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Revised and expanded numerical radius inequalities for 2x2 partitioned operators

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Abstract. More several power inequalities of the numerical radius for 2×2 partitioned operators are proved by generalizing earlier inequalities. Moreover, we give some applications of our results in estimation of numerical radius.

1. Introduction

Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathscr{H}$ with inner product $\langle \cdot, \cdot \rangle$. For $T \in \mathscr{B}(\mathscr{H})$, let $\omega(T)$ and $\|T\|$ be the numerical radius and the standard operator norm of T, respectively. Recall that

$$\omega(T) = \sup \{ |\langle Tu, u \rangle| : u \in \mathscr{H}, \ \|u\| = 1 \}$$

and

$$\|T\| = \sup \{ |\langle Tu, v \rangle| : u, v \in \mathscr{H}, \ \|u\| = \|v\| = 1 \}.$$

The direct sum of two copies of the complex Hilbert space $\mathscr{H}$ is indicated as $\mathscr{H}^2 = \mathscr{H} \oplus \mathscr{H}$. If $\Gamma_1, \Gamma_2, \Gamma_3, \Gamma_4 \in \mathscr{B}(\mathscr{H})$, then the partitioned operator $\Gamma = \begin{bmatrix} \Gamma_1 & \Gamma_2 \\ \Gamma_3 & \Gamma_4 \end{bmatrix}$ is regarded as an operator in $\mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ such that $\Gamma u = \begin{bmatrix} \Gamma_1 u_1 + \Gamma_2 u_2 \\ \Gamma_3 u_1 + \Gamma_4 u_2 \end{bmatrix}$ for every vector $u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \in \mathscr{H} \oplus \mathscr{H}$.

It is a widely recognized fact that $\omega(.)$ establishes a norm on $\mathscr{B}(\mathscr{H})$ that is equivalent to the standard operator norm $\|.\|$. In fact, for any $T \in \mathscr{B}(\mathscr{H})$,

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|. \tag{1}$$

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The inequalities in (1) are sharp. The first inequality becomes an equality when $T^2 = 0$. The second inequality becomes an equality when the operator T is normal.

The following inequality is a significant property of the numerical radius, which is called the power inequality for the numerical radius, asserts that

$$\omega\left(T^k\right) \leq \omega^k(T) \text{ for } k = 1, 2, \ldots. \tag{2}$$

For more details about the numerical radius, see [8, 9].

In [12], Kittaneh refined the second inequality in (1), and obtained the following result:

$$\omega(T) \leq \frac{1}{2}\| |T| + |T^*| \| \leq \frac{1}{2}\left(\|T\| + \left\|T^2\right\|^{\frac{1}{2}}\right) \text{ for any } T \in \mathscr{B}(\mathscr{H}), \tag{3}$$

The symbol $|T|$ represents the absolute value of the operator T written as $|T| = (T^*T)^{\frac{1}{2}}$.

Other improvements on the inequalities in (1) have been established in [13]. These improvements assert that if $T \in \mathscr{B}(\mathscr{H})$, then

$$\frac{1}{4}\left\| |T|^2 + |T^*|^2 \right\| \leq \omega^2(T) \leq \frac{1}{2}\left\| |T|^2 + |T^*|^2 \right\|. \tag{4}$$

The second part of the compound inequality in (4) is a specific instance of a more extensive form, as stated in [7].

$$\omega^{2s}(T) \leq \frac{1}{2}\left\| |T|^{2s} + |T^*|^{2s} \right\|, \text{ for } s \geq 1. \tag{5}$$

In [6], Dragomir presented a power numerical radius inequality that relates to the product of two operators:

$$\omega^s(\Gamma_1{}^*\Gamma_2) \leq \frac{1}{2}\left\| |\Gamma_2|^{2s} + |\Gamma_1|^{2s} \right\|, \text{ for } s \geq 1. \tag{6}$$

For more such inequalities, see [6, 14,16] and references therein.

In Section 2, we employ the Buzano extension of Schwarz's inequality to establish new upper bounds for the numerical radii of 2×2 partitioned operators. We provide a sufficient condition for the equality case in the first part of the compound inequality in (4). In Section 3, we establish additional power numerical radius inequalities for 2×2 partitioned operators. Our numerical radius inequalities are both sharp and more comprehensive than previous findings in the literature.

Throughout this paper, whenever the phrase "AGM inequality" is mentioned, it means the arithmetic-geometric mean inequality.

2. Improvements on numerical radius inequalities for 2×2 operator matrices

Our study commences with the utilization of the following established lemmas. The first lemma can be derived from the spectral theorem for positive operators and Jensen's inequality (see e.g., [11]).

Lemma 2.1. Given $\Gamma \in \mathscr{B}(\mathscr{H})$ be a positive operator, and let $u \in \mathscr{H}$ with $\|u\| = 1$. Then

$$\langle \Gamma u, u\rangle^s \leq \langle \Gamma^s u, u\rangle \text{ for all } s \geq 1.$$

The reverse direction of the above inequality holds if $0 < s \leq 1$.

The second lemma, which can be seen in [2], introduced a norm inequality including convex functions of positive operators.

Lemma 2.2. Given h be a non-negative, convex function on $[0, \infty)$, and let $\aleph_1, \aleph_2 \in \mathscr{B}(\mathscr{H})$ be positive operators. Then

$$\left\| h\left(\frac{\aleph_1 + \aleph_2}{2}\right)\right\| \leq \left\|\frac{h(\aleph_1) + h(\aleph_2)}{2}\right\|.$$

In particular, if $s \geq 1$, then

$$\left\| \left(\frac{\aleph_1 + \aleph_2}{2}\right)^s\right\| \leq \left\|\frac{\aleph_1^s + \aleph_2^s}{2}\right\|.$$

In [11], Kittaneh has proven the third lemma.

Lemma 2.3. Given $M \in \mathscr{B}(\mathscr{H})$, and let f and g be non-negative functions on $[0, \infty)$ which are continuous and that satisfy the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$|\langle Mu, v\rangle| \leq \|f(|M|)u\| \, \|g(|M^*|)v\| \text{ for all } u, v \in \mathscr{H}.$$

The fourth lemma, known as the Buzano extension of Schwarz's inequality, can be found in reference [5].

Lemma 2.4. Given $u, v, e \in \mathscr{H}$ with $\|e\| = 1$, we have

$$|\langle u, e\rangle \langle e, v\rangle| \leq \frac{1}{2}(\|u\| \, \|v\| + |\langle u, v\rangle|).$$

The fifth lemma, which includes two parts, is very helpful (see, e.g., [3,10]).

Lemma 2.5. Given $\aleph_1, \aleph_2 \in \mathscr{B}(\mathscr{H})$, we have

(i) $\omega\left(\begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_2 \end{bmatrix}\right) = max\{\omega(\aleph_1), \omega(\aleph_2)\}$.

(ii) $\omega\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix}\right) = max\{\omega(\aleph_1 + \aleph_2), \omega(\aleph_1 - \aleph_2)\}$.

In particular,

$$\omega\left(\begin{bmatrix} 0 & \aleph_2 \\ \aleph_2 & 0 \end{bmatrix}\right) = \omega(\aleph_2).$$

The last lemma is a simple consequence of Jensen's inequality, concerning the convexity of the function $f(x) = x^\lambda$, $\lambda \geq 1$ on $x \in [0, \infty)$.

Lemma 2.6. If $x_1, x_2, \ldots, x_n$ are non-negative real numbers, then

$(x_1 + x_2 + \cdots + x_n)^\lambda \le n^{\lambda-1}\left(x_1^\lambda + x_2^\lambda + \cdots + x_n^\lambda\right)$ for $\lambda \ge 1$.

The initial outcome can be expressed as follows.

Theorem 2.1. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ and $r \ge 1$. Then

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right) \le 2^{2r-1} max\left\{\omega^{2r}(\aleph_1), \omega^{2r}(\aleph_4)\right\} + 2^{2r-2} max\left\{\omega^r(\aleph_2\aleph_3), \omega^r(\aleph_3\aleph_2)\right\}$$
$$+2^{r-2} max\left\{\left\|\left(|\aleph_3|^2 + |\aleph_2^*|^2\right)^r\right\|, \left\|\left(|\aleph_2|^2 + |\aleph_3^*|^2\right)^r\right\|\right\}$$

