_1, \dots, t_p)$ be a graphical sequence with $\omega \geq 1$. If*

$$t_{j+\omega} \leq j \Rightarrow t_{p-j} \geq p - \omega - j \text{ for } j \in \left[1, \frac{p-\omega}{2}\right),$$

then any graph with this degree sequence is ω-path-coverable.

Lemma 2.5. [8] *Let $(t_1, \dots, t_p)$ be a graphical sequence with $\omega \geq 1$. If $t_{\omega+1} \geq p - \omega$, then any graph with this degree sequence satisfies $\alpha(H) \leq \omega$.*

3. Results

Now, we give a best possible upper bound on the AEDSI of a graph H with given radius that guarantees H is ω-connected, β-deficient, ω-Hamiltonian, ω-path-coverable and ω-edge-Hamiltonian, respectively.

We first denote $\mathcal{H}_1 = \left\{K_{\omega-1}\nabla(K_1 + K_{p-\omega}),\ K_{\frac{p-1}{2}} + K_{\frac{p+1}{2}},\ H_1\right\}$, $\mathcal{H}_2 = \left\{H_2,\ K_{\frac{p-2}{2}}\nabla(K_2 + \overline{K_{\frac{p-2}{2}}}),\ K_{\frac{p-1}{2}}\nabla\overline{K_{\frac{p+1}{2}}},\right\}$, $\mathcal{H}_3 = \left\{K_{\frac{p-2-\omega}{2}}\nabla(K_2 + \overline{K_{\frac{p-2+\omega}{2}}}),\ K_{\frac{p-\omega-1}{2}}\nabla(K_1 + \overline{K_{\frac{p+\omega-1}{2}}})\right\}$ and $\mathcal{H}_4 = \left\{H_2,\ K_{\frac{p-2+\omega}{2}}\nabla(K_2 + \overline{K_{\frac{p-2-\omega}{2}}}),\ K_{\frac{p+\omega-1}{2}}\nabla(K_1 + \overline{K_{\frac{p-\omega-1}{2}}})\right\}$, where p $(p > 2)$ is an integer, H_1 is the $\left(\frac{p}{2} - 1\right)$-regular graph and $H_2 = K_{\omega+1}\nabla(K_1 + K_{p-\omega-2})$.

Theorem 3.1. *Let H be a graph on $p \geq \omega + 1$ vertices such that $\omega \geq 1$ and with radius r. If $\xi^{sv}(H) \leq r\Gamma_1(p, \omega)$, then H is ω-connected if and only if $H \notin \mathcal{H}_1$, where*

$$\Gamma_1(p, \omega) = \begin{cases} \frac{3p^2-5p-4}{p-3} & \text{if } \omega = 1 \text{ and } p \text{ is odd,} \\ \frac{p(3p-2)}{p-2} & \text{if } \omega = 1 \text{ and } p \text{ is even,} \\ \frac{2(p-1)}{\omega-1} + \frac{2(p-1)(p-\omega)}{p-2} + 2\omega - p - 2 & \text{if } \omega \geq 2. \end{cases}$$

Proof. Sufficiency. Assume H is not ω-connected for some s in $1 \leq s \leq \frac{p-\omega+1}{2}$. Therefore according to Lemma 2.1, we know that $d_s \leq s + \omega - 2$ and $d_{p-\omega+1} \leq p - s - 1$. Notice that $1 \leq \omega \leq p - 1$. Then by inequality (1) and the fact that $\varepsilon(v_x) \geq r$, for any vertex $v_x \in V(H)$, we obtain

$$\begin{aligned} \xi^{sv}(H) &= \sum_{v_j \in V(H)} \frac{\varepsilon(v_j) \cdot D(v_j)}{\deg(v_j)} \geq r \sum_{j=1}^{p} \frac{d_j + 2\left(p - 1 - d_j\right)}{d_j} \\ &= 2r(p-1)\sum_{j=1}^{p} \frac{1}{d_j} - pr \\ &\geq 2(p-1)r\left(\frac{s}{s+\omega-2} + \frac{p-\omega-s+1}{p-s-1} + \frac{\omega-1}{p-1}\right) - pr. \end{aligned} \tag{2}$$

Now we consider the following function

$$f(x) = \frac{x}{x+\omega-2} + \frac{p-\omega-x+1}{p-x-1}$$

with $1 \le x \le \frac{1}{2}(p-\omega+1)$. So we get

$$f'(x) = (\omega-2)\Big(\frac{1}{(x+\omega-2)^2} - \frac{1}{(p-1-x)^2}\Big). \tag{3}$$

Since $1 \le x \le \frac{p-\omega+1}{2}$ and $\omega \ge 1$, we have $0 \le x+\omega-2 \le p-1-x$. Thus $\frac{1}{(x+\omega-2)^2} - \frac{1}{(p-1-x)^2} \ge 0$.

