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Some novel estimations of hadamard type inequalities for different kinds of convex functions via tempered fractional integral operator

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Abstract. In this article, Hermite-Hadamard type inequalities for $(h,m)-$convex and $s-$convex functions are established by using tempered fractional integral operators. Also, some integral inequalities related to the right and left sides of the Hermite-Hadamard inequality via tempered fractional integrals are proved.

1. Introduction

The theory of convexity plays a vital role in different fields of pure and applied sciences. Consequently, the classical concepts of convex sets and convex functions have been generalized in different directions. The concept of function is one of the basic structures of mathematics, and many researchers have focused on new function classes and made efforts to classify the space of functions. One of the types of functions defined as a product of this intense effort is the convex function, which has applications in statistics, inequality theory, convex programming, and numerical analysis. This interesting class of functions is defined as follows:
Definition 1.1. [2] Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then $f\ \mathcal{H} \to \mathbb{R}, \emptyset \neq \mathcal{H} \subseteq \mathbb{R}$ is said to be convex if

$$f\left(\xi a + (1-\xi)\, b\right) \leq \xi f\left(a\right) + (1-\xi)\, f(b)$$

for all $a,\ b \in \mathcal{H}$ and $\xi \in [0,\ 1]$.

Several research papers have been performed related to convexity and related topics in the literature, see the papers [1-5, 15].

Another aspect due to which the convexity theory has attracted many researchers is its close relation with theory of inequalities. Many famous inequalities can be obtained using the concept of convex functions. For more information related to integral inequalities, interested readers are referred to [12-19].

Among the other classical inequalities, Hermite–Hadamard's inequality, which provides us upper and lower bound fort he mean-value of a convex function, is one of the most studied inequality in the literature.

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a,\ b \in I$ with $a < b$. The following double inequalities:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f\left(a\right)+f\left(b\right)}{2}$$

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hold. This double inequality is known in the literature as the Hermite-Hadamard inequality for convex functions.

Definition 1.2. [20] Let $\mathcal{H}$ be an interval in $\mathbb{R}$. Then $f:\mathcal{H}\rightarrow\mathbb{R}, \emptyset\neq\mathcal{H}\subseteq\mathbb{R}$ is said to be $s-$convex in the second sense if the following inequality

$$f(\xi a+(1-\xi)b)\leq\xi^{s}f(a)+(1-\xi)^{s}f(b)$$

holds for all $a,\ b\in\mathcal{H}$ and $\xi\in[0,\ 1]$, $s\in(0,\ 1]$.

Definition 1.3. [13] Let $h:(0,1)\subseteq J\rightarrow\mathbb{R}$ be a non-negative function. A function $f:[0,b]\rightarrow\mathbb{R}$ is called $(h,m)-$convex function if f is non-negative and

$$f(\xi x+m(1-\xi)y)\leq h(\xi)f(x)+mh(1-\xi)f(y)$$

holds for all $x,y\in[0,b]$, $\xi\in(0,\ 1)$ and for some fixed $m\in(0,1]$.

Some new integral inequalities involving two nonnegative and integrable functions that are related to the Hermite-Hadamard type are obtained by many researchers. In [6], Pachpatte proposed some Hermite-Hadamard type inequalities involving two $log-$convex functions. An analogous result for $s-$convex functions is established by Kırmacı et al. in [8]. In [10], Sarıkaya presented some integral inequalities for $h-$convex functions. For recent results and generalizations concerning Hermite-Hadamard type inequalities for product of two functions, we can refer the paper [18] and the references given therein. It is remarkable that Sarıkaya et al. proved the following interesting inequalities of Hermite-Hadamard type involving Riemann-Liouville fractional integrals [17].

Theorem 1.1. [17]. Let f : $[a,b]\rightarrow\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_{1}[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2} \tag{1}$$

with $\alpha>0$.