Proof. Let $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$, and let $\phi = \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix}$, $\psi = \begin{bmatrix} 0 & \aleph_2 \\ \aleph_3 & 0 \end{bmatrix}$. Then

$$\left|\left\langle \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u, u \right\rangle\right|^{2r}$$
$$= \left|\left\langle (\phi + \psi) u, u \right\rangle\right|^{2r}$$
$$= \left|\left\langle \phi u, u \right\rangle + \left\langle \psi u, u \right\rangle\right|^{2r}$$
$$\le \left(\left|\left\langle \phi u, u \right\rangle\right| + \left|\left\langle \psi u, u \right\rangle\right|\right)^{2r}$$
$$\le 2^{2r-1}\left(\left|\left\langle \phi u, u \right\rangle\right|^{2r} + \left|\left\langle \psi u, u \right\rangle\right|^{2r}\right) \text{ (by Lemma 2.6)}$$
$$= 2^{2r-1}\left(\left|\left\langle \phi u, u \right\rangle\right|^{2r} + \left|\left\langle \psi u, u \right\rangle \left\langle u, \psi^* u \right\rangle\right|^{r}\right)$$
$$\le 2^{2r-1}\left(\left|\left\langle \phi u, u \right\rangle\right|^{2r} + 2^{-r}\left[\left|\left\langle \psi u, \psi^* u \right\rangle\right| + \|\psi u\| \|\psi^* u\|\right]^{r}\right) \text{ (by Lemma 2.4)}$$
$$\le 2^{2r-1}\left|\left\langle \phi u, u \right\rangle\right|^{2r} + 2^{2r-2}\left[\|\psi u\|^r \|\psi^* u\|^r + \left|\left\langle \psi u, \psi^* u \right\rangle\right|^{r}\right] \text{ (by Lemma 2.6)}$$
$$= 2^{2r-1}\left|\left\langle \phi u, u \right\rangle\right|^{2r} + 2^{2r-2}\left[\left\langle |\psi|^2 u, u \right\rangle^{\frac{r}{2}} \left\langle |\psi^*|^2 u, u \right\rangle^{\frac{r}{2}} + \left|\left\langle \psi^2 u, u \right\rangle\right|^{r}\right]$$
$$\le 2^{2r-1}\left|\left\langle \phi u, u \right\rangle\right|^{2r} + 2^{r-2}\left\langle \left(|\psi|^2 + |\psi^*|^2\right) u, u \right\rangle^{r} + 2^{2r-2}\left|\left\langle \psi^2 u, u \right\rangle\right|^{r} \text{(by the AGM inequality)}$$
$$\le 2^{2r-1}\left|\left\langle \phi u, u \right\rangle\right|^{2r} + 2^{r-2}\left\langle \left(|\psi|^2 + |\psi^*|^2\right)^r u, u \right\rangle + 2^{2r-2}\left|\left\langle \psi^2 u, u \right\rangle\right|^{r} \text{(by Lemma 2.1)}$$
$$= 2^{2r-1}\left|\left\langle \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix} u, u \right\rangle\right|^{2r} + 2^{r-2}\left\langle \begin{bmatrix} \left(|\aleph_3|^2 + |\aleph_2^*|^2\right)^r & 0 \\ 0 & \left(|\aleph_2|^2 + |\aleph_3^*|^2\right)^r \end{bmatrix} u, u \right\rangle + 2^{2r-2}\left|\left\langle \begin{bmatrix} \aleph_2\aleph_3 & 0 \\ 0 & \aleph_3\aleph_2 \end{bmatrix} u, u \right\rangle\right|^{r}$$

Thus,

$$\left|\left\langle \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u, u \right\rangle\right|^{2r} \le 2^{2r-1}\left|\left\langle \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix} u, u \right\rangle\right|^{2r} + 2^{r-2}\left\langle \begin{bmatrix} \left(|\aleph_3|^2 + |\aleph_2^*|^2\right)^r & 0 \\ 0 & \left(|\aleph_2|^2 + |\aleph_3^*|^2\right)^r \end{bmatrix} u, u \right\rangle$$
$$+2^{2r-2}\left|\left\langle \begin{bmatrix} \aleph_2\aleph_3 & 0 \\ 0 & \aleph_3\aleph_2 \end{bmatrix} u, u \right\rangle\right|^{r}.$$

By considering the supremum over all $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$ in the above inequality, we have

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right) \le 2^{2r-1} max\left\{\omega^{2r}(\aleph_1), \omega^{2r}(\aleph_4)\right\} + 2^{2r-2} max\left\{\omega^r(\aleph_2\aleph_3), \omega^r(\aleph_3\aleph_2)\right\}$$

$$+2^{r-2}max\left\{\left\|\left(|\aleph_3|^2+\left|\aleph_2^*\right|^2\right)^r\right\|,\left\|\left(|\aleph_2|^2+\left|\aleph_3^*\right|^2\right)^r\right\|\right\},$$

as required.

Theorem 2.1 presents a collection of power numerical radius inequalities for partitioned operators. The following results illustrate some of these inequalities.
If $\aleph_1 = \aleph_4$ *and* $\aleph_2 = \aleph_3$ in Theorem 2.1, then we get the following:

Corollary 2.1. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ and $r \geq 1$. Then

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix}\right) = max\left\{\omega^{2r}(\aleph_1+\aleph_2), \omega^{2r}(\aleph_1-\aleph_2)\right\} \text{ (by Lemma 2.5(ii))}$$

$$\leq 2^{2r-1}\omega^{2r}(\aleph_1) + 2^{2r-2}\omega^r\left(\aleph_2^2\right) + 2^{r-2}\left\|\left(|\aleph_2|^2+\left|\aleph_2^*\right|^2\right)^r\right\|.$$

Remark 2.1. Letting $\aleph_1 = \aleph_2$ in Corollary 2.1, we have the following inequality which is a generalization of the inequality given in [1]. To see this, note that

$$2^{2r}\omega^{2r}(\aleph_1) = \omega^{2r}(2\aleph_1) \leq 2^{2r-1}\omega^{2r}(\aleph_1) + 2^{2r-2}\omega^r\left(\aleph_1^2\right) + 2^{r-2}\left\|\left(|\aleph_1|^2+\left|\aleph_1^*\right|^2\right)^r\right\|.$$

So,

$$\omega^{2r}(\aleph_1) \leq \frac{1}{2}\omega^r\left(\aleph_1^2\right) + 2^{-r-1}\left\|\left(|\aleph_1|^2+\left|\aleph_1^*\right|^2\right)^r\right\|,$$

which is an improvement of the inequality (5). To prove this, note that

$$\omega^{2r}(\aleph_1) \leq \frac{1}{2}\omega^r\left(\aleph_1^2\right) + 2^{-r-1}\left\|\left(|\aleph_1|^2+\left|\aleph_1^*\right|^2\right)^r\right\|$$

$$\leq \frac{1}{2}\omega^{2r}(\aleph_1) + 2^{-r-1}\left\|\left(\frac{2|\aleph_1|^2+2\left|\aleph_1^*\right|^2}{2}\right)^r\right\|. \text{ (by the inequality (2))}$$

Thus,

$$\omega^{2r}(\aleph_1) \leq (2)2^{-r-1}\left(2^{r-1}\right)\left\||\aleph_1|^{2r}+\left|\aleph_1^*\right|^{2r}\right\| \text{ (Lemma 2.2)}$$

$$= \frac{1}{2}\left\||\aleph_1|^{2r}+\left|\aleph_1^*\right|^{2r}\right\|.$$

Remark 2.2. Letting $r = 2$ in Remark 2.1 yields the second inequality in (4). In fact, we have

$$\omega^4(\aleph_1) \leq \frac{1}{2}\omega^2\left(\aleph_1^2\right) + \frac{1}{8}\left\|\left(|\aleph_1|^2+\left|\aleph_1^*\right|^2\right)^2\right\|$$

$$\leq \frac{1}{2}\omega^4(\aleph_1) + \frac{1}{8}\left\||\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|^2. \text{ (by the inequality (2))}$$

Now,

$$\frac{1}{2}\omega^4(\aleph_1) \leq \frac{1}{8}\left\||\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|^2.$$

So,

$$\omega^4(\aleph_1) \leq \frac{1}{4}\left\||\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|^2.$$

Thus,

$$\omega^2(\aleph_1) \leq \frac{1}{2}\left\||\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|.$$

Theorem 2.2. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, $r \geq 1$, and let f and g be non-negative functions on $[0, \infty)$, which are continuous and that satisfy the relation $f(t)\,g(t) = t$ for all $t \in [0, \infty)$. Then

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right) \leq \frac{3^{r-1}}{2} max \left\{ \begin{array}{c} \left\| f^{4r}(|\aleph_1|) + g^{4r}\left(\left|\aleph_1^*\right|\right) + f^{4r}(|\aleph_3|) + g^{4r}\left(\left|\aleph_2^*\right|\right) + \left(|\aleph_1|^2 + \left|\aleph_2^*\right|^2\right)^r \right\|, \\ \left\| f^{4r}(|\aleph_4|) + g^{4r}\left(\left|\aleph_4^*\right|\right) + f^{4r}(|\aleph_2|) + g^{4r}\left(\left|\aleph_3^*\right|\right) + \left(|\aleph_4|^2 + \left|\aleph_3^*\right|^2\right)^r \right\| \end{array} \right\}$$

$$+6^{r-1}\omega^r\left(\begin{bmatrix} 0 & \aleph_2\aleph_4 \\ \aleph_3\aleph_1 & 0 \end{bmatrix}\right).$$

