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A class of generalized symplectic graphs based on totally isotropic subspaces in symplectic spaces over finite fields

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Abstract. Let $\mathbb{F}_q$ be a finite field of order q and $\mathbb{F}_q^{(2\nu)}$ be a 2ν-dimensional symplectic space. In the present paper, we study a class of generalized symplectic graphs Γ based on m-dimensional totally isotropic subspaces in $\mathbb{F}_q^{(2\nu)}$. It is shown that Γ is vertex-transitive, and it is a 5-Deza graph with diameter $m+1$. Moreover, we determine the parameters concerning the first subconstituent Γ_1 and it is shown that Γ_1 is also a 5-Deza graph.

1. Introduction

At the beginning of the paper, we present some necessary concepts and notations that will be used later. Let $\Gamma = (V, E)$ be a graph and u and v be two elements of the vertex set $V(\Gamma)$. If u and v are adjacent, that is u and v are joined by an edge of the graph, then they are neighbours, and we write $u \sim v$. We denote $\Gamma(u, v)$ by the common neighbors of vertices u and v in Γ. We will denote by $d_\Gamma(u, v)$ the *distance* between two vertices u and v in graph Γ, and denote by $\Gamma_i(u)$ the set of all vertices v for which $d_\Gamma(u, v) = i$. The *diameter* of the graph Γ is the maximal distance between two of its vertices. A graph is *edge-transitive*(resp. *arc-transitive* or *vertex-transitive*) if its automorphism group $Aut(\Gamma)$ acts transitively on the edge(resp. arc or vertice) set. A k-regular graph Γ is called a *strongly regular graph* with parameters (n, k, λ, μ) if Γ has precisely n vertices, any two adjacent vertices of Γ have precisely λ common neighbours, and any two nonadjacent vertices of Γ have μ common neighbours. As a generalization of strongly regular graphs, a k-regular graph on n vertices is called a *d-Deza graph* with parameters $(n, k, \{c_1, ..., c_d\})$ if every two distinct vertices of the graph have c_i common adjacent vertices, where $i = 1, \ldots, d$, see [7]. All other unexplained notions and terminology about graph theory are standard and follow mainly the reference [5].

Suppose that $\mathbb{F}_q$ is a finite field and $\mathbb{F}_q^{(2\nu)} = \{(a_1, a_2, \ldots, a_{2\nu}) : a_i \in \mathbb{F}_q, i = 1, \ldots, 2\nu\}$ is the row vector space over $\mathbb{F}_q$ of dimension 2ν, where ν is a positive integer. Let

$$K = \begin{pmatrix} \mathbf{0} & I^{(\nu)} \\ -I^{(\nu)} & \mathbf{0} \end{pmatrix}.$$

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It is easy to see that the set of $2\nu \times 2\nu$ matrices M satisfying $MKM^t = K$ forms a group with respect to the matrix multiplication, which is called *symplectic group* of degree 2ν, and denoted by $Sp_{2\nu}(\mathbb{F}_q)$. Define an action of symplectic group $Sp_{2\nu}(\mathbb{F}_q)$ on vector space $\mathbb{F}_q^{(2\nu)}$ by matrix multiplication:

$$\begin{aligned}\mathbb{F}_q^{(2\nu)} \times Sp_{2\nu}(\mathbb{F}_q) &\longrightarrow \mathbb{F}_q^{(2\nu)}\\ ((a_1, a_2, \ldots, a_{2\nu}), M) &\longmapsto (a_1, a_2, \ldots, a_{2\nu})M.\end{aligned}$$

The vector space $\mathbb{F}_q^{(2\nu)}$ with the above group action is called *symplectic space*. Let $\alpha_1, \ldots, \alpha_m$ be linearly independent vectors of $\mathbb{F}_q^{(2\nu)}$ and W be the vector subspace spanned by $\alpha_1, \ldots, \alpha_m$. For simplicity of notation, we write W as the matrix

$$W = \begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_m \end{pmatrix}$$

when no confusion can arise. It is easy to know that if matrix W(as an $m \times 2\nu$ matrix) is transformed into P with the elementary row transformations, then W and P are the same subspace. A subspace W of dimension m is *totally isotropic* if $WKW^t = \mathbf{0}$. For the sake of simplicity, let $\mathcal{M}$ denote the set of all m-dimensional totally isotropic subspaces. For the above concepts and notations about symplectic spaces we follow the monograph [16].

Recall that the *symplectic graph* relative to matrix K over a finite filed is the graph with the set $\{\langle\alpha\rangle : \alpha \in \mathbb{F}_q^{(2\nu)}\}$, where $\langle\alpha\rangle$ means the 1-dimensional subspaces generated by α, as its vertex set and $\alpha \sim \beta$ whenever $\alpha K \beta^t = 0$ for two vertices α and β. In [12], Rotman firstly studied the symplectic graph modulo 2. In 2006, Tang and Wan[14] further developed the symplectic graph over an arbitrary finite field. After that, the study related to symplectic graph has always been a hot topic. There are many remarkable results concerning symplectic graphs or generalized symplectic graphs over different algebraic structures, see[6–11, 13, 18] for example. As a development of the symplectic graphs over finite fields, authors in [13] introduced *the generalized symplectic graph of type* (m, r, t) over finite commutative rings, it has the set of m-dimensional totally isotropic free submodules, where $1 \leq m \leq \nu$, as the vertex set $V(\Gamma)$, and for two different vertices P and Q, there is an edge between them if $\mathrm{r}(PKQ^t) = r$ and the dimension of $P \cap Q$(denoted by $\dim(P \cap Q)$) is $m - t$, where $\mathrm{r}(PKQ^t) = r$ is the rank of the matrix PKQ^t. The author showed some properties of the graph, such as the transitivity, regularity and the degree of the graph and so on.

In this paper, we will focus on the generalized symplectic graph of type $(m, 0, 1)$ over a finite field, which is denoted by $\Gamma(K, m, m-1, 0)$, and we write it Γ for brevity. The graph Γ has the set $\mathcal{M}$ as the vertex set, and for vertices X and Y, $X \sim Y$ if $XKY^t = \mathbf{0}$ and $\dim(X \cap Y) = m - 1$. In particular, if $m = \nu$, then the graph is the dual polar graph constructed by dual polar spaces (see [3]), and the dual polar graphs have been extensively studied, see[1, 2, 15]. In the special case $m = 1$, the graph is exactly the complement of symplectic graph. Therefore, we will consider the case $1 < m < \nu$ in this paper.

The rest of the paper is organized as follows. In the second section, we firstly studied the basic properties of the graph $\Gamma(K, m, m-1, 0)$, where we give the vertex and arc-transitivity and the diameter of the graph. And then, we show that Γ is a 5-Deza graph. The final section is devoted to investigating the first subconstituent Γ_1 of Γ and we obtain that Γ_1 is also a 5-Deza graph.