Case 1 : $\omega \ge 2$. Then by Eq. (3), $f'(x) \ge 0$, which follows that $f(x)$ is increasing on $x \in [1, \frac{p-\omega+1}{2}]$. So $f(x) \ge f(1)$, and

$$\xi^{sv}(H) \quad \ge \quad \Big(\frac{2(p-1)}{\omega-1} + \frac{2(p-1)(p-\omega)}{p-2} + 2\omega - p - 2\Big)r = r\Gamma_1(p,\omega).$$

In combination with the conditions of the theorem, the above inequality is true if and only if we take an equal sign. So $s = 1$, and correspondingly $d_1 = \omega - 1$, $d_2 = \cdots = d_{p+1-\omega} = p-2$, $d_{p-\omega+2} = \cdots = d_p = p-1$. Therefore $H \cong K_{\omega-1}\nabla(K_1 + K_{p-\omega})$, contrary to the assumption.

Case 2 : $\omega = 1$. Then from Eq. (3), $f'(x) \le 0$. Implying that $f(x)$ is decreasing on $x \in [1, \frac{p}{2}]$. So $f_{min}(x) = f(\lfloor \frac{p}{2} \rfloor)$ as x is an integer.

Subcase 2.1 : p is odd. Thus $f(x) \ge f(\frac{p-1}{2})$. Hence

$$\xi^{sv}(H) \ge r\Big(\frac{3p^2-5p-4}{p-3}\Big) = r\Gamma_1(p,\omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Thus $s = \frac{p-1}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{p-1}{2}} = \frac{1}{2}(p-3)$, $d_{\frac{p+1}{2}} = \cdots = d_p = \frac{p-1}{2}$. So $H \cong K_{\frac{p-1}{2}} + K_{\frac{p+1}{2}}$, a contradiction.

Subcase 2.2 : p is even. Thus $f(x) \ge f(\frac{p}{2})$. So

$$\xi^{sv}(H) \quad \ge \frac{rp(3p-2)}{p-2} = r\Gamma_1(p,\omega).$$

In combination with the conditions of the theorem, the above inequality is true if and only if we take an equal sign. Therefore $s = \frac{p}{2}$, and correspondingly $d_1 = \cdots = d_p = \frac{p}{2} - 1$. Thus $H \cong H_1$, contrary to the assumption. Hence H is ω-connected.

Conversely, suppose that $H \in \mathcal{H}_1$. Then one can check that H is not ω-connected. □

Theorem 3.2. *Let H be a graph on $p \ge 10$ vertices and with radius r and matching number ν, $p \equiv \beta (mod\, 2)$ and $0 \le \beta \le p$. If*

$$\xi^{sv}(H) \le \Big(\frac{2(p-1)(p+\beta+2)}{p-\beta-2} - \beta - 2\Big)r,$$

then H is β-deficient if and only if $H \ncong (p-\nu+1)K_1\nabla K_{\nu-1}$.

Proof. Sufficiency. Assume that H is not β-deficient. Therefore according to Lemma 2.2, there exists an integer s with $1 \le s \le \frac{p+\beta-2}{2}$ such that $d_{s+1} \le s-\beta$ and $d_{p+\beta-s} \le p-s-2$. Note that $\beta+1 \le s \le p-2$ as H is connected. So by inequality (2), we have

$$\begin{aligned}\xi^{sv}(H) &\ge 2r(p-1)\sum_{j=1}^{p}\frac{1}{d_j} - pr \\ &\ge 2(p-1)r\Big(\frac{s+1}{s-\beta} + \frac{p+\beta-2s-1}{p-s-2} + \frac{s-\beta}{p-1}\Big) - pr.\end{aligned}$$

We define

$$g(x) = \frac{x+1}{x-\beta} + \frac{p+\beta-2x-1}{p-x-2} + \frac{x-\beta}{p-1}$$

with $x \in [1, \frac{1}{2}(p-2+\beta)]$. Obviously $\beta+1 \le x \le p-2$. Then

$$g'(x) = \frac{(x-\beta)^2\Big[(p-2-x)^2 - (p-1)(p-\beta-3)\Big] - (p-1)(\beta+1)(p-2-x)^2}{(p-1)(x-\beta)^2(p-2-x)^2}.$$