Proof. Let $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$, and let $\phi = \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix}$, $\psi = \begin{bmatrix} 0 & \aleph_2 \\ \aleph_3 & 0 \end{bmatrix}$. Then

$$\begin{aligned}
&\left|\left\langle \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u, u \right\rangle\right|^{2r} \\
&= \left|\left\langle (\phi + \psi)\, u, u \right\rangle\right|^{2r} \\
&= \left|\left\langle \phi u, u \right\rangle + \langle \psi u, u \rangle\right|^{2r} \\
&\leq \left(\left|\left\langle \phi u, u \right\rangle\right| + \left|\langle \psi u, u \rangle\right|\right)^{2r} \\
&= \left(\left|\left\langle \phi u, u \right\rangle\right|^2 + 2\left|\left\langle \phi u, u \right\rangle\right|\left|\langle \psi u, u \rangle\right| + \left|\langle \psi u, u \rangle\right|^2\right)^r \\
&\leq 3^{r-1}\left[\left|\left\langle \phi u, u \right\rangle\right|^{2r} + \left|\langle \psi u, u \rangle\right|^{2r} + 2^r\left|\left\langle \phi u, u \right\rangle \langle u, \psi^* u \rangle\right|^r\right] \text{ (by Lemma 2.6)} \\
&\leq 3^{r-1}\left[\begin{array}{c} \left\langle f^2\left(|\phi|\right) u,\, u \right\rangle^r \left\langle g^2\left(|\phi^*|\right) u, u \right\rangle^r + \left\langle f^2\left(|\psi|\right) u,\, u \right\rangle^r \left\langle g^2\left(|\psi^*|\right) u,\, u \right\rangle^r \\ +2^r\,(2^{-r})\left(\|\phi u\|\,\|\psi^* u\| + \left|\left\langle \phi u, \psi^* u \right\rangle\right|\right)^r \end{array} \right] \text{ (by Lemma 2.3 and Lemma 2.4)} \\
&\leq 3^{r-1}\left[\begin{array}{c} \left\langle f^{2r}\left(|\phi|\right) u,\, u \right\rangle \left\langle g^{2r}\left(|\phi^*|\right) u, u \right\rangle + \left\langle f^{2r}\left(|\psi|\right) u,\, u \right\rangle \left\langle g^{2r}\left(|\psi^*|\right) u,\, u \right\rangle \\ +2^{r-1}\left(\|\phi u\|^r \|\psi^* u\|^r + \left|\left\langle \psi\phi u, u \right\rangle\right|^r\right) \end{array} \right] \text{ (by Lemma 2.1 and Lemma 2.6)} \\
&\leq 3^{r-1}\left[\begin{array}{c} \frac{1}{2}\left(\left\langle f^{2r}\left(|\phi|\right) u,\, u \right\rangle^2 + \left\langle g^{2r}\left(|\phi^*|\right) u, u \right\rangle^2 + \left\langle f^{2r}\left(|\psi|\right) u,\, u \right\rangle^2 + \left\langle g^{2r}\left(|\psi^*|\right) u,\, u \right\rangle^2\right) \\ +2^{r-1}\left(\left\langle |\phi|^2 u, u \right\rangle^{\frac{r}{2}} \left\langle |\psi^*|^2 u, u \right\rangle^{\frac{r}{2}} + \left|\left\langle \psi\phi u, u \right\rangle\right|^r\right) \end{array} \right] \\
&\leq \frac{3^{r-1}}{2}\left\langle \left(f^{4r}\left(|\phi|\right) + g^{4r}\left(|\phi^*|\right) + f^{4r}\left(|\psi|\right) + g^{4r}\left(|\psi^*|\right)\right) u,\, u \right\rangle \\
&\quad +6^{r-1}\left(2^{-r}\left\langle \left(|\phi|^2 + |\psi^*|^2\right)^r u,\, u \right\rangle\right) + 6^{r-1}\left|\left\langle \psi\phi u, u \right\rangle\right|^r \text{ (by Lemma 2.1 and the AGM inequality)} \\
&= \frac{3^{r-1}}{2}\left\langle \left(f^{4r}\left(|\phi|\right) + g^{4r}\left(|\phi^*|\right) + f^{4r}\left(|\psi|\right) + g^{4r}\left(|\psi^*|\right) + \left(|\phi|^2 + |\psi^*|^2\right)^r\right) u,\, u \right\rangle + 6^{r-1}\left|\left\langle \psi\phi u, u \right\rangle\right|^r
\end{aligned}$$

Thus,

$$\left|\left\langle \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u, u \right\rangle\right|^{2r} \leq \frac{3^{r-1}}{2}\left\langle \left(f^{4r}\left(|\phi|\right) + g^{4r}\left(|\phi^*|\right) + f^{4r}\left(|\psi|\right) + g^{4r}\left(|\psi^*|\right) + \left(|\phi|^2 + |\psi^*|^2\right)^r\right) u,\, u \right\rangle$$
$$+6^{r-1}\left|\left\langle \psi\phi u, u \right\rangle\right|^r$$

By considering the supremum over all $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$, we obtain the desired outcome.

Theorem 2.2 presents a collection of norm inequalities pertaining to the numerical radii of partitioned operators. Below are some of these inequalities.

Let $f(t) = g(t) = t^{\frac{1}{2}}$, $\aleph_1 = \aleph_4$ and $\aleph_2 = \aleph_3$ in Theorem 2.2. Then we get the following:

Corollary 2.2. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, $r \geq 1$. Then

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix}\right)$$
$$= max\left\{\omega^{2r}(\aleph_1 + \aleph_2), \omega^{2r}(\aleph_1 - \aleph_2)\right\}$$
$$\leq \frac{3^{r-1}}{2}\left\| |\aleph_1|^{2r} + \left|\aleph_1^*\right|^{2r} + |\aleph_2|^{2r} + \left|\aleph_2^*\right|^{2r} + \left(|\aleph_1|^2 + \left|\aleph_2^*\right|^2\right)^r \right\| + 6^{r-1}\omega^r(\aleph_2\aleph_1).$$

Remark 2.3 If $\aleph_1 = \aleph_2$ in the above corollary, then

$$\omega^{2r}(\aleph_1) \leq \frac{1}{6}\left(\frac{3}{4}\right)^r \left\| 2\left(|\aleph_1|^{2r} + \left|\aleph_1^*\right|^{2r}\right) + \left(|\aleph_1|^2 + \left|\aleph_1^*\right|^2\right)^r \right\| + \frac{1}{6}\left(\frac{3}{2}\right)^r \omega^r\left(\aleph_1^2\right).$$

To see this, note that

$$2^{2r}\omega^{2r}(\aleph_1) = \omega^{2r}(2\aleph_1) \leq \frac{3^r}{6}\left\| 2\left(|\aleph_1|^{2r} + \left|\aleph_1^*\right|^{2r}\right) + \left(|\aleph_1|^2 + \left|\aleph_1^*\right|^2\right)^r \right\| + 6^{r-1}\omega^r\left(\aleph_1^2\right).$$

Thus,

$$\omega^{2r}(\aleph_1) \leq \frac{1}{6}\left(\frac{3}{4}\right)^r \left\| 2\left(|\aleph_1|^{2r} + \left|\aleph_1^*\right|^{2r}\right) + \left(|\aleph_1|^2 + \left|\aleph_1^*\right|^2\right)^r \right\| + \frac{1}{6}\left(\frac{3}{2}\right)^r \omega^r\left(\aleph_1^2\right).$$

Remark 2.4. If $r = 1$ in Remark 2.3, then we have

$$\omega^2(\aleph_1) \leq \frac{3}{8}\left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2 \right\| + \frac{1}{4}\omega^2(\aleph_1) \text{ . (by the inequality (2))}$$

Note that the last inequality is a refinement of the second part of the compound inequality in (4).

The last theorem in this section, however, can be proved using similar techniques as in the previous theorems and Lemmas 2.6, 2.4, 2.6, and 2.1, respectively.

Theorem 2.3. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ and $r \geq 1$. Then

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right) \leq 3^{r-1} max\left\{\omega^{2r}(\aleph_1), \omega^{2r}(\aleph_4)\right\} + \frac{3^r}{6} max\left\{\omega^r(\aleph_2\aleph_3), \omega^r(\aleph_3\aleph_2)\right\}$$
$$+6^{r-1}\omega^r\left(\begin{bmatrix} 0 & \aleph_2\aleph_4 \\ \aleph_3\aleph_1 & 0 \end{bmatrix}\right) + max\left\{ \begin{array}{l} \left\| \frac{1}{6}\left(\frac{3}{2}\right)^r \left(|\aleph_3|^2 + \left|\aleph_2^*\right|^2\right)^r + \frac{3^r}{6}\left(|\aleph_1|^2 + \left|\aleph_2^*\right|^2\right)^r \right\|, \\ \left\| \frac{1}{6}\left(\frac{3}{2}\right)^r \left(|\aleph_2|^2 + \left|\aleph_3^*\right|^2\right)^r + \frac{3^r}{6}\left(|\aleph_4|^2 + \left|\aleph_3^*\right|^2\right)^r \right\| \end{array} \right\}.$$

Theorem 2.3 presents a collection of norm inequalities pertaining to the numerical radii of partitioned operators. Below are some of these inequalities.