Theorem 1.2. [19] Let $\alpha\geq1$ and $f:[a,\ b]\rightarrow\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_{1}[a,\ b]$. If f is $s-$convex function on $[a,\ b]$, then the following inequality for fractional integrals hold:

$$2^{s-1}f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}\left[J_{a^{+}}^{\alpha}f(b)+J_{b^{-}}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}\left[\frac{1}{\alpha+s}+\frac{2}{\alpha+s}\left(1-\frac{1}{2^{\alpha+s}}\right)\right]$$

Theorem 1.3. [16] Let $f:[a,b]\rightarrow\mathbb{R}$ be a positive function with $0\leq a<b$ and $f\in L_{1}[a,b]$. If f is a convex function on $[a,b]$, then the following inequalities for fractional integrals hold:

$$f\left(\frac{a+b}{2}\right)\leq\frac{2^{\alpha-1}\,\Gamma(\alpha+1)}{(b-a)^{\alpha}}\left[J_{\left(\frac{a+b}{2}\right)^{+}}^{\alpha}f(b)+J_{\left(\frac{a+b}{2}\right)^{-}}^{\alpha}f(a)\right]\leq\frac{f(a)+f(b)}{2}$$

Mathematics uses a variety of tools and methods in the quest to explain physical phenomena of nature and life. Since many phenomena related to nature and life have a dynamic process, the methods put forward with the help of classical analysis are insufficient due to some limitations and disadvantages. One of the most effective methods for explaining, discussing and developing dynamic processes is fractional analysis, whose origins go back as far as classical analysis. It has succeeded in bringing a new momentum not only to mathematics but also to many disciplines with effective applications (see [21-26]). Fractional analysis is a field that tries to achieve this movement by introducing new fractional derivative and integral operators. In particular, researchers who argue that real world problems cannot be explained only by power laws have introduced fractional derivative and integral operators, which include the exponential function and its generalized versions in their kernels. These new operators differ in their kernel structures, such as singularity, locality and general form. In this context, we will continue by introducing two integral operators that have an important place in fractional analysis. Among the operators defined here, especially the tempered fractional integral operator is a useful operator that has attracted the attention of many researchers due to the advantages it offers in applications.

Definition 1.4. [11] Let $f \in L_1[a,b]$. The Riemann Liouville integrals $I^{\alpha}_{a^+}f$ and $I^{\alpha}_{b^-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$I^{\alpha}_{a^+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x f(\xi)(x-\xi)^{\alpha-1}d\xi, \quad x > a \tag{2}$$

and

$$I^{\alpha}_{b^-}f(x) = \frac{1}{\Gamma(\alpha)}\int_x^b f(\xi)(\xi-x)^{\alpha-1}d\xi, \quad b > x \tag{3}$$

The tempered fractional integral was first studied by Buschman [5], but Li et al. [9] and Meerschaert et al. [10] have described the associated tempered fractional calculus more explicitly as following:

Definition 1.5. [5,7,10] Let $[a,b]$ be a real interval and $\zeta \geq 0, \alpha > 0$. Then, for a function $f \in L_1[a,b]$, the left and right tempered fractional integral, respectively, defined by

$$ {}^{\tau}_{(c)^+}I^{\alpha,\zeta}_{b}f(b) = \frac{1}{\Gamma(\alpha)}\int_c^x (x-\xi)^{\alpha-1}e^{-\zeta(x-\xi)}f(t)\,d\xi \tag{4}$$

and

$$ {}^{\tau}_{(x)^-}I^{\alpha,\zeta}_{a}f(a) = \frac{1}{\Gamma(\alpha)}\int_x^b (\xi-x)^{\alpha-1}e^{-\zeta(\xi-x)}f(\xi)\,d\xi \tag{5}$$

where $\Gamma(\alpha)$ is the gamma function.

Remark 1.1. If we take $\zeta = 0$ in the Equations (4) and (5), then we have the left and right RL operators (2) and (3) respectively.