Since $1 \le \beta+1 \le x \le p-2$, we have $0 \le p-2-x \le p-3 < p-1$ and $p-2-x \le p-\beta-3$. So $(p-2-x)^2 - (p-1)(p-\beta-3) < 0$, and consequently $g'(x) < 0$ when $x \in [1, \frac{p+\beta-2}{2}]$. Implying that $g(x)$ is decreasing on $x \in [1, \frac{p+\beta-2}{2}]$. Thus $g(x) \ge g(\frac{p+\beta-2}{2})$. So

$$\xi^{sv}(H) \ge \Big(\frac{2(p-1)(p+\beta+2)}{p-\beta-2} - \beta - 2\Big)r.$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Hence $s = \frac{p+\beta-2}{2}$. Note that $\frac{p-\beta}{2} = \nu$. Then $\beta = p-2\nu$ and $s = p-\nu-1$. So $d_1 = \cdots = d_{p-\nu+1} = \nu-1$, $d_{p-\nu+2} = \cdots = d_p = p-1$. Thus $H \cong (p-\nu+1)K_1 \nabla K_{\nu-1}$, contradicting the assumption. Therefore H is β-deficient.

Conversely, suppose that $H \cong (p-\nu+1)K_1 \nabla K_{\nu-1}$. Then one can check that H is not β-deficient. □

Theorem 3.3. *Let H be a graph on p (p is sufficiently large) vertices and with radius r, $0 \le \omega \le p-3$. If $\xi^{sv}(H) \le r\Gamma_2(p,\omega)$, then H is ω-Hamiltonian if and only if $H \notin \mathcal{H}_2$, where*

$$\Gamma_2(p,\omega) = \begin{cases} 2(p+2-\frac{4}{p}) & \text{if } \omega = 0 \text{ and } p \text{ is even}, \\ 2p+1 & \text{if } \omega = 0 \text{ and } p \text{ is odd}, \\ \frac{2(p-1)}{\omega+1} + \frac{2(p-1)(p-\omega-2)}{p-2} + 2\omega - p + 2 & \text{if } \omega \ge 1. \end{cases}$$

Proof. Sufficiency. Assume H is not ω-Hamiltonian for some s in $1 \le s < \frac{p-\omega}{2}$. Therefore according to Lemma 2.3 (*i*), we know that $d_s \le s+\omega$ and $d_{p-s-\omega} \le p-s-1$. Thus by inequality (2), we have

$$\begin{aligned}\xi^{sv}(H) &\ge 2r(p-1)\sum_{j=1}^{p}\frac{1}{d_j} - pr \\ &\ge 2(p-1)r\Big(\frac{s}{s+\omega} + \frac{p-2s-\omega}{p-s-1} + \frac{s+\omega}{p-1}\Big) - pr.\end{aligned}$$

Since s is an integer, we have $1 \le s \le \frac{p-\omega-1}{2}$. Denote

$$h(x) = \frac{x}{x+\omega} + \frac{p-2x-\omega}{p-x-1} + \frac{x+\omega}{p-1}$$

with $1 \le x \le \frac{1}{2}(p-1-\omega)$. Thus

$$h'(x) = \frac{\omega(p-1)(p-1-x)^2 - (p-1)(p+\omega-2)(x+\omega)^2 + (x+\omega)^2(p-1-x)^2}{(p-1)(x+\omega)^2(p-1-x)^2}.$$

Case 1 : $\omega \ge 1$. Denote

$$\eta(x) = \omega(p-1)(p-1-x)^2 - (p-1)(p+\omega-2)(x+\omega)^2 + (x+\omega)^2(p-1-x)^2$$

with $x \in [1, \frac{1}{2}(p-1-\omega)]$. Simplifying the expression of $\eta(x)$, we obtain

$$\begin{aligned}\eta(x) &= x^4 - 2(p-\omega-1)x^3 + (p-4\omega p+4\omega+\omega^2-1)x^2 - 2\omega(p-1)(2\omega+p-2)x \\ &\quad +\omega(p-1)\Big((p-1)^2-\omega(\omega-1)\Big).\end{aligned}$$