2. The generalized symplectic graph Γ

In this section we discuss the basic properties and parameters of the graph Γ, which will play a crucial role in the study of subconstituents of Γ.

Lemma 2.1. [16, Theorem 3.7] *The symplectic group $Sp_{2\nu}(\mathbb{F}_q)$ acts transitively on the vertex set $\mathcal{M}$.*

Lemma 2.2. [16, Corollary 3.19] *The number of vertices of graph Γ is $\frac{\prod_{i=\nu-m+1}^{\nu}(q^{2i}-1)}{\prod_{i=1}^{m}(q^i-1)}$.*

By [13, Theorem 2.2], the graph Γ is arc-transitive and thus it is a vertex transitive regular graph. Therefore, to study the diameter of Γ, without loss of generality, it suffices to consider the longest path which starts with a fixed vertex. We introduce the following notations. Let M be a vertex of Γ given by

$$M = (I^{(m)}, \mathbf{0}^{(m\times(2\nu-m))})$$

and $S(r,t)$ denotes the vertex set composed of vertices $X \in V(\Gamma)$ such that $\mathrm{r}(MKX^t) = r$ and $\dim(M \cap X) = t$. For any vertex

$$X = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} A & B & C & D \\ A' & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{l} \\ m-t \\ t \end{array} \in S(r,t),$$

where $\mathrm{r}(A') = t$. It is easy to check that $\mathrm{r}(MKX^t) = \mathrm{r}\begin{pmatrix} C \\ \mathbf{0} \end{pmatrix} = r$. So $r \le m - t$. Let Γ_i be the subgraph induced by the vertex set $\Gamma_i(M)$, and it is clear that $V(\Gamma_1) = S(0, m-1)$.

Lemma 2.3. *Suppose that there are two vertices of $S(r,t)$ and $S(r',t')$, say P and Q respectively, such that $P \sim Q$. We have $\mid t - t' \mid \le 1$, and*

(*i*) *if $t - t' = 1$, then $r' - r = 1$ or 0;*

(*ii*) *if $t - t' = -1$, then $r - r' = 1$ or 0;*

(*iii*) *if $t - t' = 0$, then $r - r' = \pm 1$ or 0.*

Proof. Since P is adjacent to Q, clearly, $\dim(P \cap Q) = m - 1$ and $\mathrm{r}(PKQ^t) = 0$. Suppose that $P \cap Q = W$ and $P = \binom{W}{\alpha}$, $Q = \binom{W}{\beta}$, where α is not in Q and β is not in P. Furthermore, assume that $\dim(M \cap W) = t_0$, then without loss of generality, let

$$P = \begin{pmatrix} W \\ \alpha \end{pmatrix} = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1 & B_2 & B_3 & B_4 \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} \end{array} \begin{array}{l} \\ t_0 \\ m-t_0-1 \\ 1 \end{array},$$

where $\mathrm{r}(A) = t_0$, $\begin{pmatrix} B_2 & B_3 & B_4 \\ \alpha_2 & \alpha_3 & \alpha_4 \end{pmatrix} \ne \mathbf{0}$, $A\begin{pmatrix} B_3 \\ \alpha_3 \end{pmatrix} = \mathbf{0}$. It is easy to see from elementary matrix transformations that there is matrix $S \in GL_m(\mathbb{F}_q)$ such that $AS = (I^{(t_0)}, \mathbf{0})$. Let $T = \begin{pmatrix} S & & & \\ & I^{(\nu-m)} & & \\ & & (S^{-1})^t & \\ & & & I^{(\nu-m)} \end{pmatrix}$. Clearly, T is an element of $Sp_{2\nu}(\mathbb{F}_q)$ such that

$$PT = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} AS & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1S & B_2 & B_3(S^{-1})^t & B_4 \\ \alpha_1 S & \alpha_2 & \alpha_3(S^{-1})^t & \alpha_4 \end{pmatrix} \end{array}.$$

Further, by performing a series of elementary matrix transformations, PT has the form of

$$PT = \begin{array}{c} \begin{array}{cccccc} t_0 & m-t_0 & \nu-m & t_0 & m-t_0 & \nu-m \end{array} \\ \begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_{12} & B_2 & \mathbf{0} & B_{32} & B_4 \\ \mathbf{0} & \alpha_{12} & \alpha_2 & \mathbf{0} & \alpha_{32} & \alpha_4 \end{pmatrix} \end{array}.$$

Since (M, P) and $(MT, PT) = (M, PT)$ are in the same orbit by [17, Theorem 2.1], where M and MT are the same as subspaces, hence we have $\dim(M \cap PT) = \dim(MT \cap PT) = \dim(M \cap P) = t$ and $r(MK(PT)^t) = r((MT)K(PT)^t) = r(MKP^t) = r$, where $r = r\begin{pmatrix} B_{32} \\ \alpha_{32} \end{pmatrix}$. Since $\dim(M \cap W) = t_0$, it clear that $t = t_0$ or $t_0 + 1$.

Similarly, QT can be written as

$$QT = \begin{array}{c} \begin{array}{cccccc} t_0 & m-t_0 & \nu-m & t_0 & m-t_0 & \nu-m \end{array} \\ \begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & B_{12} & B_2 & \mathbf{0} & B_{32} & B_4 \\ \mathbf{0} & \beta_{12} & \beta_2 & \mathbf{0} & \beta_{32} & \beta_4 \end{pmatrix} \end{array},$$

where $\dim(M \cap QT) = t' = t_0$ or $t_0 + 1$ and $r(MKQ^t) = r' = r\begin{pmatrix} B_{32} \\ \beta_{32} \end{pmatrix}$. Since $r((PT)K(QT)^t) = r(PKQ^t) = 0$, we have

$$\alpha_{12}\beta_{32}^t - \alpha_{32}\beta_{12}^t + \alpha_2\beta_4^t - \alpha_4\beta_2^t = 0. \tag{1}$$

Obviously, $| \; t - t' \; | \leq 1$, and so we will break into the following several possible cases depending on the values of t and t'.

(i) Firstly, if $t - t' = 1$, that is, $t = t_0 + 1$ and $t' = t_0$, then we can take a matrix representation of PT such that $\alpha_{12} \neq \mathbf{0}$ and $(\alpha_2, \alpha_{32}, \alpha_4) = \mathbf{0}$. Clearly $r = r(B_{32})$. It follows at once from the equation (1) that $\alpha_{12}\beta_{32}^t = 0$, and $r' = r(B_{32})$ or $r' = r(B_{32}) + 1$, which is determined by β_{32}. It follows that $r' - r = 1$ or 0.

(ii) Then suppose that $t - t' = -1$, it is easy to show that $r - r' = 1$ or 0, the proof follows in exactly the same way as above.