Definition 1.6. [12] For the real numbers, $\alpha > 0$ and $x, \zeta \geq 0$, we define the $\zeta-$Incomplete gamma function by

$$I_{\alpha}(\alpha, b) = \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{b}} \mathrm{x}^{\alpha-1}\mathrm{e}^{-\zeta \mathrm{t}}\mathrm{dx}$$

If $\zeta = 1$, it reduces to the incomplete gamma function.

$$I(\alpha, b) = \frac{1}{\Gamma(\alpha)}\int_0^{\mathrm{b}} \mathrm{x}^{\alpha-1}\mathrm{e}^{-\mathrm{x}}\mathrm{dx}, \ \alpha > 0$$

Remark 1.2. For the reel numbers $\alpha > 0$ and $x, \zeta \geq 0$, we have

1. $I_{\zeta(b-a)}(\alpha, 1) = \int_0^1 \mathrm{x}^{\alpha-1}\mathrm{e}^{-\zeta(\mathrm{b-a})\mathrm{x}}\mathrm{dx} = \frac{1}{(b-a)^{\alpha}}I_{\alpha}(\alpha, b-a)$
2. $\int_0^1 I_{\alpha(b-a)}(\alpha, x)\,\mathrm{dx} = \frac{I_{\alpha}(\alpha, b-a)}{(b-a)^{\alpha}} - \frac{I_{\alpha}(\alpha+1, b-a)}{(b-a)^{\alpha+1}}$

2. Hermite-Hadamard Type Inequalities for Tempered Fractional Integral Operators

Now, we are in a position to establish some generalized inequalities of Hermite-Hadamard type involving tempered fractional integrals for $(h,m)-$convex and $s-$convex functions in the second sense.

Theorem 2.1. Let $f : (0,\infty) \to \mathbb{R}$ be $(h,m)-$convex function where $0 < a < b < \infty$ such that $f \in L_1[a,b]$. Then, we have the following inequality for tempered fractional integral operators

$$\begin{aligned} \frac{1}{h(1/2)}f\left(\frac{a+b}{2}\right) &\leq \frac{2^{\alpha}\Gamma(\alpha)}{\mathrm{I}_{\alpha}(\alpha, m\mathrm{b}-\mathrm{a})(mb-a)^{\alpha}}\left[{}^{\tau}_{\left(\frac{a+mb}{2}\right)^+}I^{\alpha,\zeta}_{b}f(mb) + m^{\alpha}\,{}^{\tau}_{\left(\frac{a+mb}{2}\right)^-}I^{\alpha,\zeta}_{a}f\left(\frac{a}{m}\right)\right] \\ &\leq \frac{f(a)+f(b)}{I_{\alpha}(\alpha, mb-a)}\varphi + \frac{\left[f\left(\frac{a}{m}\right)+f\left(\frac{b}{m}\right)\right]}{I_{\alpha}(\alpha, mb-a)}\omega. \end{aligned} \tag{6}$$

with $\zeta \geq 0, \alpha > 0$ and $m \in (0,1)$ where $\varphi = \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(b-a)}} h\left(\frac{\xi}{2}\right) d\xi$ and $\omega = \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(mb-a)}} h\left(\frac{2-\xi}{2}\right) d\xi$.

Proof Since f is $(h,m)-$convex function on $[a,b]$, for $\xi \in [0,1]$ with setting $\xi = \frac{1}{2}$, we can write

$$f\left(\frac{x+my}{2}\right) \leq h\left(\frac{1}{2}\right)[f(x)+f(y)]. \tag{7}$$

By changing of the variables such as $x = \frac{\xi}{2}a + m\left(\frac{2-\xi}{2}\right)b$ and $y = \left(\frac{2-\xi}{2}\right)\frac{a}{m} + \frac{\xi}{2}b$, then we get

$$\frac{1}{h(1/2)} f\left(\frac{a+b}{2}\right) \leq f\left(\frac{\xi}{2}a + m\left(\frac{2-\xi}{2}\right)b\right) + f\left(\left(\frac{2-\xi}{2}\right)\frac{a}{m} + \frac{\xi}{2}b\right). \tag{8}$$