Take the first and second derivatives of $\eta(x)$ on $x \in [1, \frac{p-\omega-1}{2}]$ respectively, we have

$$\eta'(x) = 4x^3 - 6(p-\omega-1)x^2 + 2(p-4\omega p+4\omega+\omega^2-1)x - 2\omega(p-1)(p+2\omega-2),$$

and

$$\eta''(x) = 2\Big[6x^2 - 6(p-\omega-1)x + p - 4\omega p + 4\omega + \omega^2 - 1\Big] \doteq 2\rho(x).$$

Then $\rho'(x) = 6\Big[2x-(p-\omega-1)\Big] \le 0$ as $x \le \frac{p-\omega-1}{2}$. So $\rho(x)$ is decreasing on $1 \le x \le \frac{1}{2}(p-\omega-1)$. It follows that $\rho(x) \le \rho(1)$. Note that $\rho(1) = -(4\omega+5)p + \omega^2 + 10\omega + 11 < 0$ when p is sufficiently large. So $\rho(x) < 0$ and consequently $\eta''(x)$ is negative. Therefore $\eta(x)$ is concave up for $x \in [1, \frac{1}{2}(p-1-\omega)]$. Hence $\eta(x)$ attains its minimum value at $x = 1$ or $x = \lfloor \frac{p-\omega-1}{2} \rfloor$. Direct calculations yield $\eta(1) \le \eta(\lfloor \frac{p-\omega-1}{2} \rfloor)$. Thus $\eta(x) \ge \eta(1)$. So

$$\xi^{sv}(H) \ge r\Big(\frac{2(p-1)}{\omega+1} + \frac{2(p-1)(p-\omega-2)}{p-2} + 2\omega - p + 2\Big) = r\Gamma_2(p,\omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Hence $s = 1$ and correspondingly $d_1 = \omega+1$, $d_2 = \cdots = d_{p-\omega-1} = p-2$, $d_{p-\omega} = \cdots = d_p = p-1$. Thus $H \cong K_{\omega+1}\nabla(K_1 + K_{p-\omega-2}) = H_2$, a contradiction.

Case 2 : $\omega = 0$. Therefore $1 \le x \le \frac{1}{2}(p-1)$ and

$$h'(x) = \frac{(p-1-x)^2 - (p-1)(p-2)}{(p-1)(p-1-x)^2}.$$

Since $1 \le x \le p-1$, we have $0 \le p-1-x \le p-2 < p-1$. So $(p-1-x)^2 - (p-1)(p-2) < 0$. Thus $h'(x) < 0$ when $1 \le x \le \frac{1}{2}(p-1)$, implying that $h(x)$ is decreasing on $x \in [1, \frac{p-1}{2}]$. Hence $h_{min}(x) = h(\lfloor \frac{p-1}{2} \rfloor)$ as $x \in \mathbb{Z}$.

Subcase 2.1 : p is even. It follows that $h(x) \ge h(\frac{p-2}{2})$. Thus

$$\xi^{sv}(H) \ge 2r\Big(p + 2 - \frac{4}{p}\Big) = r\Gamma_2(p,\omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Hence $s = \frac{p-2}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{p-2}{2}} = \frac{1}{2}(p-2)$, $d_{\frac{p}{2}} = d_{\frac{p+2}{2}} = \frac{p}{2}$, $d_{\frac{p+4}{2}} = \cdots = d_p = p-1$. Thus $H \cong K_{\frac{p-2}{2}}\nabla(K_2 + \overline{K_{\frac{p-2}{2}}})$, contrary to the assumption.

Subcase 2.2 : p is odd. Thus $h(x) \ge h(\frac{p-1}{2})$. Then

$$\xi^{sv}(H) \ge r(2p+1) = r\Gamma_2(p,\omega).$$

In combination with the conditions of the theorem, the above inequality is true if and only if we take an equal sign. Hence $s = \frac{p-1}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{p+1}{2}} = \frac{1}{2}(p-1)$, $d_{\frac{p+3}{2}} = \cdots = d_p = p-1$. Thus $H \cong K_{\frac{p-1}{2}}\nabla\overline{K_{\frac{p+1}{2}}}$, a contradiction. Therefore H is ω-Hamiltonian.

Conversely, suppose that $H \in \mathcal{H}_2$. Then one can check that H is not ω-Hamiltonian. $\square$

The following corollary can be obtained directly from Theorem 3.3 by setting $\omega = 0$. Note that in this case $\mathcal{H}_2 = \left\{K_{\frac{p-2}{2}} \nabla (K_2 + \overline{K_{\frac{p-2}{2}}}),\ K_{\frac{p-1}{2}} \nabla \overline{K_{\frac{p+1}{2}}}\right\}$.