(iii) Finally, suppose that $t - t' = 0$. Then there are the following two subcases. If $t = t' = t_0 + 1$, then we can take suitable matrix representations of PT and QT such that $(\alpha_2, \alpha_{32}, \alpha_4) = (\beta_2, \beta_{32}, \beta_4) = \mathbf{0}$, $\alpha_{12} \neq \mathbf{0}$ and $\beta_{12} \neq \mathbf{0}$. It follows that $r = r' = r(B_{32})$. If $t = t' = t_0$, then similarly it is easy to see from (1) that there are three possibilities of $r - r'$, i.e., $r - r' = 0, 1$ or -1. □

Proposition 2.4. *There exists edges between $S(r,t)$ and $S(r',t')$ for each one of the conditions (i)-(iii) in Lemma 2.3 except for the cases $t - t' = r' - r = \pm 1$ when $r + t = m$.*

Proof. Suppose that $r + t = m$ and $t - t' = r' - r = \pm 1$. Without loss of generality, take $t - t' = r' - r = -1$ for example, let

$$P = \begin{array}{cc} \begin{array}{cccccc} t & r & \nu-m & t & r & \nu-m \end{array} & \\ \begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha_1 & \alpha_2 & \mathbf{0} & \alpha_3 & \alpha_4 \\ \mathbf{0} & A_1 & A_2 & \mathbf{0} & A_3 & A_4 \end{pmatrix} & \begin{array}{l} t \\ 1 \\ r-1 \end{array} \end{array} \in S(r,t),$$

where $r\begin{pmatrix} \alpha_3 \\ A_3 \end{pmatrix} = r$. Suppose by way of contradiction that there is a vertex $Q \in S(r', t')$ such that $P \sim Q$. Then

$$Q = \begin{array}{cc} \begin{array}{cccccc} t & r & \nu-m & t & r & \nu-m \end{array} & \\ \begin{pmatrix} I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_1' & A_2' & \mathbf{0} & A_3' & A_4' \end{pmatrix} & \begin{array}{l} t \\ 1 \\ r-1 \end{array} \end{array} \in S(r', t') = S(r-1, t+1),$$

where $\beta \neq 0$, $r(A_3') = r' = r - 1$ and $(\mathbf{0}, A_1', A_2', A_3', \mathbf{0}, A_4')$ is $(r-1)$-dimensional subspace of

$$\begin{array}{c} \begin{array}{cccccc} t & r & \nu-m & t & r & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & \alpha_1 & \alpha_2 & \mathbf{0} & \alpha_3 & \alpha_4 \\ \mathbf{0} & A_1 & A_2 & \mathbf{0} & A_3 & A_4 \end{pmatrix} \end{array}.$$

Without loss of generality, we assume that $(\mathbf{0}, \alpha_1, \alpha_2, \mathbf{0}, \alpha_3, \alpha_4)$ is in P but not in Q. Since $P \sim Q$, we have

$$\begin{pmatrix} \alpha_3 \\ A_3 \end{pmatrix} \beta^t = 0. \tag{2}$$

where the matrix $\left(\begin{smallmatrix} \alpha_3 \\ A_3 \end{smallmatrix}\right)$ has column full rank, it turns out that $\beta = \mathbf{0}$, this contradicts with $\beta \neq 0$. Hence there is no such vertex Q in $S(r', t')$ with the property that P is adjacent to Q.

It is clear that there exists edges between $S(r, t)$ and $S(r', t')$ for other cases. $\square$

Lemma 2.3 provides us with significant implication to explore the following partition of vertex sets $\Gamma_i, i \geq 2$, and from which we can easily obtain the diameter of graph Γ.

Theorem 2.5. *Let $i \geq 2$. Then $V(\Gamma_i) = \bigcup\limits_{r=0}^{i-1} S(r, m-i) \cup S(i-1, m-(i-1))$.*

Proof. Induct on the integer i. If $i = 2$ and P is any vertex of $V(\Gamma_1) = S(0, m-1)$, then by Lemma 2.3, the vertices adjacent to P are in $S(1, m-1), S(0, m-1), S(0, m-2), S(1, m-2)$ or $S(0, m)$, where $S(0, m)$ is the vertex M. Therefore, the vertices in $S(1, m-1), S(0, m-2)$ and $S(1, m-2)$ are at distance 2 from the vertex M.

Now assume that $i \geq 3$ and the conclusion holds for i. For any $X \in V(\Gamma_{i+1})$, there is vertex $P \in V(\Gamma_i)$ such that $P \sim X$. If P is in $S(i-1, m-(i-1))$, then it is easy to see from Lemma 2.3 and Proposition 2.4 that X belongs to $S(i-1, m-i)$, $S(i-1, m-i+1)$ or $S(i-2, m-i+1)$, and by induction the vertices in $S(i-1, m-i)$ and $S(i-1, m-i+1)$ are at distance i, and the vertices in set $S(i-2, m-i+1)$ are at distance $i-1$ from M, which is a contradiction. Therefore, $P \in S(r, m-i)$, where $0 \leq r \leq i-1$. Then by Lemma 2.3 again, we have $X \in S(r, m-i+1) \cup S(r, m-i) \cup S(r+1, m-i) \cup S(r-1, m-i) \cup S(r+1, m-i-1) \cup S(r, m-i-1) \cup S(r-1, m-i+1)$. By induction the vertices in set $S(r, m-i+1)$ are at distance $i-1$ and the vertices in $S(r, m-i)$, $S(r+1, m-i)$ and $S(r-1, m-i)$ are at distance i from M. Hence $X \in S(r+1, m-i-1)$, $S(r, m-i-1)$, or $S(r+1, m-i)$(if $r = i-1$). Thus we see that $V(\Gamma_{i+1}) = \bigcup_{r=0}^{i} S(r, m-i-1) \cup S(i, m-i)$. $\square$

It is easy to see that Theorem 2.5 yields the next corollary.

Corollary 2.6. *Suppose that $P \in S(r, t)$, Then*

$$d_\Gamma(M, P) = \begin{cases} m-t, \text{if } r+t \leq m-1 \\ r+1, \text{if } r+t = m. \end{cases}$$

Theorem 2.7. *The diameter of the graph Γ is $m+1$.*

Proof. In Corollary 2.6, let $t = 0$, it is easy to see that the diameter of Γ is less than or equal to $m+1$. In fact, there exist

$$P = \begin{array}{c} \begin{matrix} \nu & m & \nu - m \end{matrix} \\ \begin{pmatrix} \mathbf{0} & I & \mathbf{0} \end{pmatrix} \end{array} \in S(m, 0)$$

such that $d_\Gamma(M, P) = m+1$ by Corollary 2.6. $\square$

Lemma 2.8. (1) *$V(\Gamma_1)$ consists of vertices of form*

$$\Gamma(k, \gamma; b, \alpha, \beta) = \begin{array}{c} \begin{matrix} k-1 & 1 & m-k & \nu-m & m & \nu-m \end{matrix} \\ \begin{pmatrix} \mathbf{0} & b & \mathbf{0} & \alpha & \mathbf{0} & \beta \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \in S(0, m-1), \tag{3}$$

where $(\alpha, \beta) \neq \mathbf{0}$ and in particular, $\gamma = \emptyset$ (disappear) in the special case $k = m$.