Multiplying both sides of (8) by $\frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(mb-a)}}$, then integrating the resulting inequality with respect to ξ over $[0,1]$, we obtain

$$\frac{1}{h(1/2)} f\left(\frac{a+b}{2}\right) \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(mb-a)}} d\xi \leq \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(mb-a)}} f\left(\frac{\xi}{2}a + m\left(\frac{2-\xi}{2}\right)b\right) d\xi + \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(b-a)}} f\left(\left(\frac{2-\xi}{2}\right)\frac{a}{m} + \frac{\xi}{2}b\right) d\xi.$$

As consequence, we provide

$$\begin{aligned} & \frac{1}{h(1/2)} f\left(\frac{a+b}{2}\right) \mathrm{I}_\alpha(\alpha, m\mathrm{b}-\mathrm{a}) \\ \leq\ & \frac{2^\alpha}{(mb-a)^\alpha} \int_{\frac{a+mb}{2}}^{mb} (mb-x)^{\alpha-1} e^{\zeta\,(mb-x)} f(x)\,dx + \frac{2^\alpha m^\alpha}{(mb-a)^\alpha} \int_{a/m}^{\frac{a+mb}{2}} (x-a)^{\alpha-1} e^{\zeta\,(x-a)} f(x)\,dx \end{aligned}$$

and the first inequality is proved.
For the proof of the second ineqaulity in (6), we first note that if f is $(h,m)-$convex function, then for $\xi \in [0,1]$, we can write

$$f\left(\frac{\xi}{2}a + m\left(\frac{2-\xi}{2}\right)b\right) \leq h\left(\frac{\xi}{2}\right) f(a) + mh\left(\frac{2-\xi}{2}\right) f\left(\frac{b}{m}\right)$$

and

$$f\left(\left(\frac{2-\xi}{2}\right)\frac{a}{m} + \frac{\xi}{2}b\right) \leq mh\left(\frac{2-\xi}{2}\right) f\left(\frac{a}{m}\right) + h\left(\frac{\xi}{2}\right) f(b).$$

By adding the above inequalities, we have

$$f\left(\frac{\xi}{2}a + m\left(\frac{2-\xi}{2}\right)b\right) + f\left(\left(\frac{2-\xi}{2}\right)\frac{a}{m} + \frac{\xi}{2}b\right) \leq [f(a)+f(b)]\, h\left(\frac{\xi}{2}\right) + \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right] h\left(\frac{2-\xi}{2}\right). \tag{9}$$

Multiplying both sides of (9) by $\frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(b-a)}}$, then integrating the resulting inequality with respect to ξ over $[0,1]$, we obtain

$$\begin{aligned} & \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(mb-a)}} f\left(\frac{\xi}{2}a + m\left(\frac{2-\xi}{2}\right)b\right) d\xi + \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(b-a)}} f\left(\left(\frac{2-\xi}{2}\right)\frac{a}{m} + \frac{\xi}{2}b\right) d\xi \\ \leq\ & [f(a)+f(b)]\,\varphi + \left[f\left(\frac{a}{m}\right) + f\left(\frac{b}{m}\right)\right]\omega \end{aligned}$$

where $\varphi = \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(b-a)}} h\left(\frac{\xi}{2}\right) d\xi$ and $\omega = \int_0^1 \frac{\xi^{\alpha-1}}{e^{\zeta \frac{\xi}{2}(mb-a)}} h\left(\frac{2-\xi}{2}\right) d\xi$.
The proof is completed.

Remark 2.1. Inequalities (6) become the inequalities (1) by choosing $\zeta = 0$, $\alpha = m = 1$ and $\mathrm{h}(\xi) = \xi$.