Corollary 3.4. *Let H be a graph on p (p is sufficiently large) vertices and with radius r. If $\xi^{sv}(H) \le r\Gamma_2(p, 0)$, then H is Hamiltonian if and only if $H \notin \mathcal{H}_2$, where*

$$\Gamma_2(p,0) = \begin{cases} 2(p + 2 - \frac{4}{p}) & \text{if } p \text{ is even,} \\ 2p + 1 & \text{if } p \text{ is odd.} \end{cases}$$

Theorem 3.5. *Let H be a graph on $p \ge 3$ vertices and with radius r, $\omega \in [1, p - 3]$. If $\xi^{sv}(H) \le r\Gamma_3(p, \omega)$, then H is ω-path-coverable if and only if $H \notin \mathcal{H}_3$, where*

$$\Gamma_3(p,\omega) = \begin{cases} \frac{2(p-1)(p+\omega-2)}{p-\omega-2} + \frac{8(p-1)}{p-\omega} - \omega - 2 & \text{if } p - \omega \text{ is even,} \\ \frac{2(p-1)(p+\omega+1)}{p-\omega-1} - \omega - 1 & \text{if } p - \omega \text{ is odd.} \end{cases}$$

Proof. Sufficiency. Assume H is not ω-path-coverable for some s in $1 \le s < \frac{p-\omega}{2}$. Thus from Lemma 2.4, we know that $d_{s+\omega} \le s$ and $d_{p-s} \le p - s - \omega - 1$. Hence by inequality (2), we have

$$\begin{aligned} \xi^{sv}(H) &\ge 2r(p-1)\sum_{j=1}^{p} \frac{1}{d_j} - pr \\ &\ge 2r(p-1)\Big(\frac{s+\omega}{s} + \frac{p-2s-\omega}{p-s-\omega-1} + \frac{s}{p-1}\Big) - pr. \end{aligned}$$

Note that $1 \le s \le \frac{p-\omega-1}{2}$. Define

$$\varphi(x) = \frac{x+\omega}{x} + \frac{p-2x-\omega}{p-x-\omega-1} + \frac{x}{p-1}$$

with $x \in [1, \frac{p-\omega-1}{2}]$. Then

$$\varphi'(x) = -\frac{\omega}{x^2} - \frac{p-2-\omega}{(p-1-\omega-x)^2} + \frac{1}{p-1}.$$

Note that $0 \le p-\omega-1-x \le p-\omega-2 < p-1$. Hence $\frac{p-\omega-2}{(p-\omega-1-x)^2} \ge \frac{1}{p-\omega-2}$, and consequently $-\frac{p-\omega-2}{(p-\omega-1-x)^2} + \frac{1}{p-1} \le -\frac{1}{p-\omega-2} + \frac{1}{p-1} < 0$. So $\varphi'(x) < 0$ implies $\varphi(x)$ is decreasing for $x \in [1, \frac{1}{2}(p-\omega-1)]$. Thus $\varphi_{min}(x) = \varphi(\lfloor \frac{p-\omega-1}{2} \rfloor)$ as $x \in \mathbb{Z}$.

Case 1 : $p - \omega$ is even. Thus $\varphi(x) \ge \varphi(\frac{p-\omega-2}{2})$. Therefore

$$\xi^{sv}(H) \ge r\Big(\frac{2(p-1)(p+\omega-2)}{p-\omega-2} + \frac{8(p-1)}{p-\omega} - \omega - 2\Big) = r\Gamma_3(p,\omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Hence $s = \frac{p-\omega-2}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{\omega+p-2}{2}} = \frac{1}{2}(p-2-\omega)$, $d_{\frac{p+\omega}{2}} = d_{\frac{p+\omega+2}{2}} = \frac{p-\omega}{2}$, $d_{\frac{p+\omega+4}{2}} = \cdots = d_p = p-1$. Thus $H \cong K_{\frac{p-2-\omega}{2}} \nabla (K_2 + \overline{K_{\frac{p-2+\omega}{2}}})$, contradicting the assumption.