(2) *$V(\Gamma_2)$ consists of vertices of forms*

(*i*)

$$\begin{array}{c} \begin{array}{cccccccc} k-1 & 1 & m-k & \nu-m & k-1 & 1 & m-k & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & b & \mathbf{0} & \alpha & \mathbf{0} & 1 & \gamma & \beta \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\gamma^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \in S(1, m-1); \quad (4)$$

(*ii*)

$$\begin{array}{c} \begin{array}{cccccccc} k-1 & 1 & l-k-1 & 1 & m-l & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & a_1 & \mathbf{0} & b_1 & \mathbf{0} & \alpha_1 & \mathbf{0} & \beta_1 \\ \mathbf{0} & a_2 & \mathbf{0} & b_2 & \mathbf{0} & \alpha_2 & \mathbf{0} & \beta_2 \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_1^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_2^t & \mathbf{0} & \gamma_3^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \in S(0, m-2), \quad (5)$$

where $\mathrm{r}\begin{pmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{pmatrix} = 2$, *and* $\alpha_1\beta_2^t - \beta_1\alpha_2^t = 0$*;*

(*iii*)

$$\begin{array}{c} \begin{array}{cccccccccccc} k-1 & 1 & l-k-1 & 1 & m-l & \nu-m & k-1 & 1 & l-k-1 & 1 & m-l & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & a_1 & \mathbf{0} & b_1 & \mathbf{0} & \alpha_1 & \mathbf{0} & c & c\gamma_1 & d & \delta & \beta_1 \\ \mathbf{0} & a_2 & \mathbf{0} & b_2 & \mathbf{0} & \alpha_2 & \mathbf{0} & 0 & \mathbf{0} & 0 & \mathbf{0} & \beta_2 \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\gamma_1^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\gamma_2^t & \mathbf{0} & -\gamma_3^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}, \quad (6)$$

where $\delta = c\gamma_2 + d\gamma_3$*,* $(\alpha_2, \beta_2) \neq \mathbf{0}$*,* $(c, d) \neq \mathbf{0}$*,* $a_2c + b_2d + \alpha_2\beta_1^t - \beta_2\alpha_1^t = 0$*. And clearly the vertices of form* (6) *belong to* $S(1, m-2)$*.*

Proof. (1) Suppose that $X \in V(\Gamma_1) = S(0, m-1)$. That means $\dim(X \cap M) = m-1$ and $MKX^t = \mathbf{0}$. So we assume that

$$X = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} \xi & \alpha & \eta & \beta \\ A & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{l} 1 \\ m-1 \end{array},$$

where the rank of A is $m-1$, $(\alpha, \eta, \beta) \neq \mathbf{0}$, and $A\eta^t = \mathbf{0}$. Obviously, there exists $T \in GL_{m-1}(\mathbb{F}_q)$ such that

$$TA = \begin{pmatrix} I^{(k-1)} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^t & I^{(m-k)} \end{pmatrix}.$$

Thus, there is $T' = \begin{pmatrix} 1 & \\ & T \end{pmatrix} \in GL_m(\mathbb{F}_q)$ such that

$$T'X = \begin{array}{c} \begin{array}{cccccc} k-1 & 1 & m-k & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} \xi_1 & \xi_2 & \xi_3 & \alpha & \eta & \beta \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \begin{array}{l} 1 \\ k-1 \\ m-k \end{array},$$

where $\xi = (\xi_1, \xi_2, \xi_3)$. It follows from $MKX^t = \mathbf{0}$ that $\eta = 0$, and further by elementary rank transformations of matrix, $\xi = (\xi_1, \xi_2, \xi_3)$ can be transformed into $(\mathbf{0}, b, \mathbf{0})$, where $b \in \mathbb{F}_q$, consequently X is of the form (3).

(2) By Theorem 2.5, the vertex set $V(\Gamma_2) = \bigcup_{i=0}^{1} S(i, m-2) \cup S(1, m-1)$. By using the proof technique detailed as shown above, it is not hard to obtain the forms of vertices of $V(\Gamma_2)$. The details of the proof are omitted here for brevity. □

Let $\Gamma(k,\gamma) = \{\Gamma(k,\gamma;b,\alpha,\beta) : (b,\alpha,\beta) \in \mathbb{F}_q^{(2\nu-2m+1)}\}$, the vertex set is determined by k and γ. This provides us with a partition of the vertices of $V(\Gamma_1)$ into disjoint sets.

Theorem 2.9. *Γ is a $\frac{(q^{2\nu-2m+1}-q)(q^m-1)}{(q-1)^2}$-regular graph.*

Proof. The result follows directly from [13, Theorem 2.3]. □

Theorem 2.10. (1) *Assume that X is any vertex of $S(1,m-1)$, then $|\Gamma_1(X)\cap S(0,m-1)| = \frac{q^{2(\nu-m)}-q}{q-1}$;*

(2) *Assume that X is any vertex of $S(r,t)$, then $|\Gamma_1(X)\cap S(r-1,t+1)| = \frac{(q^r-1)(q^{m-t-r}-1)}{(q-1)^2}$, where $r\geq 1, r+t\leq m$. In particular, if $X\in S(r,m-r)$, where $r\geq 2$, then $|\Gamma_1(X)\cap S(r-1,m-r+1)| = 0$;*

(3) *Assume that X is any vertex of $S(0,t)$, then $|\Gamma_1(X)\cap S(0,t+1)| = (\frac{q^{m-t}-1}{q-1})^2$, where $0\leq t\leq m-2$.*

Proof. (1) Supposed that X is a vertex of the form (4) in $S(1,m-1)$ and $Y = \Gamma(k_1,\gamma_1;b_1,\alpha_1,\beta_1)$ is the vertex adjacent to X in $S(0,m-1)$. If $X\sim Y$, then $k=k_1, \gamma=\gamma_1$, and

$$\begin{cases} b_1+\alpha_1\beta^t-\beta_1\alpha^t = 0, \\ (\alpha_1,\beta_1)\neq \mathbf{0}. \end{cases}$$

Thus it can be seen that the number of the choices of Y is $\frac{q^{2(\nu-m)}-q}{q-1}$.