If $\aleph_1 = \aleph_4$ and $\aleph_2 = \aleph_3$ in Theorem 2.3, then we get the following:

Corollary 2.3. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$ and $r \geq 1$. Then

$$\omega^{2r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix}\right)$$
$$= max\left\{\omega^{2r}(\aleph_1 + \aleph_2), \omega^{2r}(\aleph_1 - \aleph_2)\right\}$$
$$\leq 3^{r-1}\omega^{2r}(\aleph_1) + \left\| \frac{1}{6}\left(\frac{3}{2}\right)^r \left(|\aleph_2|^2 + \left|\aleph_2^*\right|^2\right)^r + \frac{3^r}{6}\left(|\aleph_1|^2 + \left|\aleph_2^*\right|^2\right)^r \right\| + \frac{3^r}{6}\omega^r\left(\aleph_2^2\right) + 6^{r-1}\omega^r(\aleph_2\aleph_1)$$

Remark 2.5. If $r = 1$ and $\aleph_1 = \aleph_2$ in Corollary 2.3, then we have
$\omega^2(\aleph_1) \le \frac{1}{4}\left\| \left|\aleph_1^2\right| + \left|\aleph_1^{*2}\right| \right\| + \frac{1}{2}\omega\left(\aleph_1^2\right).$
To see this, note that
$4\omega^2(\aleph_1) = \omega^2(2\aleph_1) \le \omega^2(\aleph_1) + \left\|\frac{1}{4}\left(|\aleph_1|^2 + \left|\aleph_1^*\right|^2\right) + \frac{1}{2}\left(|\aleph_1|^2 + \left|\aleph_1^*\right|^2\right)\right\| + \frac{1}{2}\omega\left(\aleph_1^2\right) + \omega\left(\aleph_1^2\right).$
So,
$3\omega^2(\aleph_1) \le \frac{3}{4}\left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\| + \frac{3}{2}\omega\left(\aleph_1^2\right).$
Thus,
$\omega^2(\aleph_1) \le \frac{1}{4}\left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\| + \frac{1}{2}\omega\left(\aleph_1^2\right).$
Here, this inequality becomes an equality if $\aleph_1^2 = 0$. That is, if $\aleph_1^2 = 0$, then in view of the last inequality and the first inequality in (4), we get
$\omega^2(\aleph_1) = \frac{1}{4}\left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\|.$

3. Generalized power numerical radius inequalities for 2 × 2 partitioned operators

This section commences with the proof of a lemma that relies on the Buzano extension of Schwarz's inequality.

Lemma 3.1. Let $u, v, e \in \mathscr{H}$ with $\|e\| = 1$, $r \ge 1$, and $z \in \mathbb{C}$. Then
$|\langle u, e\rangle \langle e, v\rangle|^{2r} \le \frac{3^{r-1}}{4^r}\left(\left(\frac{2+|z|}{1+|z|}\right)\|u\|^{2r}\|v\|^{2r} + \left(2^r + \frac{|z|}{1+|z|}\right)\|u\|^r\|v\|^r|\langle u, v\rangle|^r\right).$

Proof. Using the Cauchy-Schwarz inequality, we have

$$
\begin{aligned}
|\langle u, v\rangle|^{2r} &= \left(\frac{1+|z|}{1+|z|}\right)|\langle u, v\rangle|^{2r} \\
&= \left(\frac{1}{1+|z|}\right)|\langle u, v\rangle|^{2r} + \left(\frac{|z|}{1+|z|}\right)|\langle u, v\rangle|^{2r} \\
&\le \left(\frac{1}{1+|z|}\right)\|u\|^{2r}\|v\|^{2r} + \left(\frac{|z|}{1+|z|}\right)\|u\|^r\|v\|^r|\langle u, v\rangle|^r.
\end{aligned}
$$

Using Lemma 2.4, we get

$$
\begin{aligned}
|\langle u, e\rangle \langle e, v\rangle|^{2r} &\le \left(\frac{\|u\|\,\|v\| + |\langle u, v\rangle|}{2}\right)^{2r} \\
&= \frac{1}{4^r}\left(\|u\|^2\|v\|^2 + 2\,\|u\|\,\|v\|\,\,|\langle u, v\rangle| + |\langle u, v\rangle|^2\right)^r \\
&\le \frac{3^{r-1}}{4^r}\left(\|u\|^{2r}\|v\|^{2r} + |\langle u, v\rangle|^{2r} + 2^r\|u\|^r\|v\|^r|\langle u, v\rangle|^r\right) \text{(by Lemma 2.6)} \\
&\le \frac{3^{r-1}}{4^r}\left(\begin{array}{c} \|u\|^{2r}\|v\|^{2r} + \left(\frac{1}{1+|z|}\right)\|u\|^{2r}\|v\|^{2r} + \left(\frac{|z|}{1+|z|}\right)\|u\|^r\|v\|^r|\langle u, v\rangle|^r \\ +2^r\|u\|^r\|v\|^r|\langle u, v\rangle|^r \end{array}\right) \\
&= \frac{3^{r-1}}{4^r}\left(\left(\frac{2+|z|}{1+|z|}\right)\|u\|^{2r}\|v\|^{2r} + \left(2^r + \frac{|z|}{1+|z|}\right)\|u\|^r\|v\|^r|\langle u, v\rangle|^r\right),
\end{aligned}
$$

as required.

Remark 3.1. The inequality in Lemma 3.1 is more stricter than the inequality in [4, Lemma 3.1] when $r = 1$ and $z = 0$. To show this, note that

$$\begin{aligned}|\langle u,e\rangle\langle e,v\rangle|^2 &\le \frac{1}{4}\left(2\|u\|^2\|v\|^2 + 2\|u\|\|v\|\ |\langle u,v\rangle|\right)\\ &\le \frac{1}{4}\left(2\|u\|^2\|v\|^2 + \|u\|^2\|v\|^2 + \|u\|\|v\|\ |\langle u,v\rangle|\right)\\ &= \frac{1}{4}\left(3\|u\|^2\|v\|^2 + \|u\|\|v\|\ |\langle u,v\rangle|\right).\end{aligned}$$

The above lemma enable us to prove the following theorem.

Theorem 3.1. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} \in \mathscr{B}(\mathscr{H}\oplus\mathscr{H})$, $z \in \mathbb{C}$, and $r \ge 1$. Then

$$\begin{aligned}&\omega^{4r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right)\\ &\le 2^{4r-1} max\left\{\omega^{4r}(\aleph_1)\ ,\omega^{4r}(\aleph_4)\right\} + (12)^{r-1}\left(\frac{2+|z|}{1+|z|}\right) max\left\{\left\||\aleph_3|^{4r} + \left|\aleph_2^*\right|^{4r}\right\|, \left\||\aleph_2|^{4r} + \left|\aleph_3^*\right|^{4r}\right\|\right\}\\ &\quad +(12)^{r-1}\left(2^r + \frac{|z|}{1+|z|}\right) max\left\{\left\||\aleph_3|^{2r} + \left|\aleph_2^*\right|^{2r}\right\|, \left\||\aleph_2|^{2r} + \left|\aleph_3^*\right|^{2r}\right\|\right\}\ max\left\{\omega^r(\aleph_2\aleph_3), \omega^r(\aleph_3\aleph_2)\right\}.\end{aligned}$$

Proof. Let $u \in \mathscr{H}\oplus\mathscr{H}$ with $\|u\| = 1$, and let $\phi = \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix}$, $\psi = \begin{bmatrix} 0 & \aleph_2 \\ \aleph_3 & 0 \end{bmatrix}$. Then

$$\begin{aligned}&\left|\left\langle\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u,u\right\rangle\right|^{4r}\\ &= \left|\left\langle \phi u,u\right\rangle + \left\langle \psi u,u\right\rangle\right|^{4r}\\ &\le \left(\left|\left\langle \phi u,u\right\rangle\right| + \left|\left\langle \psi u,u\right\rangle\right|\right)^{4r}\\ &\le 2^{4r-1}\left(\left|\left\langle \phi u,u\right\rangle\right|^{4r} + \left|\left\langle \psi u,u\right\rangle\left\langle u,\ \psi^* u\right\rangle\right|^{2r}\right) \text{ (by Lemma 2.6)}\\ &\le 2^{4r-1}\left|\left\langle \phi u,u\right\rangle\right|^{4r} + 2^{4r-1}\left(\frac{3^{r-1}}{4^r}\right)\begin{pmatrix} \left(\frac{2+|z|}{1+|z|}\right)\|\psi u\|^{2r}\|\psi^* u\|^{2r} \\ +\left(2^r + \frac{|z|}{1+|z|}\right)\|\psi u\|^{r}\|\psi^* u\|^{r}\left|\left\langle \psi u, \psi^* u\right\rangle\right|^r \end{pmatrix} \text{(by Lemma 3.1)}\\ &= 2^{4r-1}\left|\left\langle \phi u,u\right\rangle\right|^{4r} + 2^{2r-1}\left(3^{r-1}\right)\begin{pmatrix} \left(\frac{2+|z|}{1+|z|}\right)\left\langle|\psi|^2 u,u\right\rangle^r\left\langle|\psi^*|^2 u,u\right\rangle^r \\ +\left(2^r + \frac{|z|}{1+|z|}\right)\left\langle|\psi|^2 u,u\right\rangle^{\frac{r}{2}}\left\langle|\psi^*|^2 u,u\right\rangle^{\frac{r}{2}}\left|\left\langle \psi^2 u,u\right\rangle\right|^r \end{pmatrix}\\ &\le 2^{4r-1}\left|\left\langle \phi u,u\right\rangle\right|^{4r} + \frac{(12)^r}{6}\begin{pmatrix} \left(\frac{2+|z|}{1+|z|}\right)\left\langle|\psi|^{2r} u,u\right\rangle\left\langle|\psi^*|^{2r} u,u\right\rangle + \\ \left(2^r + \frac{|z|}{1+|z|}\right)\left\langle|\psi|^{2r} u,u\right\rangle^{\frac{1}{2}}\left\langle|\psi^*|^{2r} u,u\right\rangle^{\frac{1}{2}}\left|\left\langle \psi^2 u,u\right\rangle\right|^r \end{pmatrix}\\ &\le 2^{4r-1}\left|\left\langle \phi u,u\right\rangle\right|^{4r} + \frac{(12)^r}{6}\begin{pmatrix} \left(\frac{2+|z|}{2+2|z|}\right)\left\langle\left(|\psi|^{4r} + |\psi^*|^{4r}\right)u,u\right\rangle \\ +\left(2^{r-1} + \frac{|z|}{2+2|z|}\right)\left\langle\left(|\psi|^{2r} + |\psi^*|^{2r}\right)u,u\right\rangle\left|\left\langle \psi^2 u,u\right\rangle\right|^r \end{pmatrix}\end{aligned}$$