Case 2 : $p-\omega$ is odd. Therefore $\varphi(x) \geq \varphi(\frac{p-\omega-1}{2})$. Thus

$$\xi^{sv}(H) \geq r\Big(\frac{2(p-1)(p+\omega+1)}{p-\omega-1} - \omega - 1\Big) = r\Gamma_3(p,\omega).$$

In combination with the conditions of the theorem, the above inequality is true if and only if we take an equal sign. Hence $s = \frac{p-\omega-1}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{p+\omega-1}{2}} = \frac{1}{2}(p-1-\omega)$, $d_{\frac{\omega+p+1}{2}} = \frac{p-\omega-1}{2}$, $d_{\frac{p+\omega+3}{2}} = \cdots = d_p = p-1$. Thus $H \cong K_{\frac{p-\omega-1}{2}} \nabla (K_1 + \overline{K_{\frac{p+\omega-1}{2}}})$, contrary to the assumption. Hence H is ω-path-coverable.

Conversely, suppose that $H \in \mathcal{H}_3$. Then one can check that H is not ω-path-coverable. □

The following corollary can be obtained directly from Theorem 3.5 by setting $\omega = 1$. Evidently, in this case $\mathcal{H}_3 = \Big\{K_{\frac{p-3}{2}} \nabla (K_2 + \overline{K_{\frac{p-1}{2}}}),\ K_{\frac{p-2}{2}} \nabla (K_1 + \overline{K_{\frac{p}{2}}})\Big\}$.

Corollary 3.6. *Let H be a graph on $p \geq 3$ vertices and with radius r. If $\xi^{sv}(H) \leq r\Gamma_3(p,1)$, then H is traceable if and only if $H \notin \mathcal{H}_3$, where*

$$\Gamma_3(p,1) = \begin{cases} \frac{2p^2+p-13}{p-3} & \text{if } p \text{ is odd,} \\ \frac{2p^2}{p-2} & \text{if } p \text{ is even.} \end{cases}$$

Theorem 3.7. *Let H be a graph on p (p is sufficiently large) vertices and with radius r, $\omega \in [0, p-3]$. If $\xi^{sv}(H) \leq r\Gamma_4(p,\omega)$, then H is ω-edge-Hamiltonian if and only if $H \notin \mathcal{H}_4$, where*

$$\Gamma_4(p,\omega) = \begin{cases} \frac{2(p-1)(p-\omega-2)}{p+\omega-2} + \frac{8(p-1)}{p+\omega} + \omega - 2 & \text{if } \omega \in [0,1] \text{ and } p+\omega \text{ is even,} \\ \frac{2(p-1)(p-\omega+1)}{p+\omega-1} + \omega - 1 & \text{if } \omega \in [0,1] \text{ and } p+\omega \text{ is odd,} \\ \frac{2(p-1)}{\omega+1} + \frac{2(p-1)(p-\omega-2)}{p-2} + 2\omega - p + 2 & \text{if } \omega \geq 2. \end{cases}$$

Proof. Sufficiency. Assume H is not ω-edge-Hamiltonian for some s in $\omega+1 \leq s < \frac{p+\omega}{2}$. Then from Lemma 2.3 (*ii*), we see that $d_{s-\omega} \leq s$, $d_{p-s} \leq p-s+\omega-1$. Therefore by inequality (2), we have

$$\begin{aligned} \xi^{sv}(H) &\geq 2r(p-1)\sum_{j=1}^{p}\frac{1}{d_j} - pr \\ &\geq 2r(p-1)\Big(\frac{s-\omega}{s} + \frac{p-2s+\omega}{p-s+\omega-1} + \frac{s}{p-1}\Big) - pr. \end{aligned}$$

Note that $1 \leq s \leq \frac{p+\omega-1}{2}$. We define

$$\psi(x) = \frac{x-\omega}{x} + \frac{p-2x+\omega}{p-x+\omega-1} + \frac{x}{p-1}$$

with $\omega+1 \leq x \leq \frac{1}{2}(\omega-1+p)$. Then

$$\psi'(x) = \frac{\omega(p-1)(p+\omega-1-x)^2 - (p-1)x^2(\omega-2+p) + x^2(\omega-1+p-x)^2}{x^2(p-1)(p+\omega-1-x)^2}. \tag{4}$$

Denote

$$\zeta(x) = \omega(p-1)(p+\omega-1-x)^2 - (p-1)x^2(\omega-2+p) + x^2(\omega-1+p-x)^2$$

with $x \in [\omega+1, \frac{1}{2}(\omega+p-1)]$. It can be deformed to

$$\begin{aligned}\zeta(x) &= x^4 - 2(p+\omega-1)x^3 + \Big[(p+\omega-1)^2 - (p-1)(p-2)\Big]x^2 \\ &\quad -2\omega(p-1)(p+\omega-1)x + \omega(p-1)(p+\omega-1)^2.\end{aligned}$$