(2) Assume that

$$X = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1 & B_2 & B_3 & B_4 \\ C_1 & C_2 & \mathbf{0} & C_3 \end{pmatrix} \end{array} \begin{array}{l} \\ t \\ r \\ m-t-r \end{array}$$

is any vertex of $S(r,t)$, where $\mathrm{r}(A)=t$, $\mathrm{r}(B_3)=r$. Clearly, the set $\Gamma_1(X)\cap S(r-1,t+1)$ consists of vertices of form

$$Y = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \alpha & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1' & B_2' & B_3' & B_4' \\ C_1 & C_2 & \mathbf{0} & C_3 \end{pmatrix} \end{array} \begin{array}{l} \\ t \\ 1 \\ r-1 \\ m-t-r \end{array},$$

where $\mathrm{r}\binom{A}{\alpha} = t+1$ and (B_1',B_2',B_3',B_4') is a subspace of (B_1,B_2,B_3,B_4) with dimension $r-1$. It follows from [16, Corollary 1.8] that (B_1',B_2',B_3',B_4') has $\frac{q^r-1}{q-1}$ choices. For convenience, we may assume that

$$(B_1,B_2,B_3,B_4) = \begin{pmatrix} \beta_1 & \beta_2 & \beta_3 & \beta_4 \\ B_1' & B_2' & B_3' & B_4' \end{pmatrix},$$

where $(\beta_1,\beta_2,\beta_3,\beta_4)\notin Y$. Since $Y\sim X$ and $YKY^t=\mathbf{0}$, we have

$$\begin{cases} \beta_3\alpha^t = 0, \\ B_3'\alpha^t = \mathbf{0}. \end{cases}$$

Note that $\mathrm{r}\begin{pmatrix} \beta_3 \\ B_3' \end{pmatrix} = \mathrm{r}(B_3) = r$ and α has $m-t$ unknowns, thus α has $\frac{q^{m-t-r}-1}{q-1}$ choices. Hence it is easy to see

$$|S(r-1,t+1)\cap\Gamma_1(X)| = \frac{q^r-1}{q-1}\cdot\frac{q^{m-t-r}-1}{q-1}.$$

(3) Suppose that

$$X = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1 & B_2 & \mathbf{0} & B_3 \end{pmatrix} \end{array} \begin{array}{l} t \\ m-t \end{array} \in S(0,t),$$

where the rank of A is t. Then the vertices of $\Gamma_1(X) \cap S(0, t+1)$ have the form of

$$Y = \begin{array}{c} \begin{array}{cccc} m & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} A & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \alpha & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ B_1' & B_2' & \mathbf{0} & B_3' \end{pmatrix} \end{array} \begin{array}{l} d \\ 1 \\ m-d-1 \end{array},$$

where $(B_1', B_2', \mathbf{0}, B_3')$ is a subspace of $(B_1, B_2, \mathbf{0}, B_3)$ of dimension $m-t-1$. It is easy to check that both $(B_1', B_2', \mathbf{0}, B_3')$ and α have $\frac{q^{m-t}-1}{q-1}$ choices. Hence there are $(\frac{q^{m-t}-1}{q-1})^2$ vertices in $\Gamma_1(X) \cap S(0, t+1)$. $\square$

Lemma 2.11. (1) *For any* $X \in V(\Gamma_1)$*, the common number of* M *and* X *is* $|\,\Gamma(M,X)\,| = \frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1}$*;*

(2) *For any* $X \in V(\Gamma_2)$*,* $|\,\Gamma(M,X)\,| = \frac{q^{2\nu-2m}-q}{q-1}, (q+1)^2$ *or* 1.

Proof. (1) Assume that $X = \Gamma(k,\gamma; b, \alpha, \beta) \in V(\Gamma_1)$ with the form (3). It is known that the set $\Gamma(M,X)$ of common neighbors of M and X is in $V(\Gamma_1) = S(0, m-1)$, and the vertices adjacent to X in $V(\Gamma_1)$ are also adjacent to M. Suppose that $Y = \Gamma(k', \gamma'; b', \alpha', \beta')$ such that $Y \in S(0, m-1)$ and $X \sim Y$.

(i) Firstly, consider the case where $k \neq k'$ and assume that $k > k'$. Then we assume that

$$X = \begin{array}{c} \begin{array}{cccccccc} k'-1 & 1 & k-k'-1 & 1 & m-k & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & 0 & \mathbf{0} & b & \mathbf{0} & \alpha & \mathbf{0} & \beta \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & 1 & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \gamma & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \in \Gamma(k,\gamma),$$

$$Y = \begin{array}{c} \begin{array}{cccccccc} k'-1 & 1 & k-k'-1 & 1 & m-k & \nu-m & m & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & b' & \mathbf{0} & 0 & \mathbf{0} & \alpha' & \mathbf{0} & \beta' \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_1 & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & c & \mathbf{0} & 1 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \gamma_2 & \mathbf{0} & \mathbf{0} & I & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array} \in \Gamma(k',\gamma'),$$

where $\gamma' = (\gamma_1, c, \gamma_2)$. Let $\xi_X^{(k)}$ be the k-th row of the matrix X. Since $X \sim Y$, we have that for any $b' \in \mathbb{F}_q$, there exists $a = b' + bc \in \mathbb{F}_q$ such that

$$\xi_X^{(1)} + a\xi_X^{(k'+1)} = \xi_Y^{(1)} + b\xi_Y^{(k)}.$$

Thus $(\alpha', \beta') = (\alpha, \beta)$, this means that b' can be any element of $\mathbb{F}_q$. Therefore, $|\,\Gamma_1(X) \cap \Gamma(k', \gamma')\,| = q$ for any $X \in \Gamma(k,\gamma)$.

(ii) Secondly, consider the case that $k = k'$ and $\gamma \neq \gamma'$. Then clearly $(\alpha', \beta') = (\alpha, \beta)$ and the dimension of $X + Y$ is $m+1$. This implies that $\dim(X \cap Y) = m-1$. It is obvious that for any $b' \in \mathbb{F}_q$, we have $XKY^t = \mathbf{0}$. It turns out that $|\,\Gamma_1(X) \cap \Gamma(k, \gamma')\,| = q$ for any X of $\Gamma(k,\gamma)$.

(iii) Finally, suppose that $k = k', \gamma = \gamma'$. Then the elements α' and β' of Y need satisfy the following conditions,

$$\begin{cases} \beta'\alpha^t - \alpha'\beta^t = 0, \\ (\alpha', \beta') \neq \mathbf{0}. \end{cases}$$

It is follows that $|\Gamma_1(X)\cap\Gamma(k,\gamma)| = \frac{q(q^{2\nu-2m-1}-1)}{q-1}$.