(by the AGM inequality and Lemma 2.1)

$$= 2^{4r-1}\left|\left\langle\begin{bmatrix}\aleph_1 & 0\\ 0 & \aleph_4\end{bmatrix}u,u\right\rangle\right|^{4r} + (12)^{r-1}\left(\frac{2+|z|}{1+|z|}\right)\left\langle\begin{bmatrix}|\aleph_3|^{4r}+\left|\aleph_2^*\right|^{4r} & 0\\ 0 & |\aleph_2|^{4r}+\left|\aleph_3^*\right|^{4r}\end{bmatrix}u,u\right\rangle$$
$$+(12)^{r-1}\left(2^r+\frac{|z|}{1+|z|}\right)\left\langle\begin{bmatrix}|\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r} & 0\\ 0 & |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r}\end{bmatrix}u,u\right\rangle\left|\left\langle\begin{bmatrix}\aleph_2\aleph_3 & 0\\ 0 & \aleph_3\aleph_2\end{bmatrix}u,u\right\rangle\right|^{r}.$$

Thus,

$$\left|\left\langle\begin{bmatrix}\aleph_1 & \aleph_2\\ \aleph_3 & \aleph_4\end{bmatrix}u,u\right\rangle\right|^{4r}$$
$$\leq 2^{4r-1}\left|\left\langle\begin{bmatrix}\aleph_1 & 0\\ 0 & \aleph_4\end{bmatrix}u,u\right\rangle\right|^{4r} + (12)^{r-1}\left(\frac{2+|z|}{1+|z|}\right)\left\langle\begin{bmatrix}|\aleph_3|^{4r}+\left|\aleph_2^*\right|^{4r} & 0\\ 0 & |\aleph_2|^{4r}+\left|\aleph_3^*\right|^{4r}\end{bmatrix}u,u\right\rangle$$
$$+(12)^{r-1}\left(2^r+\frac{|z|}{1+|z|}\right)\left\langle\begin{bmatrix}|\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r} & 0\\ 0 & |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r}\end{bmatrix}u,u\right\rangle\left|\left\langle\begin{bmatrix}\aleph_2\aleph_3 & 0\\ 0 & \aleph_3\aleph_2\end{bmatrix}u,u\right\rangle\right|^{r}.$$

By considering the supremum over all $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$ in the last inequality, we have the desired outcome.

Now, we will introduce some special cases of Theorem 3.1.
If $\aleph_1 = \aleph_4$ *and* $\aleph_2 = \aleph_3$ in Theorem 3.1, then we get the following:

Corollary 3.1. Let $\Gamma = \begin{bmatrix}\aleph_1 & \aleph_2\\ \aleph_2 & \aleph_1\end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, $z \in \mathbb{C}$, and $r \geq 1$. Then

$$\omega^{4\mathrm{r}}\left(\begin{bmatrix}\aleph_1 & \aleph_2\\ \aleph_2 & \aleph_1\end{bmatrix}\right) = max\left\{\omega^{4\mathrm{r}}(\aleph_1+\aleph_2), \omega^{4\mathrm{r}}(\aleph_1-\aleph_2)\right\} \text{ (by Lemma 2.5(ii))}$$
$$\leq 2^{4r-1}\omega^{4\mathrm{r}}(\aleph_1) + (12)^{r-1}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_2|^{4\mathrm{r}}+\left|\aleph_2^*\right|^{4\mathrm{r}}\right\|$$
$$+(12)^{r-1}\left(2^r+\frac{|z|}{1+|z|}\right)\left\| |\aleph_2|^{2\mathrm{r}}+\left|\aleph_2^*\right|^{2\mathrm{r}}\right\|\omega^r\left(\aleph_2^2\right).$$

Remark 3.2. If $\aleph_1 = \aleph_2$ in Corollary 3.1, then we have

$$\omega^{4\mathrm{r}}(\aleph_1) \leq \frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_1|^{4\mathrm{r}}+\left|\aleph_1^*\right|^{4\mathrm{r}}\right\| + \frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_1|^{2\mathrm{r}}+\left|\aleph_1^*\right|^{2\mathrm{r}}\right\|\omega^r\left(\aleph_1^2\right).$$

To see this, note that from Corollary 3.1 with $\aleph_1 = \aleph_2$ we have

$$2^{4r}\omega^{4\mathrm{r}}(\aleph_1) = \omega^{4\mathrm{r}}(2\aleph_1) \leq 2^{4r-1}\omega^{4\mathrm{r}}(\aleph_1) + (12)^{r-1}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^{4\mathrm{r}}+\left|\aleph_1^*\right|^{4\mathrm{r}}\right\|$$
$$+(12)^{r-1}\left(2^r+\frac{|z|}{1+|z|}\right)\left\| |\aleph_1|^{2\mathrm{r}}+\left|\aleph_1^*\right|^{2\mathrm{r}}\right\|\omega^r\left(\aleph_1^2\right).$$

So,

$$\frac{2^{4r}}{2}\omega^{4\mathrm{r}}(\aleph_1) \leq (12)^{r-1}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^{4\mathrm{r}}+\left|\aleph_1^*\right|^{4\mathrm{r}}\right\| + (12)^{r-1}\left(2^r+\frac{|z|}{1+|z|}\right)\left\| |\aleph_1|^{2\mathrm{r}}+\left|\aleph_1^*\right|^{2\mathrm{r}}\right\|\omega^r\left(\aleph_1^2\right).$$

Thus,

$$\omega^{4\mathrm{r}}(\aleph_1) \leq \frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_1|^{4\mathrm{r}}+\left|\aleph_1^*\right|^{4\mathrm{r}}\right\| + \frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_1|^{2\mathrm{r}}+\left|\aleph_1^*\right|^{2\mathrm{r}}\right\|\omega^r\left(\aleph_1^2\right).$$

Now, if $r = 1$ in the last inequality, then we get a refinement of the inequality (5) with $s = 2$ as shown below:

$$\begin{aligned}
\omega^4(\aleph_1) &\le \frac{1}{8}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{8}\left(\frac{2+3|z|}{1+|z|}\right)\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \\
&\le \frac{1}{8}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{8}\left(\frac{2+3|z|}{1+|z|}\right)\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega^2(\aleph_1) \text{ (by the inequality (2))} \\
&\le \frac{1}{8}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{16}\left(\frac{2+3|z|}{1+|z|}\right)\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\|^2 \text{ (by the inequality (4))} \\
&= \frac{1}{8}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{16}\left(\frac{2+3|z|}{1+|z|}\right)\left\| \left(|\aleph_1|^2+|\aleph_1^*|^2\right)^2\right\| \\
&= \frac{1}{8}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{16}\left(\frac{2+3|z|}{1+|z|}\right)\left\| \left(\frac{2|\aleph_1|^2+2|\aleph_1^*|^2}{2}\right)^2\right\| \\
&\le \frac{1}{8}\left(\frac{2+|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{8}\left(\frac{2+3|z|}{1+|z|}\right)\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| \text{ (by Lemma 2.2 with } s=2) \\
&= \frac{1}{2}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\|.
\end{aligned}$$

Remark 3.3. The inequality in Theorem 3.1 is more stricter than the inequality in [4, Theorem 3.1] when $z \in \mathbb{C}$ with $|z| = 1$ and $r = 1$. That is,

$$\begin{aligned}
&\omega^4\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right) \\
&\le 8max\left\{\omega^4(\aleph_1), \omega^4(\aleph_4)\right\} + \frac{3}{2}max\left\{\left\| |\aleph_3|^4 + |\aleph_2^*|^4\right\|, \left\| |\aleph_2|^4 + |\aleph_3^*|^4\right\|\right\} \\
&+\frac{5}{2}max\left\{\left\| |\aleph_3|^2 + |\aleph_2^*|^2\right\|, \left\| |\aleph_2|^2 + |\aleph_3^*|^2\right\|\right\} max\left\{\omega(\aleph_2\aleph_3), \omega(\aleph_3\aleph_2)\right\} \\
&\le 8max\left\{\omega^4(\aleph_1), \omega^4(\aleph_4)\right\} + 3max\left\{\left\| |\aleph_3|^4 + |\aleph_2^*|^4\right\|, \left\| |\aleph_2|^4 + |\aleph_3^*|^4\right\|\right\} \\
&+ max\left\{\left\| |\aleph_3|^2 + |\aleph_2^*|^2\right\|, \left\| |\aleph_2|^2 + |\aleph_3^*|^2\right\|\right\} max\left\{\omega(\aleph_2\aleph_3), \omega(\aleph_3\aleph_2)\right\}.
\end{aligned}$$

Remark 3.4. Let $z \in \mathbb{C}$ with $|z| = 1$ and $r = 1$ in Remark 3.2. Then we have an improvement of the inequality in [15, Theorem 2.1], namely

$$\omega^4(\aleph_1) \le \frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{5}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \le \frac{3}{8}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{8}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right).$$