Take the first, second and third derivatives of $\zeta(x)$ respectively, we get

$$\begin{aligned}\zeta'(x) &= 4x^3 - 6(p+\omega-1)x^2 + 2\Big[(p+\omega-1)^2 - (p-1)(p-2)\Big]x \\ &\quad -2\omega(p-1)(p+\omega-1),\end{aligned}$$

and

$$\zeta''(x) = 2\Big[6x^2 - 6(\omega-1+p)x + (\omega-1+p)^2 - (p-1)(p-2)\Big],$$

and

$$\zeta'''(x) = 12\Big[2x - (p+\omega-1)\Big].$$

It is easy to know that $\zeta'''(x) \le 0$ as $x \le \frac{1}{2}(p+\omega-1)$, which implies that $\zeta''(x)$ is decreasing for $\omega+1 \le x \le \frac{1}{2}(\omega+p-1)$. So $\zeta''(x) \le \zeta''(\omega+1)$. By direct calculation,

$$\phi(\omega) := \zeta''(\omega+1) = 2\Big[\omega^2 + 2(5-2p)\omega + 11 - 5p\Big].$$

Note that $\phi'(\omega) = 4(\omega+5-2p) \le 4[(p-3)+5-2p] = 4(2-p) < 0$ when $p \ge 5$. Thus $\phi(\omega)$ is decreasing on $\omega \in [0, p-3]$. So $\phi(\omega) \le \phi(0)$. Evidently, $\phi(0) = 2(11-5p) < 0$ when $p \ge 5$. Hence $\zeta''(\omega+1) = \phi(\omega) < 0$ and consequently $\zeta''(x) < 0$. Implying that $\zeta'(x)$ is decreasing on $\omega+1 \le x \le \frac{1}{2}(\omega-1+p)$. Thus $\zeta'(x) \le \zeta'(\omega+1)$. Note that

$$\zeta'(\omega+1) = (8-4p) + 2\omega^2(3-2p) + 2\omega(5-p) - 2p^2\omega < 0,$$

when $p \ge 5$ and $\omega \ge 0$. So $\zeta'(x) < 0$, and consequently $\zeta(x)$ is decreasing on $\omega+1 \le x \le \frac{1}{2}(\omega-1+p)$. Thus $\zeta(\frac{p+\omega-1}{2}) \le \zeta(x) \le \zeta(\omega+1)$. Evidently,

$$\begin{aligned}\zeta\Big(\frac{p+\omega-1}{2}\Big) &= -\frac{1}{8}\Big[2(p-1)(p+\omega-1)^3 - (p+\omega+1)^2(p+\omega-1)^2 \\ &\quad - 2(\omega+1)(p-1)(p+\omega+1)^2\Big] < 0,\end{aligned}$$

and

$$\zeta(\omega+1) = \omega(p-1)(p-2)^2 - (\omega+1)^2(p-1)(p+\omega-2) + (\omega+1)^2(p-2)^2 > 0,$$

when p is sufficiently large. Also notice that $\zeta(x)$ is continuous for $x \in [\omega+1, \frac{p+\omega-1}{2}]$. Then by the zero point theorem, there exists some σ in $\sigma \in (\omega+1, \frac{p+\omega-1}{2})$ such that $\zeta(\sigma) = 0$. So $\zeta(x) \ge 0$ when $x \in [\omega+1, \sigma)$ and $\zeta(x) < 0$ when $x \in [\sigma, \frac{p+\omega-1}{2}]$. By Eq. (4), $\psi'(x) \ge 0$ when $x \in [\omega+1, \sigma)$ and $\psi'(x) < 0$ when $x \in [\sigma, \frac{p+\omega-1}{2}]$. Consequently, $\psi(x)$ is increasing on $x \in [\omega+1, \sigma)$ and decreasing on $x \in [\sigma, \frac{p+\omega-1}{2}]$. Therefore $\psi(x) \ge \min\Big\{\psi(\omega+1), \psi(\lfloor\frac{p+\omega-1}{2}\rfloor)\Big\}$.

Case 1 : $\omega \ge 2$. Then by direct computation, $\psi_{min}(x) = \psi(\omega+1)$. Thus

$$\xi^{sv}(H) \ge r\Big(\frac{2(p-1)}{\omega+1} + \frac{2(p-1)(p-\omega-2)}{p-2} + 2\omega - p + 2\Big) = r\Gamma_4(p,\omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Hence $s = \omega+1$, and correspondingly $d_1 = \omega+1$, $d_2 = \cdots = d_{p-\omega-1} = p-2$, $d_{p-\omega} = \cdots = d_p = p-1$. Thus $H \cong K_{\omega+1}\nabla(K_1 + K_{p-2-\omega}) = H_2$, contrary to the assumption.