Theorem 2.2. Let $f:[a,b]\to\mathbb{R}^{+}$ be $s-$convex function in the second sense on $[a,b]$ with $a<b$ such that $f\in L_1[a,b]$. Then, we have the following inequality for tempered fractional integral operators

$$f\left(\frac{a+b}{2}\right)\leq\frac{\Gamma(\alpha)}{2^{\mathrm{s}}\mathrm{I}_{\alpha}(\alpha,m\mathrm{b}-\mathrm{a})(b-a)^{\alpha}}\left[{}^{\tau}_{\left(\frac{a+b}{2}\right)^{+}}I^{\alpha,2\zeta}_{b}f(b)+{}^{\tau}_{\left(\frac{a+b}{2}\right)^{-}}I^{\alpha,2\zeta}_{a}f(a)\right]\leq\frac{f(a)+f(b)}{2^{\mathrm{s}}}(\mu+\eta)\tag{10}$$

with $\zeta\geq 0$, $\alpha>0$, $s\in[0,1]$ where $\mu=\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}\xi^s d\xi$ and $\eta=\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}(1-\xi)^s d\xi$.
Proof: Since f is a $s-$convex function in the second sense on $[a,b]$, for $\xi\in[0,1]$ with take $\xi=\frac{1}{2}$, we have

$$f\left(\frac{x+y}{2}\right)\leq\frac{f(x)+f(y)}{2^{\mathrm{s}}}.$$

By changing of the variables as $x=\xi a+(1-\xi)b$ and $y=(1-\xi)b+\xi a$, we get

$$2^{\mathrm{s}}f\left(\frac{a+b}{2}\right)\leq f(\xi a+(1-\xi)b)+f((1-\xi)b+\xi a).\tag{11}$$

Multiplying both sides of (11) by $\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}$, then integrating the resulting inequality with respect to ξ over $[0,1]$, we obtain

$$2^{\mathrm{s}}f\left(\frac{a+b}{2}\right)\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}d\xi\leq\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}f(\xi a+(1-\xi)b)\,d\xi+\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}f((1-\xi)b+\xi a)\,d\xi.$$

As consequence, we obtain

$$2^{\mathrm{s}}f\left(\frac{a+b}{2}\right)\mathrm{I}_{\alpha}(\alpha,\mathrm{b}-\mathrm{a})\leq\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\int_a^b(b-x)^{\alpha-1}e^{\zeta\ (b-x)}f(x)\,dx+\frac{\Gamma(\alpha)}{2(b-a)^{\alpha}}\int_a^b(x-a)^{\alpha-1}e^{\zeta\ (x-a)}f(x)\,dx$$

and the first inequality is proved.
For the proof of the second ineqaulity in (10), we first note that if f is $s-$convex function, then for $\xi\in[0,1]$

$$f(\xi a+(1-\xi)b)\leq\xi^s f(a)+(1-\xi)^s f(b)$$

and

$$f((1-\xi)b+\xi a)\leq\xi^s f(b)+(1-\xi)^s f(a).$$

By addition, we have

$$f(\xi a+(1-\xi)b)+f((1-\xi)b+\xi a)\leq[f(a)+f(b)]\left[\xi^s+(1-\xi)^s\right].\tag{12}$$

Multiplying both sides of (12) by $\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}$, then integrating the resulting inequality with respect to ξ over $[0,1]$, we obtain

$$\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}f(\xi a+(1-\xi)b)\,d\xi+\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \frac{\xi}{2}(b-a)}}f((1-\xi)b+\xi a)\,d\xi\leq[f(a)+f(b)](\mu+\eta)$$

where $\mu=\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}\xi^s d\xi$ and $\eta=\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta\ \xi(b-a)}}(1-\xi)^s d\xi$.