To see this, note that

$$\begin{aligned}
&\frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{5}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \\
&= \frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{3}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) + \frac{2}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \\
&\le \frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{3}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega^2(\aleph_1) + \frac{2}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \text{ (by the inequality (2))} \\
&\le \frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{3}{32}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\|^2 + \frac{2}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \text{ (by the inequality (4))} \\
&\le \frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{3}{16}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{2}{16}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right) \text{ (by Lemma 2.2)} \\
&= \frac{3}{8}\left\| |\aleph_1|^4+|\aleph_1^*|^4\right\| + \frac{1}{8}\left\| |\aleph_1|^2+|\aleph_1^*|^2\right\| \omega\left(\aleph_1^2\right).
\end{aligned}$$

Theorem 3.2. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, $r \ge 1$, and $z \in \mathbb{C}$. Then

$$\omega^{4r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right)$$

$$\leq 4^{r-1}\left(\begin{array}{c} \max\left\{\omega^{4r}(\aleph_1),\omega^{4r}(\aleph_4)\right\}+\frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r \max\left\{\begin{array}{c}\left\| |\aleph_3|^{4r}+\left|\aleph_2^*\right|^{4r}\right\|, \\ \left\| |\aleph_2|^{4r}+\left|\aleph_3^*\right|^{4r}\right\|\end{array}\right\} \\ +\frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r \max\left\{\begin{array}{c}\left\| |\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r}\right\|, \\ \left\| |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r}\right\|\end{array}\right\}\max\left\{\omega^r(\aleph_2\aleph_3),\omega^r(\aleph_3\aleph_2)\right\} \\ +\frac{6^r}{2}\max\left\{\omega^{2r}(\aleph_1),\omega^{2r}(\aleph_4)\right\}\left(\frac{1}{2}\max\left\{\begin{array}{c}\left\| |\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r}\right\|, \\ \left\| |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r}\right\|\end{array}\right\}+\max\left\{\omega^r(\aleph_2\aleph_3),\omega^r(\aleph_3\aleph_2)\right\}\right) \\ +\frac{8^r}{4}\left(\begin{array}{c}\frac{1}{2}\max\left\{\begin{array}{c}\left\| |\aleph_1|^{2r}+\left|\aleph_2^*\right|^{2r}\right\|, \\ \left\| |\aleph_4|^{2r}+\left|\aleph_3^*\right|^{2r}\right\|\end{array}\right\} \\ +\omega^r\left(\begin{bmatrix} 0 & \aleph_2\aleph_4 \\ \aleph_3\aleph_1 & 0 \end{bmatrix}\right)\end{array}\right)\left(\begin{array}{c}\max\left\{\omega^{2r}(\aleph_1),\ \omega^{2r}(\aleph_4)\right\} \\ +\omega^{2r}\left(\begin{bmatrix} 0 & \aleph_2 \\ \aleph_3 & 0 \end{bmatrix}\right)\end{array}\right) \end{array}\right).$$

Proof. Let $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$, and let $\phi = \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix}$, $\psi = \begin{bmatrix} 0 & \aleph_2 \\ \aleph_3 & 0 \end{bmatrix}$. Then

$$\left|\left\langle \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u, u\right\rangle\right|^{4r}$$

$$\leq \left(\left|\langle \phi u, u\rangle\right| + \left|\langle \psi u, u\rangle\right|\right)^{4r}$$

$$= \begin{bmatrix} \left|\langle \phi u, u\rangle\right|^4 + \left|\langle \psi u, u\rangle\right|^4 + 6\left|\langle \phi u, u\rangle\right|^2\left|\langle \psi u, u\rangle\right|^2 \\ +4\left|\langle \phi u, u\rangle\right|^3\left|\langle \psi u, u\rangle\right| + 4\left|\langle \phi u, u\rangle\right|\left|\langle \psi u, u\rangle\right|^3 \end{bmatrix}^r$$

$$= \begin{bmatrix} \left|\langle \phi u, u\rangle\right|^4 + \left|\langle \psi u, u\rangle\langle u, \psi^* u\rangle\right|^2 + 6\left|\langle \phi u, u\rangle\right|^2\left|\langle \psi u, u\rangle\langle u, \psi^* u\rangle\right| \\ +4\left|\langle \phi u, u\rangle\right|^2\left|\langle \phi u, u\rangle\langle u, \psi^* u\rangle\right| + 4\left|\langle \phi u, u\rangle\langle u, \psi^* u\rangle\right|\left|\langle \psi u, u\rangle\right|^2 \end{bmatrix}^r$$

$$= \begin{bmatrix} \left|\langle \phi u, u\rangle\right|^4 + \left|\langle \psi u, u\rangle\langle u, \psi^* u\rangle\right|^2 + 6\left|\langle \phi u, u\rangle\right|^2\left|\langle \psi u, u\rangle\langle u, \psi^* u\rangle\right| \\ +4\left|\langle \phi u, u\rangle\langle u, \psi^* u\rangle\right|\left(\left|\langle \phi u, u\rangle\right|^2 + \left|\langle \psi u, u\rangle\right|^2\right) \end{bmatrix}^r$$

$$\leq 4^{r-1}\begin{bmatrix} \left|\langle \phi u, u\rangle\right|^{4r} + \left|\langle \psi u, u\rangle\langle u, \psi^* u\rangle\right|^{2r} + 6^r\left|\langle \phi u, u\rangle\right|^{2r}\left|\langle \psi u, u\rangle\langle u, \psi^* u\rangle\right|^r \\ +4^r\left|\langle \phi u, u\rangle\langle u, \psi^* u\rangle\right|^r\left(\left|\langle \phi u, u\rangle\right|^2 + \left|\langle \psi u, u\rangle\right|^2\right)^r \end{bmatrix} \text{(by Lemma 2.6)}$$

$$\leq 4^{r-1}\left[\begin{array}{c} \left|\left\langle \phi u,u\right\rangle\right|^{4r}+\frac{3^{r-1}}{4^r}\left(\begin{array}{c}\left(\frac{2+|z|}{1+|z|}\right)\left\|\psi u\right\|^{2r}\left\|\psi^* u\right\|^{2r}\\ +\left(2^r+\frac{|z|}{1+|z|}\right)\left\|\psi u\right\|^{r}\left\|\psi^* u\right\|^{r}\left|\left\langle \psi u,\psi^* u\right\rangle\right|^{r}\end{array}\right)\\ +3^r\left|\left\langle \phi u,u\right\rangle\right|^{2r}\left(\left\|\psi u\right\|\left\|\psi^* u\right\|+\left|\left\langle \psi u,\psi^* u\right\rangle\right|\right)^r\\ +2^r\left(\left\|\phi u\right\|\left\|\psi^* u\right\|+\left|\left\langle \phi u,\psi^* u\right\rangle\right|\right)^r\left(\left(\left|\left\langle \phi u,u\right\rangle\right|^2+\left|\left\langle \psi u,u\right\rangle\right|^2\right)\right)^r\end{array}\right] \text{(by Lemma 3.1 and Lemma 2.4)}$$

$$\leq 4^{r-1}\left[\begin{array}{c} \left|\left\langle \phi u,u\right\rangle\right|^{4r}+\frac{3^{r-1}}{4^r}\left(\begin{array}{c}\left(\frac{2+|z|}{1+|z|}\right)\left\langle \left|\psi\right|^{2r}u,u\right\rangle\left\langle \left|\psi^*\right|^{2r}u,u\right\rangle\\ +\left(2^r+\frac{|z|}{1+|z|}\right)\sqrt{\left\langle \left|\psi\right|^{2r}u,u\right\rangle\left\langle \left|\psi^*\right|^{2r}u,u\right\rangle}\left|\left\langle \psi^2 u,u\right\rangle\right|^{r}\end{array}\right)\\ +\frac{6^r}{2}\left|\left\langle \phi u,u\right\rangle\right|^{2r}\left(\sqrt{\left\langle \left|\psi\right|^{2r}u,u\right\rangle\left\langle \left|\psi^*\right|^{2r}u,u\right\rangle}+\left|\left\langle \psi^2 u,u\right\rangle\right|^r\right)\\ +\frac{8^r}{4}\left(\sqrt{\left\langle \left|\phi\right|^{2r}u,u\right\rangle\left\langle \left|\psi^*\right|^{2r}u,u\right\rangle}+\left|\left\langle \psi\phi u,u\right\rangle\right|^r\right)\left(\left|\left\langle \phi u,u\right\rangle\right|^{2r}+\left|\left\langle \psi u,u\right\rangle\right|^{2r}\right)\end{array}\right]$$