Case 2 : $\omega \in [0, 1]$. Thus $\psi_{min}(x) = \psi(\lfloor \frac{p+\omega-1}{2} \rfloor)$.

Subcase 2.1 : $p + \omega$ is even. Hence $\psi(x) \geq \psi(\frac{p+\omega-2}{2})$. So

$$\xi^{sv}(H) \geq r\Big(\frac{2(p-1)(p-\omega-2)}{p+\omega-2} + \frac{8(p-1)}{p+\omega} + \omega - 2\Big) = r\Gamma_4(p, \omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Thus $s = \frac{p-2+\omega}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{1}{2}(p-2-\omega)} = \frac{1}{2}(p - 2 + \omega)$, $d_{\frac{1}{2}(p-\omega)} = d_{\frac{1}{2}(p+2-\omega)} = \frac{1}{2}(p + \omega)$, $d_{\frac{1}{2}(p+4-\omega)} = \cdots = d_p = p - 1$. Hence $H \cong K_{\frac{p-2+\omega}{2}} \nabla (K_2 + \overline{K_{\frac{p-2-\omega}{2}}})$, contrary to the assumption.

Subcase 2.2 : $p + \omega$ is odd. Thus $\psi(x) \geq \psi(\frac{p+\omega-1}{2})$. So

$$\xi^{sv}(H) \geq r\Big(\frac{2(p-1)(p-\omega+1)}{p+\omega-1} + \omega - 1\Big) = r\Gamma_4(p, \omega).$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Therefore $s = \frac{p+\omega-1}{2}$, and correspondingly $d_1 = \cdots = d_{\frac{p-\omega-1}{2}} = \frac{1}{2}(p + \omega - 1)$, $d_{\frac{p+1-\omega}{2}} = \frac{1}{2}(p - 1 + \omega)$, $d_{\frac{1}{2}(p+3-\omega)} = \cdots = d_p = p - 1$. Thus $H \cong K_{\frac{p+\omega-1}{2}} \nabla (K_1 + \overline{K_{\frac{p-\omega-1}{2}}})$, contradicting the assumption. Hence H is ω-edge-Hamiltonian.

Conversely, suppose that $H \in \mathcal{H}_4$. Then one can check that H is not ω-edge-Hamiltonian. □

Theorem 3.8. *Let H be a graph on p vertices and with radius r. If $\xi^{sv}(H) \leq \big(\frac{2(p-1)(\omega+1)}{p-\omega-1} + p - 2\omega - 2\big)r$, then H satisfies $\alpha(H) \leq \omega$ if and only if $H \ncong \overline{K_{\omega+1}} \nabla K_{p-\omega-1}$.*

Proof. Sufficiency. Assume that $\alpha(H) > \omega$. Thus from Lemma 2.5, $d_{\omega+1} \leq p - \omega - 1$. So by inequality (2), we have

$$\begin{aligned} \xi^{sv}(H) &\geq 2r(p-1)\sum_{j=1}^{p} \frac{1}{d_j} - pr \\ &\geq 2r(p-1)\Big(\frac{\omega+1}{p-1-\omega} + \frac{p-1-\omega}{p-1}\Big) - pr \\ &= \Big(\frac{2(p-1)(\omega+1)}{p-\omega-1} + p - 2\omega - 2\Big)r. \end{aligned}$$

In combination with the conditions of the theorem, the inequality above can only be true if equality holds. Thus $d_1 = \cdots = d_{\omega+1} = p - 1 - \omega$, $d_{\omega+2} = \cdots = d_p = p - 1$. Therefore $H \cong \overline{K_{\omega+1}} \nabla K_{p-\omega-1}$, contrary to the assumption. Hence H satisfies $\alpha(H) \leq \omega$.

Conversely, suppose that $H \cong \overline{K_{\omega+1}} \nabla K_{p-\omega-1}$. Then one can check that $\alpha(H) > \omega$. □

4. Conclusion

In this paper, by employing the adjacent eccentric distance sum index, we present sufficient conditions for a graph with given radius to possess certain properties. These results partially solved the problems raised at the end of [13]. How to apply the method in this paper to other eccentricity-based topological indices, such as CEI, ECI and EDS, is still unknown. These are all interesting questions for future study.

Statements and Declarations

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