The proof is completed.
Remark 2.1. Inequalities (10) become the inequalities (1) by choosing $\zeta=0$, $s=1$ and $\alpha=1$.

3. New Findings via Tempered Fractional Integrals

In this section, we give an identity which use to assist us for proving our results as follows:
Lemma 3.1 Let $f:[a,b]\to\mathbb{R}$ is a twice differentiable function such that $f''\in L_1[a,b]$. Then, we have

$$\begin{aligned}&\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[{}^{\tau}_{\left(\frac{a+b}{2}\right)^{+}}I^{(\alpha,2\zeta)}_{b}f(b)+{}^{\tau}_{\left(\frac{a+b}{2}\right)^{-}}I^{(\alpha,2\zeta)}_{a}f(a)\right]-\mathrm{I}_{\zeta(b-a)}(\alpha,1)\,f\left(\frac{a+b}{2}\right)\\ =\;&\frac{(b-a)^2}{8}\int_0^1\left[\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right]\times\left[f''\left(\frac{\xi}{2}a+\frac{2-\xi}{2}b\right)+f''\left(\frac{\xi}{2}b+\frac{2-\xi}{2}a\right)\right]d\xi\end{aligned}$$

with $\zeta\ \geq\ 0,\alpha\ >\ 0,\xi\in[0,1]$.
Proof. By making the use of integrating by parts for right hand side of the equality and Remark 1.2 (a), we obtain

$$l_1=\frac{2}{a-b}\left[\mathrm{I}_{\zeta(b-a)}(\alpha,1)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\ 1)\right]\ f'\left(\frac{a+b}{2}\right)+\frac{2}{b-a}\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta(b-a)\xi}}f'\left(\frac{\xi}{2}a+\frac{2-\xi}{2}b\right)d\xi.$$

Thus, we get

$$\begin{aligned}l_1&=\frac{2}{a-b}\left[\mathrm{I}_{\zeta(b-a)}(\alpha,1)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\ 1)\right]\ f'\left(\frac{a+b}{2}\right)\\ &+\frac{2}{b-a}\left[\mathrm{I}_{\zeta(b-a)}(\alpha,1)\,f\left(\frac{a+b}{2}\right)\frac{2}{a-b}+\frac{2^{\alpha+1}\Gamma(\alpha)}{(b-a)^{1+\alpha}}\int_{\frac{a+b}{2}}^{b}(b-x)^{\alpha-1}e^{-2\zeta(b-x)}f'(x)\,dx\right]\end{aligned}$$

Finally, it is easy to obtain

$$\begin{aligned}l_1&=\frac{2}{a-b}\left[\mathrm{I}_{\zeta(b-a)}(\alpha,1)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\ 1)\right]\ f'\left(\frac{a+b}{2}\right)\\ &-\frac{4}{(b-a)^2}\mathrm{I}_{\zeta(b-a)}(\alpha,1)\,f\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+2}\Gamma(\alpha)}{(b-a)^{2+\alpha}}{}^{\tau}_{\left(\frac{a+b}{2}\right)^{+}}I^{(\alpha,2\zeta)}_{b}f(b)\end{aligned}$$

Similarly, we get

$$\begin{aligned}l_2&=\int_0^1\left(\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right)f''\left(\frac{\xi}{2}b+\frac{2-\xi}{2}a\right)d\xi\\ &=\frac{2}{b-a}\left[\mathrm{I}_{\zeta(b-a)}(\alpha,1)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\ 1)\right]\ f'\left(\frac{a+b}{2}\right)\\ &-\frac{4}{(b-a)^2}\mathrm{I}_{\zeta(b-a)}(\alpha,1)\,f\left(\frac{a+b}{2}\right)+\frac{2^{\alpha+2}\Gamma(\alpha)}{(b-a)^{2+\alpha}}{}^{\tau}_{\left(\frac{a+b}{2}\right)^{-}}I^{(\alpha,2\zeta)}_{a}f(a)\end{aligned}$$