(by Lemma 2.1, and Lemma 2.6)

$$\leq 4^{r-1}\left[\begin{array}{c} \left|\left\langle \phi u,u\right\rangle\right|^{4r}+\frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\langle \left(\left|\psi\right|^{4r}+\left|\psi^*\right|^{4r}\right)u,u\right\rangle\\ +\frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\langle \left(\left|\psi\right|^{2r}+\left|\psi^*\right|^{2r}\right)u,u\right\rangle\left|\left\langle \psi^2 u,u\right\rangle\right|^r\\ +\frac{6^r}{2}\left|\left\langle \phi u,u\right\rangle\right|^{2r}\left(\left\langle \frac{1}{2}\left(\left|\psi\right|^{2r}+\left|\psi^*\right|^{2r}\right)u,\ u\right\rangle+\left|\left\langle \psi^2 u,u\right\rangle\right|^r\right)\\ +\frac{8^r}{4}\left(\left\langle \frac{1}{2}\left(\left|\phi\right|^{2r}+\left|\psi^*\right|^{2r}\right)u,u\right\rangle+\left|\left\langle \psi\phi u,u\right\rangle\right|^r\right)\left(\left|\left\langle \phi u,u\right\rangle\right|^{2r}+\left|\left\langle \psi u,u\right\rangle\right|^{2r}\right)\end{array}\right]$$

(by the AGM inequality and Lemma 2.1)

$$= 4^{r-1}\left[\begin{array}{l} \left|\left\langle \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix} u,u\right\rangle\right|^{4r} + \frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^{r}\left\langle \begin{bmatrix} |\aleph_3|^{4r}+\left|\aleph_2^*\right|^{4r} & 0 \\ 0 & |\aleph_2|^{4r}+\left|\aleph_3^*\right|^{4r} \end{bmatrix} u,u\right\rangle \\ +\frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^{r}\left\langle \begin{bmatrix} |\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r} & 0 \\ 0 & |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r} \end{bmatrix} u,u\right\rangle \left|\left\langle \begin{bmatrix} \aleph_2\aleph_3 & 0 \\ 0 & \aleph_3\aleph_2 \end{bmatrix} u,u\right\rangle\right|^{r} \\ +\frac{6^r}{2}\left|\left\langle \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix} u,u\right\rangle\right|^{2r}\left(\begin{array}{c} \frac{1}{2}\left\langle \begin{bmatrix} |\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r} & 0 \\ 0 & |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r} \end{bmatrix} u,u\right\rangle \\ +\left|\left\langle \begin{bmatrix} \aleph_2\aleph_3 & 0 \\ 0 & \aleph_3\aleph_2 \end{bmatrix} u,u\right\rangle\right|^{r} \end{array}\right) \\ +\frac{8^r}{4}\left(\begin{array}{c} \frac{1}{2}\left\langle \begin{bmatrix} |\aleph_1|^{2r}+\left|\aleph_2^*\right|^{2r} & 0 \\ 0 & |\aleph_4|^{2r}+\left|\aleph_3^*\right|^{2r} \end{bmatrix} u,u\right\rangle \\ +\left|\left\langle \begin{bmatrix} 0 & \aleph_2\aleph_4 \\ \aleph_3\aleph_1 & 0 \end{bmatrix} u,u\right\rangle\right|^{r} \end{array}\right)\left(\left|\left\langle \phi u,u\right\rangle\right|^{2r}+\left|\left\langle \psi u,u\right\rangle\right|^{2r}\right) \end{array}\right].$$

Thus,

$$\left|\left\langle \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} u,u\right\rangle\right|^{4r}$$

$$\leq 4^{r-1}\left[\begin{array}{l} \left|\left\langle \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix} u,u\right\rangle\right|^{4r} + \frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^{r}\left\langle \begin{bmatrix} |\aleph_3|^{4r}+\left|\aleph_2^*\right|^{4r} & 0 \\ 0 & |\aleph_2|^{4r}+\left|\aleph_3^*\right|^{4r} \end{bmatrix} u,u\right\rangle \\ +\frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^{r}\left\langle \begin{bmatrix} |\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r} & 0 \\ 0 & |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r} \end{bmatrix} u,u\right\rangle \left|\left\langle \begin{bmatrix} \aleph_2\aleph_3 & 0 \\ 0 & \aleph_3\aleph_2 \end{bmatrix} u,u\right\rangle\right|^{r} \\ +\frac{6^r}{2}\left|\left\langle \begin{bmatrix} \aleph_1 & 0 \\ 0 & \aleph_4 \end{bmatrix} u,u\right\rangle\right|^{2r}\left(\begin{array}{c} \frac{1}{2}\left\langle \begin{bmatrix} |\aleph_3|^{2r}+\left|\aleph_2^*\right|^{2r} & 0 \\ 0 & |\aleph_2|^{2r}+\left|\aleph_3^*\right|^{2r} \end{bmatrix} u,u\right\rangle \\ +\left|\left\langle \begin{bmatrix} \aleph_2\aleph_3 & 0 \\ 0 & \aleph_3\aleph_2 \end{bmatrix} u,u\right\rangle\right|^{r} \end{array}\right) \\ +\frac{8^r}{4}\left(\begin{array}{c} \frac{1}{2}\left\langle \begin{bmatrix} |\aleph_1|^{2r}+\left|\aleph_2^*\right|^{2r} & 0 \\ 0 & |\aleph_4|^{2r}+\left|\aleph_3^*\right|^{2r} \end{bmatrix} u,u\right\rangle \\ +\left|\left\langle \begin{bmatrix} 0 & \aleph_2\aleph_4 \\ \aleph_3\aleph_1 & 0 \end{bmatrix} u,u\right\rangle\right|^{r} \end{array}\right)\left(\left|\left\langle \phi u,u\right\rangle\right|^{2r}+\left|\left\langle \psi u,u\right\rangle\right|^{2r}\right) \end{array}\right]$$

By considering the supremum over all $u \in \mathscr{H} \oplus \mathscr{H}$ with $\|u\| = 1$ in the final inequality, we obtain the desired outcome.

In the subsequent statements, we present specific instances of the aforementioned theorem.

Remark 3.5. If $\aleph_1 = \aleph_4$ *and* $\aleph_2 = \aleph_3$ in Theorem 3.2, then we have

$$\omega^{4\mathrm{r}}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix}\right)$$

$$= max\left\{\omega^{4\mathrm{r}}\left(\aleph_1+\aleph_2\right), \omega^{4\mathrm{r}}\left(\aleph_1-\aleph_2\right)\right\}$$

$$\leq 4^{r-1}\begin{bmatrix} \omega^{4r}(\aleph_1)+\frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_2|^{4r}+\left|\aleph_2^*\right|^{4r}\right\| \\ +\frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_2|^{2r}+\left|\aleph_2^*\right|^{2r}\right\|\omega^r\left(\aleph_2^2\right) \\ +\frac{6^r}{2}\omega^{2r}(\aleph_1)\left(\frac{1}{2}\left\| |\aleph_2|^{2r}+\left|\aleph_2^*\right|^{2r}\right\|+\omega^r\left(\aleph_2^2\right)\right) \\ +\frac{8^r}{4}\left(\frac{1}{2}\left\| |\aleph_1|^{2r}+\left|\aleph_2^*\right|^{2r}\right\|+\omega^r(\aleph_2\aleph_1)\right)\left(\omega^{2r}(\aleph_1)+\omega^{2r}(\aleph_2)\right) \end{bmatrix}.$$

Remark 3.6. If $\aleph_1 = \aleph_2$ in Remark 3.5, then we have

$$\omega^{4r}(\aleph_1)\leq 4^{-r-1}\begin{bmatrix} \omega^{4r}(\aleph_1)+\frac{1}{6}\left(\frac{2+|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_1|^{4r}+\left|\aleph_1^*\right|^{4r}\right\| \\ +\frac{1}{6}\left(2^r+\frac{|z|}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left\| |\aleph_1|^{2r}+\left|\aleph_1^*\right|^{2r}\right\|\omega^r\left(\aleph_1^2\right) \\ +\left(\frac{6^r+8^r}{2}\right)\omega^{2r}(\aleph_1)\left(\frac{1}{2}\left\| |\aleph_1|^{2r}+\left|\aleph_1^*\right|^{2r}\right\|+\omega^r\left(\aleph_1^2\right)\right) \end{bmatrix}.$$

Using simple steps, it can be shown that this inequality (when $r = 1$) is more stricter than the inequality (5) when $s = 2$.

Now, we will prove the following helpful lemma.

Lemma 3.2. Let $u, v, e \in \mathscr{H}$ with $\|e\| = 1$, $r \geq 1$, and $z \in \mathbb{C}$. Then

$$|\langle u,e\rangle\langle e,v\rangle|^{2r} \leq \frac{1}{3(1+|z|)}\left(\frac{3}{4}\right)^r\left[\|u\|^{2r}\|v\|^{2r}+|\langle u,v\rangle|^{2r}+2^r\|u\|^r\|v\|^r|\langle u,v\rangle|^r\right]$$
$$+\left(\frac{|z|}{2(1+|z|)}\right)\left[\|u\|^r\|v\|^r+|\langle u,v\rangle|^r\right]|\langle u,e\rangle\langle e,v\rangle|^r.$$