Since $V(\Gamma_1)$ is divided into $\frac{q^m-1}{q-1}$ different sets of form $\Gamma(k,\gamma)$,

$$|\Gamma(M,X)| = q(\frac{q^m-1}{q-1}-1)+\frac{q(q^{2\nu-2m-1}-1)}{q-1}=\frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1}.$$

(2) If X is any element of $S(1,m-1)$, then by Theorem 2.10(1), $|\Gamma_1(X)\cap S(0,m-1)| = \frac{q^{2(\nu-m)}-q}{q-1}$. If X is any element of $S(1,m-2)$, then by Theorem 2.10(2), $|\Gamma_1(X)\cap S(0,m-1)| = 1$. Finally, if $X\in S(0,m-2)$, then by Theorem 2.10(3), $|\Gamma_1(X)\cap S(0,m-1)| = (q+1)^2$. □

It is clear that for any vertex of Γ such that $d_\Gamma(M,X)\geq 3$, $\Gamma(M,X)=\emptyset$. Since Γ is vertices-transitive, by Lemma 2.2, Theorem 2.9 and Lemma 2.11, we have the following result.

Theorem 2.12. *Γ is a 5-Deza graph with parameters $(n,k,\{\lambda_i, i=1,\ldots,5\})$, where*

$$n=\frac{\prod\limits_{i=\nu-m+1}^{\nu}(q^{2i}-1)}{\prod\limits_{i=1}^{m}(q^i-1)},\qquad k=\frac{(q^{2\nu-2m+1}-q)(q^m-1)}{(q-1)^2},$$

$$\lambda_1=\frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1},\quad \lambda_2=\frac{q^{2\nu-2m}-q}{q-1},\lambda_3=(q+1)^2,\lambda_4=1,\lambda_5=0.$$

3. Results about subconstituents Γ_1 and Γ_2

We mainly discuss the structure of the first subconstituents Γ_1 in this section, and we also give some information about Γ_2.

Lemma 3.1. *Γ_1 is a $\frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1}$-regular graph.*

Proof. Let X be any vertex of graph Γ_1. Then it is easy to see from Lemma 2.11 the number of $\Gamma(M,X)$ is the number of the neighbors of X in Γ_1. Therefore the degree of X is $\frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1}$, and Γ_1 is regular. □

Lemma 3.2. (1) *Let X and Y be the two adjacent vertices of $V(\Gamma_1)$ and $\Gamma_1(X,Y)$ be the common neighbors of vertices X and Y in Γ_1. Then*

$$|\Gamma_1(X,Y)| = \frac{q^{m+1}-3q+2}{q-1},\frac{q^{2(\nu-m)-1}-q}{q-1}\ \text{or}\ \frac{q^{2(\nu-m)}+q^{m+1}-q^2-q}{q-1}.$$

(2) *Let X and Y be the two nonadjacent vertices of $V(\Gamma_1)$. Then*

$$|\Gamma_1(X,Y)| = \frac{q^{2(\nu-m)-1}-q}{q-1},2q\ \text{or}\ 0.$$

Proof. (1) Let $X=\Gamma(k,\gamma;b,\alpha,\beta)$ and $Y=\Gamma(k',\gamma';b',\alpha',\beta')$ such that $X\sim Y$.

Firstly, consider the case $\gamma\neq\gamma'$. From the proof of Lemma 2.11, it can be concluded that $(\alpha,\beta)=(\alpha',\beta')$ and there exist the following q vertices

$$\Gamma(k,\gamma_0;b_0,\alpha,\beta), b_0\in\mathbb{F}_q$$

in $\Gamma(k,\gamma_0)$ adjacent to both X and Y, where $\Gamma(k,\gamma_0)\neq\Gamma(k,\gamma),\Gamma(k,\gamma_0)\neq\Gamma(k',\gamma')$. In addition, $|\Gamma_1(X,Y)\cap\Gamma(k,\gamma)| = |\Gamma_1(X,Y)\cap\Gamma(k',\gamma')| = q-1$. Hence $|\Gamma_1(X,Y)| = q(\frac{q^m-1}{q-1}-2)+2(q-1)=\frac{q^{m+1}-3q+2}{q-1}$.

And then, let $k=k',\gamma=\gamma'$ and P be vertex of $V(\Gamma_1)$ adjacent to both X and Y. We only need to consider the following different subcases.

(a) If $P=\Gamma(k,\gamma;x,\xi,\eta)\in\Gamma(k,\gamma)$ which is of the form (3), then we have

$$\begin{cases}\alpha\eta^t-\beta\xi^t=0,\\ \alpha'\eta^t-\beta'\xi^t=0,\\ (\xi,\eta)\neq\mathbf{0},\end{cases}$$

where $\alpha'\beta^t-\beta'\alpha^t=0$. If $\mathrm{r}\begin{pmatrix}\alpha&\beta\\ \alpha'&\beta'\end{pmatrix}=2$, then $\mid\Gamma_1(X,Y)\cap\Gamma(k,\gamma)\mid=\frac{q^{2(\nu-m)-1}-q}{q-1}$. If $\mathrm{r}\begin{pmatrix}\alpha&\beta\\ \alpha'&\beta'\end{pmatrix}=1$, then $\mid\Gamma_1(X,Y)\cap\Gamma(k,\gamma)\mid=\frac{q^{2(\nu-m)}-q}{q-1}$.

(b) Suppose that $l\neq k$ and $P=\Gamma(l,\gamma_0;x,\xi,\eta)\in\Gamma(l,\gamma_0)$. Clearly, $\gamma\neq\gamma_0$. We can assume, without loss of generality, that $k>l$, and

$$X=\begin{array}{c}\begin{array}{cccccccc} l-1 & 1 & k-l-1 & 1 & m-k & \nu-m & m & \nu-m\end{array}\\ \begin{pmatrix}\mathbf{0}&0&\mathbf{0}&b&\mathbf{0}&\alpha&\mathbf{0}&\beta\\ I&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&1&\mathbf{0}&0&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\eta&I&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array}\in\Gamma(k,\gamma),$$

$$Y=\begin{array}{c}\begin{array}{cccccccc} l-1 & 1 & k-l-1 & 1 & m-k & \nu-m & m & \nu-m\end{array}\\ \begin{pmatrix}\mathbf{0}&0&\mathbf{0}&b'&\mathbf{0}&\alpha'&\mathbf{0}&\beta'\\ I&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&1&\mathbf{0}&0&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&I&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\mathbf{0}&\mathbf{0}&\eta&I&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array}\in\Gamma(k,\gamma),$$

$$P=\begin{array}{c}\begin{array}{cccccccc} l-1 & 1 & k-l-1 & 1 & m-k & \nu-m & m & \nu-m\end{array}\\ \begin{pmatrix}\mathbf{0}&x&\mathbf{0}&0&\mathbf{0}&\xi&\mathbf{0}&\eta\\ I&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\gamma_1&I&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&c&\mathbf{0}&1&\mathbf{0}&\mathbf{0}&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&\gamma_2&\mathbf{0}&\mathbf{0}&I&\mathbf{0}&\mathbf{0}&\mathbf{0}\end{pmatrix}\end{array},\gamma_0=(\gamma_1,c,\gamma_2).$$

If $\mathrm{r}\begin{pmatrix}\alpha&\beta\\ \alpha'&\beta'\end{pmatrix}=2$, then there is no vertex P such that $P\sim X, P\sim Y$ in $\Gamma(l,\gamma_0)$. Now assumet that $\mathrm{r}\begin{pmatrix}\alpha&\beta\\ \alpha'&\beta'\end{pmatrix}=1$. Since $P\sim X$ and $P\sim Y$, it follows that $(\xi,\eta)=d(\alpha,\beta)$, where d is any nonzero element of $\mathbb{F}_q$. Meanwhile, x is any element of $\mathbb{F}_q$. Therefore, $\mid\Gamma_1(X,Y)\cap\Gamma(l,\eta_0)\mid=\frac{q(q-1)}{q-1}=q$.