Finally, by adding l_1 and l_2 and multiplying each side of the resulting identity by $\frac{(b-a)^2}{8}$, we get the desired result.
Theorem 3.1. Let $f:[a,b]\to\mathbb{R}^{+}$ be a differentiable mapping on $[a,b]$ with $a<b$ and $f''\in L_1[a,b]$. If $\left|f''\right|$ is $s-$convex function in the second sense, then the following inequality for tempered fractional integral

$$\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[{}^{\tau}_{\left(\frac{a+b}{2}\right)^{+}}I^{(\alpha,2\zeta)}_{b}f(b)+{}^{\tau}_{\left(\frac{a+b}{2}\right)^{-}}I^{(\alpha,2\zeta)}_{a}f(a)\right]-\mathrm{I}_{\zeta(b-a)}(\alpha,1)\,f\left(\frac{a+b}{2}\right)\right|\leq\frac{(b-a)^2}{2^{s+3}}\left[\left|f''(a)\right|+\left|f''(b)\right|\right](\boldsymbol{\tau}+\varsigma)$$

holds for $\zeta\ \geq\ 0,\alpha\ >\ 0,\ s\in[0,1]$ where $\tau=\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta(b-a)\xi}}(2-\xi)^s d\xi$ and $\varsigma=\int_0^1\frac{\xi^{\alpha-1}}{e^{\zeta(b-a)\xi}}(\xi)^s d\xi$.

Proof. From Lemma 3.1 and using $s-$convexity of $\left|f''\right|$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[{}_{\left(\frac{a+b}{2}\right)^{+}}^{\tau}I_{b}^{(\alpha,2\zeta)}f(b)+{}_{\left(\frac{a+b}{2}\right)^{-}}^{\tau}I_{a}^{(\alpha,2\zeta)}f(a)\right]-\mathrm{I}_{\zeta(b-a)}(\alpha,1)f\left(\frac{a+b}{2}\right)\right| \\
\leq\ &\left|\frac{(b-a)^{2}}{8}\int_{0}^{1}\left[\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right]\times\left[f''\left(\frac{\xi}{2}a+\frac{2-\xi}{2}b\right)+f''\left(\frac{\xi}{2}b+\frac{2-\xi}{2}a\right)\right]d\xi\right| \\
\leq\ &\frac{(b-a)^{2}}{8}\int_{0}^{1}\left[\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right]\times\left[\left(\left|f''(a)\right|+\left|f''(b)\right|\right)\left[\left(\frac{2-\xi}{2}\right)^{s}+\left(\frac{\xi}{2}\right)^{s}\right]\right]d\xi
\end{aligned}$$

By calculating the above integrals, we have the desired result. This completes the proof.

Theorem 3.2. Let $f:(0,\infty)\to\mathbb{R}$ be a twice differentiable mapping on $[a,b]$ with $0<a<b<\infty$ and $f''\in L_{1}[a,b]$. If $\left|f''\right|$ is $(h,m)-$convex function, then the following inequality for tempered fractional integral holds:

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[{}_{\left(\frac{a+b}{2}\right)^{+}}^{\tau}I_{b}^{(\alpha,2\zeta)}f(b)+{}_{\left(\frac{a+b}{2}\right)^{-}}^{\tau}I_{a}^{(\alpha,2\zeta)}f(a)\right]-\mathrm{I}_{\zeta(b-a)}(\alpha,1)f\left(\frac{a+b}{2}\right)\right| \\
\leq\ &\frac{(b-a)^{2}}{8}\left[\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\varphi+\left[\left|f''\left(\frac{a}{m}\right)\right|+\left|f''\left(\frac{b}{m}\right)\right|\right]\omega\right]
\end{aligned}$$

with $\zeta\geq0,\alpha>0,\ m\in(0,1)$ where $\varphi=\int_{0}^{1}\frac{\xi^{\alpha-1}}{e^{\zeta\frac{\xi}{2}(b-a)}}h\left(\frac{\xi}{2}\right)d\xi$ and $\omega=\int_{0}^{1}\frac{\xi^{\alpha-1}}{e^{\zeta\frac{\xi}{2}(mb-a)}}h\left(\frac{2-\xi}{2}\right)d\xi$.