Proof. We have

$$\begin{aligned}
&|\langle u,e\rangle\langle e,v\rangle|^{2r}\\
&=\left(\frac{1}{1+|z|}\right)|\langle u,e\rangle\langle e,v\rangle|^{2r}+\left(\frac{|z|}{1+|z|}\right)|\langle u,e\rangle\langle e,v\rangle|^{2r}\\
&\leq\left(\frac{1}{1+|z|}\right)\left[\frac{1}{2}(\|u\|\,\|v\|+|\langle u,v\rangle|)\right]^{2r}+\left(\frac{|z|}{1+|z|}\right)|\langle u,e\rangle\langle e,v\rangle|^r|\langle u,e\rangle\langle e,v\rangle|^r \text{ (Lemma 2.4)}\\
&\leq\frac{1}{2^{2r}}\left(\frac{1}{1+|z|}\right)\left[\|u\|^2\|v\|^2+|\langle u,v\rangle|^2+2\,\|u\|\,\|v\|\,|\langle u,v\rangle|\right]^r\\
&\quad+\left(\frac{|z|}{1+|z|}\right)\left(\frac{1}{2}(\|u\|\,\|v\|+|\langle u,v\rangle|)\right)^r|\langle u,e\rangle\langle e,v\rangle|^r \text{ (Lemma 2.4)}\\
&\leq\frac{1}{2^{2r}}\left(\frac{1}{1+|z|}\right)\left(3^{r-1}\right)\left[\|u\|^{2r}\|v\|^{2r}+|\langle u,v\rangle|^{2r}+2^r\|u\|^r\|v\|^r|\langle u,v\rangle|^r\right]\\
&\quad+\left(\frac{|z|}{1+|z|}\right)2^{-r}\left(2^{r-1}\right)\left[\|u\|^r\|v\|^r+|\langle u,v\rangle|^r\right]|\langle u,e\rangle\langle e,v\rangle|^r \text{ (Lemma 2.6)}
\end{aligned}$$

$$= \frac{1}{3}\left(\frac{1}{1+|z|}\right)\left(\frac{3}{4}\right)^r\left[\|u\|^{2r}\|v\|^{2r} + |\langle u, v\rangle|^{2r} + 2^r\|u\|^r\|v\|^r|\langle u, v\rangle|^r\right]$$
$$+\frac{1}{2}\left(\frac{|z|}{1+|z|}\right)\left[\|u\|^r\|v\|^r + |\langle u, v\rangle|^r\right]|\langle u, e\rangle\langle e, v\rangle|^r.$$

Now, we can prove the last theorem in this paper using similar techniques as in the previous theorems and Lemma 2.6, Lemma 3.2, Lemma 2.1, the AGM inequality, respectively.

Theorem 3.3. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, $r \geq 1$, and $z \in \mathbb{C}$. Then

$$\omega^{4r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_3 & \aleph_4 \end{bmatrix}\right)$$
$$\leq 2^{4r-1} max\left\{\omega^{4r}(\aleph_1), \omega^{4r}(\aleph_4)\right\} + \left(\frac{1}{1+|z|}\right)(12)^{r-1}\, max\left\{\left\| |\aleph_3|^{4r} + \left|\aleph_2^*\right|^{4r}\right\|, \left\| |\aleph_2|^{4r} + \left|\aleph_3^*\right|^{4r}\right\|\right\}$$
$$+\left(\frac{1}{1+|z|}\right)\frac{(12)^r}{6}\, max\left\{\omega^{2r}(\aleph_3\aleph_2), \omega^{2r}(\aleph_2\aleph_3)\right\}$$
$$+\left(\frac{1}{1+|z|}\right)\frac{(24)^r}{12} max\left\{\left\| |\aleph_3|^{2r} + \left|\aleph_2^*\right|^{2r}\right\|, \left\| |\aleph_2|^{2r} + \left|\aleph_3^*\right|^{2r}\right\|\right\}\, max\left\{\omega^r(\aleph_2\aleph_3), \omega^r(\aleph_3\aleph_2)\right\}$$
$$+\left(\frac{|z|}{1+|z|}\right)4^{2r-1}\left(\begin{array}{c}\frac{1}{2}\, max\left\{\left\| |\aleph_3|^{2r} + \left|\aleph_2^*\right|^{2r}\right\|, \left\| |\aleph_2|^{2r} + \left|\aleph_3^*\right|^{2r}\right\|\right\} \\ + max\left\{\omega^r(\aleph_2\aleph_3), \omega^r(\aleph_3\aleph_2)\right\}\end{array}\right)\omega^{2r}\left(\begin{bmatrix} 0 & \aleph_2 \\ \aleph_3 & 0 \end{bmatrix}\right).$$

Now, we will introduce some special cases of Theorem 3.3.
If $\aleph_1 = \aleph_4$ *and* $\aleph_2 = \aleph_3$ in Theorem 3.3, then we get the following:

Corollary 3.3. Let $\Gamma = \begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix} \in \mathscr{B}(\mathscr{H} \oplus \mathscr{H})$, $r \geq 1$, and $z \in \mathbb{C}$. Then

$$\omega^{4r}\left(\begin{bmatrix} \aleph_1 & \aleph_2 \\ \aleph_2 & \aleph_1 \end{bmatrix}\right) = max\left\{\omega^{4r}(\aleph_1+\aleph_2), \omega^{4r}(\aleph_1-\aleph_2)\right\} \quad \text{(by lemma 2.5(ii))}$$
$$\leq 2^{4r-1}\omega^{4r}(\aleph_1) + \left(\frac{1}{1+|z|}\right)(12)^{r-1}\left\| |\aleph_2|^{4r} + \left|\aleph_2^*\right|^{4r}\right\|$$
$$+\frac{1}{6}\left(\frac{1}{1+|z|}\right)(12)^r\,\omega^{2r}\left(\aleph_2^2\right) + \frac{1}{12}\left(\frac{1}{1+|z|}\right)(24)^r\left\| |\aleph_2|^{2r} + \left|\aleph_2^*\right|^{2r}\right\|\omega^r\left(\aleph_2^2\right)$$
$$+\left(\frac{|z|}{1+|z|}\right)4^{2r-1}\left(\frac{1}{2}\left\| |\aleph_2|^{2r} + \left|\aleph_2^*\right|^{2r}\right\| + \omega^r\left(\aleph_2^2\right)\right)\omega^{2r}(\aleph_2).$$

Remark 3.7. If $\aleph_1 = \aleph_2$, $z \in \mathbb{C}$ with $|z| = 2$, and $r = 1$ in Corollary 3.3, then we get the inequality given in [14, Theorem 3], namely

$$\omega^4(\aleph_1) \leq \frac{1}{24}\left\| |\aleph_1|^4 + \left|\aleph_1^*\right|^4\right\| + \frac{1}{12}\left(\omega^2\left(\aleph_1^2\right) + \left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\|\omega\left(\aleph_1^2\right)\right)$$
$$+\frac{1}{3}\omega^2(\aleph_1)\left(\frac{1}{2}\left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\| + \omega\left(\aleph_1^2\right)\right).$$

Remark 3.8. If $\aleph_1 = \aleph_2$, and $r = 1$ in Corollary 3.3, then we get a refinement of the inequality (5) with $s = 2$ as shown below:

$$\omega^4(\aleph_1) \leq \frac{1}{8}\left(\frac{1}{1+|z|}\right)\left\| |\aleph_1|^4 + \left|\aleph_1^*\right|^4\right\| + \frac{1}{4}\left(\frac{1}{1+|z|}\right)\left(\omega^2\left(\aleph_1^2\right) + \left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\|\omega\left(\aleph_1^2\right)\right)$$
$$+\frac{1}{2}\left(\frac{|z|}{1+|z|}\right)\omega^2(\aleph_1)\left(\frac{1}{2}\left\| |\aleph_1|^2 + \left|\aleph_1^*\right|^2\right\| + \omega\left(\aleph_1^2\right)\right).$$

$$\begin{aligned}
&\leq \frac{1}{8}\left(\frac{1}{1+|z|}\right)\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| + \frac{1}{4}\left(\frac{1}{1+|z|}\right)\left(\omega^2\left(\aleph_1^2\right)+\left\| |\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|\omega^2(\aleph_1)\right) \\
&\quad + \frac{1}{2}\left(\frac{|z|}{1+|z|}\right)\omega^2(\aleph_1)\left(\frac{1}{2}\left\| |\aleph_1|^2+\left|\aleph_1^*\right|^2\right\| + \omega^2(\aleph_1)\right). \text{ (by the inequality (2))} \\
&\leq \frac{1}{8}\left(\frac{1}{1+|z|}\right)\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| + \frac{1}{4}\left(\frac{1}{1+|z|}\right)\left(\frac{1}{2}\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| + \frac{1}{2}\left\| |\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|^2\right) \\
&\quad + \frac{1}{2}\left(\frac{|z|}{1+|z|}\right)\left(\frac{1}{2}\right)\left\| |\aleph_1|^2+\left|\aleph_1^*\right|^2\right\|^2 \quad \text{(by the inequalities (4) and (6) with } s=2) \\
&= \frac{1}{4}\left(\frac{1}{1+|z|}\right)\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| + \frac{1}{8}\left(\frac{1+2|z|}{1+|z|}\right)\left\|\left(|\aleph_1|^2+\left|\aleph_1^*\right|^2\right)^2\right\| \\
&= \frac{1}{4}\left(\frac{1}{1+|z|}\right)\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| + \frac{1}{8}\left(\frac{1+2|z|}{1+|z|}\right)\left\|\left(\frac{2|\aleph_1|^2+2\left|\aleph_1^*\right|^2}{2}\right)^2\right\| \\
&\leq \frac{1}{4}\left(\frac{1}{1+|z|}\right)\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| + \frac{1}{4}\left(\frac{1+2|z|}{1+|z|}\right)\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\| \text{ (by Lemma 2.2 with } s=2) \\
&= \frac{1}{2}\left\| |\aleph_1|^4+\left|\aleph_1^*\right|^4\right\|.
\end{aligned}$$

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