(c) Suppose that $P\in\Gamma(k,\gamma_0)$, where $\gamma_0\neq\gamma$. Similarly, if $\mathrm{r}\begin{pmatrix}\alpha&\beta\\ \alpha'&\beta'\end{pmatrix}=2$, then there is no vertex P such that $P\sim X, P\sim Y$ in $\Gamma(k,\gamma_0)$, and there are q common neighbors of X and Y in $\Gamma(k,\gamma_0)$ for the case of $\mathrm{r}\begin{pmatrix}\alpha_1&\beta_1\\ \alpha_2&\beta_2\end{pmatrix}=1$.

Above all, we obtain

$$\mid\Gamma_1(X,Y)\mid=\frac{q^{m+1}-3q+2}{q-1},\frac{q^{2\nu-2m-1}-q}{q-1}\ \text{ or}$$

$$\frac{q^{2\nu-2m}-q}{q-1}+(\frac{q^m-1}{q-1}-1)q=\frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1}.$$

(2) Let $X=\Gamma(k,\gamma;b,\alpha,\beta)$ and $Y=\Gamma(k',\gamma';b',\alpha',\beta')$ be two nonadjacent vertices and P be a vertex of $V(\Gamma_1)$ such that $X\sim P$ and $Y\sim P$.

(a) Suppose that $k=k',\gamma=\gamma'$ and $P=\Gamma(k,\gamma;x,\xi,\eta)\in\Gamma(k,\gamma)$. Then it is easy to see from $XKP^t=YKP^t=\mathbf{0}$ and $M\cap P=m-1$ that P satisfies the following conditions,

$$\begin{cases}\alpha\eta^t-\beta\xi^t=0,\\ \alpha'\eta^t-\beta'\xi^t=0,\\ (\xi,\eta)\neq\mathbf{0}.\end{cases}$$

Since $\beta'\alpha^t - \alpha'\beta^t \neq 0$, it is obvious that $r\begin{pmatrix} \alpha & \beta \\ \alpha' & \beta' \end{pmatrix} = 2$. It turns out that the number of the choices of P satisfying above conditions is $\frac{q^{2\nu-2m-1}-q}{q-1}$. Now suppose that $P \in \Gamma(l, \gamma_0)$, where $k \neq l$. Then there is no vertex P such that $X \sim P, Y \sim P$ in $\Gamma(k, \eta_0)$. Thus,

$$| \Gamma_1(X, Y) |= \frac{q^{2\nu-2m-1} - q}{q - 1}.$$

(b) In the next part of the proof we assume that $\gamma \neq \gamma'$. If $P = \Gamma(l, \gamma_0; x, \xi, \eta) \in \Gamma(l, \gamma_0)$, where $\gamma_0 \neq \gamma, \gamma'$, then it is concluded from $P \sim X$ that $(\xi, \eta) = (\alpha, \beta)$. Furthermore, by $P \sim Y$, we have $(\xi, \eta) = (\alpha', \beta')$, which implies that $X \sim Y$, this contradicts $X \nsim Y$. Thus there is no common neighbors of X and Y in $\Gamma(l, \gamma_0)$. If $P = \Gamma(k', \gamma'; x, \xi, \eta) \in \Gamma(k', \gamma')$, then $(\xi, \eta) = (\alpha, \beta)$. Suppose that $\beta'\alpha^t - \alpha'\beta^t = 0$. Then X and Y have q common neighbors in $\Gamma(k', \gamma')$ since x is any element of $\mathbb{F}_q$. Meanwhile, there are also q common neighbors of X and Y in $\Gamma(k, \gamma)$. Suppose that $\beta'\alpha^t - \alpha'\beta^t \neq 0$. Then it is clear that there is no such vertex P with the property $P \sim X$ and $P \sim Y$.

Thus, by (a) and (b), we see that $| \Gamma_1(X_1, X_2) |= \frac{q^{2\nu-2m-1}-q}{q-1}, 2q$ or 0. □

By Lemma 3.1 and Lemma 3.2, it is immediately seen the following theorem.

Theorem 3.3. Γ_1 *is a* 5*-Deza graph with parameters* $(n, k, \{\lambda_i, i = 1, \ldots, 5\})$*, where*

$$n = \frac{(q^{2\nu-2m+1}-1)(q^m-1)}{(q-1)^2}, k = \frac{q^{2\nu-2m}+q^{m+1}-q^2-q}{q-1}, \lambda_1 = \frac{q^{m+1}-3q+2}{q-1},$$
$$\lambda_2 = \frac{q^{2(\nu-m)-1}-q}{q-1}, \lambda_3 = \frac{q^{2(\nu-m)}+q^{m+1}-q^2-q}{q-1}, \lambda_4 = 2q, \lambda_5 = 0.$$

Finally, we present some results for the second subconstituents of Γ. For simplicity, let Γ_{21}, Γ_{22}, and Γ_{23} be the subgraphs induced by $S(1, m-1), S(0, m-2)$, and $S(1, m-2)$, respectively.

Theorem 3.4. (1) *Let* $\Gamma_{21}(k, \gamma)$ (*see the form* (4)) *be the subset of* $S(1, m-1)$ *determined by* k *and* γ*. Then* $V(\Gamma_{21})$ *can be partitioned into* $\frac{q^m-1}{q-1}$ *connected components. Each graph induced by* $\Gamma_{21}(k, \gamma)$ *is a* $q^{2\nu-2m}$*-regular subgraph with* $q^{2\nu-2m+1}$ *vertices. Moreover, the distance of* Γ_{21} *is at most* 3.