Proof. From Lemma 3.1 and by using $(h,m)-$convexity of $\left|f''\right|$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[{}_{\left(\frac{a+b}{2}\right)^{+}}^{\tau}I_{b}^{(\alpha,2\zeta)}f(b)+{}_{\left(\frac{a+b}{2}\right)^{-}}^{\tau}I_{a}^{(\alpha,2\zeta)}f(a)\right]-\mathrm{I}_{\zeta(b-a)}(\alpha,1)f\left(\frac{a+b}{2}\right)\right| \\
\leq\ &\left|\frac{(b-a)^{2}}{8}\int_{0}^{1}\left[\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right]\times\left[f''\left(\frac{\xi}{2}a+\frac{2-\xi}{2}b\right)+f''\left(\frac{\xi}{2}b+\frac{2-\xi}{2}a\right)\right]d\xi\right| \\
\leq\ &\frac{(b-a)^{2}}{8}\int_{0}^{1}\left[\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right]\times\left[\left[\left|f''(a)\right|+\left|f''(b)\right|\right]h\left(\frac{\xi}{2}\right)\right]d\xi \\
+\ &\frac{(b-a)^{2}}{8}\int_{0}^{1}\left[\xi.\mathrm{I}_{\zeta(b-a)}(\alpha,\xi)-\mathrm{I}_{\zeta(b-a)}(\alpha+1,\xi)\right]\times\left[\left[\left|f''\left(\frac{a}{m}\right)\right|+\left|f''\left(\frac{b}{m}\right)\right|\right]h\left(\frac{2-\xi}{2}\right)\right]d\xi
\end{aligned}$$

By taking into account the followings in the resulting inequality, $\varphi=\int_{0}^{1}\frac{\xi^{\alpha-1}}{e^{\zeta\frac{\xi}{2}(b-a)}}h\left(\frac{\xi}{2}\right)d\xi$ and $\omega=\int_{0}^{1}\frac{\xi^{\alpha-1}}{e^{\zeta\frac{\xi}{2}(mb-a)}}h\left(\frac{2-\xi}{2}\right)d\xi$, we obtain

$$\begin{aligned}
&\left|\frac{2^{\alpha-1}\Gamma(\alpha)}{(b-a)^{\alpha}}\left[{}_{\left(\frac{a+b}{2}\right)^{+}}^{\tau}I_{b}^{(\alpha,2\zeta)}f(b)+{}_{\left(\frac{a+b}{2}\right)^{-}}^{\tau}I_{a}^{(\alpha,2\zeta)}f(a)\right]-\mathrm{I}_{\zeta(b-a)}(\alpha,1)f\left(\frac{a+b}{2}\right)\right| \\
\leq\ &\frac{(b-a)^{2}}{8}\left[\left[\left|f''(a)\right|+\left|f''(b)\right|\right]\varphi+\left[\left|f''\left(\frac{a}{m}\right)\right|+\left|f''\left(\frac{b}{m}\right)\right|\right]\omega\right]
\end{aligned}$$

This completes the proof.

4. Conclusion

In this paper, we have investigated some novel Hermite Hadamard-type inequalities in the context of tempered fractional integrals in the light of the incomplete gamma function. Through our work, we have tried to advance the theoretical foundations of fractional analysis, improve our understanding of convex functions in the framework of fractional analysis, and inspire further study.

Integral inequalities form a very important branch of analysis and are combined with various types of fractional integrals, but the main motivation of this study is to provide new integral inequalities involving different types of convex functions via tempered fractional integrals.

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