(2) *Let* $\Gamma_{22}(k, l, \gamma_1, \gamma_2, \gamma_3)$ (*see the form* (5)) *be the subset of* $S(0, m-2)$ *determined by* $(k, l, \gamma_1, \gamma_2, \gamma_3)$*. Then* $V(\Gamma_{22})$ *can be partitioned into* $\frac{q^{2m-1}-q^m-q^{m-1}+1}{(q-1)^2(q+1)}$ *pairwise disjoint sets, and* Γ_{22} *is isomorphic to the generalized symplectic graph* $\Gamma(K_1, 2, 1, 0)$ *based on* $\mathbb{F}_q^{2(\nu-m+1)}$*, where*

$$K_1 = \begin{pmatrix} 0 & I^{(\nu-m+1)} \\ -I^{(\nu-m+1)} & 0 \end{pmatrix}.$$

(3) *Let* $\Gamma_{23}(k, l, \gamma_1, \gamma_2, \gamma_3)$ (*see the form* (6)) *be the subset of* $S(1, m-2)$ *determined by* $(k, l, \gamma_1, \gamma_2, \gamma_3)$*. Then* $V(\Gamma_{23})$ *can be partitioned into* $\frac{q^{2m-1}-q^m-q^{m-1}+1}{(q-1)^2(q+1)}$ *connected components.*

Proof. (1) Since $\Gamma_{21}(k, \gamma)$ is determined by k and γ, it is easy to see from the form (4) that $1 \leq k \leq m$ and η has q^{m-k} choices, which implies that $V(\Gamma_{21})$ can be partitioned into $\frac{q^m-1}{q-1}$ different sets. Let $X \in \Gamma_{21}(k, \gamma)$ and $Y \in \Gamma_{21}(k', \gamma')$. Note that $\gamma \neq \gamma'$ if $\Gamma_{21}(k, \gamma) \neq \Gamma_{21}(k', \gamma')$. Consequently, it is clear that the dimension of $X \cap Y$ is not $m-1$, and hence there is no edge between different sets $\Gamma_{21}(k, \gamma)$ and $\Gamma_{21}(k', \gamma')$.

Obviously, there are $q^{2\nu-2m+1}$ vertices in each set $\Gamma_{21}(k, \gamma)$ since $(b, \alpha, \beta) \in \mathbb{F}_q^{2\nu-2m+1}$. Let X be any vertex of form (4) in $\Gamma_{21}(k, \gamma)$ and $P \in S(1, m-1)$ such that $P \sim X$, and clearly $P \in \Gamma_{21}(k, \gamma)$. Thus, let

$$P = \begin{array}{c} \begin{array}{cccccccc} k-1 & 1 & m-k & \nu-m & k-1 & 1 & m-k & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & b' & \mathbf{0} & \alpha' & \mathbf{0} & 1 & \gamma & \beta' \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\gamma^t & I & \mathbf{0} & \mathbf{0} & 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}.$$

It follows that

$$b - b' - \alpha'\beta^t + \beta'\alpha^t = 0.$$

This implies that the degree of X is $q^{2\nu-2m}$.

Now we consider the distance between any two vertices of $\Gamma_{21}(k,\gamma)$. Let

$$X_i = \begin{array}{c} \begin{array}{cccccccc} k-1 & 1 & m-k & \nu-m & k-1 & 1 & m-k & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & b_i & \mathbf{0} & \alpha_i & \mathbf{0} & 1 & \gamma & \beta_i \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\gamma^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}, i = 1,2,$$

be any two vertices of $\Gamma_{21}(k,\gamma)$.

(a) If $b_1 - b_2 + \alpha_1\beta_2^t - \beta_1\alpha_2^t = 0$, then we obviously have that $X_1 \sim X_2$ and $d_\Gamma(X_1, X_2) = 1$.

(b) If $b_1 - b_2 + \alpha_1\beta_2^t - \beta_1\alpha_2^t \neq 0$ and $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$, then $X_1 \nsim X_2$, and it is evidently that the linear equations

$$\begin{cases} -b_1 + b - \beta\alpha_1^t + \alpha\beta_1^t = 0, \\ -b_2 + b - \beta\alpha_2^t + \alpha\beta_2^t = 0, \end{cases}$$

with respect to (b, α, β) always have solution. Thus there always exists vertex X of form (4) such that $X \sim X_1$ and $X \sim X_2$, and so $d_\Gamma(X_1, X_2) = 2$.

(c) Finally, we assume that $b_1 - b_2 + \alpha_1\beta_2^t - \beta_1\alpha_2^t \neq 0$ and $(\alpha_1, \beta_1) = (\alpha_2, \beta_2)$. Let

$$X_i' = \begin{array}{c} \begin{array}{cccccccc} k-1 & 1 & m-k & \nu-m & k-1 & 1 & m-k & \nu-m \end{array} \\ \begin{pmatrix} \mathbf{0} & b_i' & \mathbf{0} & \alpha_i' & \mathbf{0} & 1 & \eta & \beta_i' \\ I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\eta^t & I & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \end{array}, i = 1,2.$$

Obviously, the following equations

$$\begin{cases} -b_1' + b_1 - \alpha_1'\beta_1^t + \beta_1'\alpha_1^t = 0, \\ b_2' - b_2 - \beta_2'\alpha_2^t + \alpha_2'\beta_2^t = 0, \\ -b_1' + b_2' - \beta_2'(\alpha_1')^t + \alpha_2'(\beta_1')^t = 0, \end{cases}$$

with respect to $(b_1', \alpha_1', \beta_1', b_2', \alpha_2', \beta_2')$ always have solution such that $X_1' \neq X_2'$, and this implies that $X_1 \sim X_1'$, $X_2 \sim X_2'$, and $X_1' \sim X_2'$. Therefore the distance between X_1 and X_2 in $\Gamma_{21}(i,\eta)$ is 3.

(2) Since $(\gamma_1, \gamma_2, \gamma_3)$ has $q^{2m-k-l-1}$ possibilities for given integers k and l, it is not hard to check that there are $\Sigma_{k=1}^{m-1}\Sigma_{l=2}^{m} q^{2m-k-l-1} = \frac{q^{2m-1}-q^m-q^{m-1}+1}{(q-1)^2(q+1)}$ sets of form $\Gamma_{22}(k,l,\gamma_1,\gamma_2,\gamma_3)$. Clearly, there is a one-to-one correspondence between vertices of $\Gamma_{22}(k,l,\gamma_1,\gamma_2,\gamma_3)$ and vertices of $\Gamma(K_1,2,1,0)$, and so $\Gamma_{22} \cong \Gamma(K_1,2,1,0)$.

(3) Using an analogous argument as above we obtain that $V(\Gamma_{23})$ can be partitioned into $\frac{q^{2m-1}-q^m-q^{m-1}+1}{(q-1)^2(q+1)}$ pairwise disjoint sets. Let $X \in \Gamma_{23}(k,l,\gamma_1,\gamma_2,\gamma_3)$ and $Y \in \Gamma_{23}(k',l',\gamma_1',\gamma_2',\gamma_3')$. It is not hard to see that the dimension of $X \cap Y$ is not $m-1$ when $\Gamma_{23}(k,l,\gamma_1,\gamma_2,\gamma_3) \neq \Gamma_{23}(k',l',\gamma_1',\gamma_2',\gamma_3')$, and consequently $X \nsim Y$. Therefore, each subgraph induced by $\Gamma_{23}(k',l',\gamma_1',\gamma_2',\gamma_3')$ is a connected component of Γ_{23}. $\square$

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