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Some Milne's rule type inequalities for convex functions with their computational analysis on quantum calculus

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Abstract. In this paper, we establish some new Milne's type inequalities for the differentiable convex functions in quantum calculus (q-calculus). We prove q-integral identity first and then we prove some new Milne's type inequalities for q-differentiable convex functions. These inequalities play an important role in Open-Newton's Cotes formulas. Furthermore, we give the computational analysis of these inequalities for convex functions and prove that the bounds of this paper are better than the existing ones. Ultimately, we provide some mathematical examples to show the validity of newly establish inequalities in q-calculus.

1. Introduction

A function $f : [a, b] \subset \mathbb{R} \to \mathbb{R}$ is said to be convex, if

$$f((1-t)b + ta) \leq (1-t)f(b) + tf(a), \quad \forall\, a, b \in I \,\&\, t \in [0, 1].$$

The theory of convexity refers to the shape of the graph of a function and can be described as a function whose graph lies below or on the line segment connecting any two points on the graph. Convexity offers several advantages in mathematics and optimization problems. Convex functions are mathematically controllable and have well-defined properties. These properties make it easier to analyze and understand the behavior of the function. Convexity often provides validity in optimization problems, meaning that small changes in the problem formulation or data do not significantly affect the optimal solution. This validity is valuable in practical applications where uncertainties and variations exist. Convexity plays an important role in various fields such as engineering, economics, statistics, machine learning, and operations research. Mathematical inequalities are mathematical statements that compare two values or expressions, indicating their relative sizes or relationships. They provide a way to express and analyze the differences or relationships between quantities. The concept of inequalities has been a fundamental part of mathematics for a long time, with records of their use dating back to ancient civilizations. However, the systematic study of inequalities and their properties gained prominence in the 17th and 18th centuries. Mathematicians such as Pierre de

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Fermat, Isaac Newton, and Joseph Louis Lagrange made notable contributions to this field. Inequalities play a crucial role in various areas of mathematics. They are extensively used in calculus, real analysis, number theory, optimization, and functional analysis, among other branches. In calculus, inequalities are essential for studying limits, continuity, and convergence. In optimization and mathematical modeling, they are used to find the maximum or minimum values satisfying certain constraints. The field of inequalities was comprehensively and systematically studied by Hardy, Littlewood, and Polya, and their findings were compiled in the book "Inequalities". In recent years, mathematical inequalities and their applications have rapidly developed and had a considerable impact on various modern mathematical disciplines such as information theory, game theory, integral operator theory, error analysis, and approximation theory. Several well-known inequalities, such as Hermite-Hadamard, Simpson's, Ostrowski, and Gruss, provide bounds for quadrature rules. Several inequalities can be obtained directly from the applications of convex functions, one of which is the Hermite-Hadamard inequality for convex functions.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}. \tag{1}$$

This inequality has several applications in various areas of mathematics, including analysis, optimization theory, and economics. It provides a powerful tool for proving numerous other inequalities and has implications for the study of convex functions. Due to its numerous applications, mathematicians began to study it and produced new results. In [1, 2], the bounds of the trapezoidal and midpoint types inequalities are produced by using differentiable convexity from (1), respectively. A new Bullen's type inequality was established in [3], by using inequality (1).

Milne's rule type inequalities for classical and q-calculus were established by mathematicians named Joseph P. Milne and Mourad E. H. Ismail respectively. Joseph P. Milne is known for his work in numerical analysis and the development of approximation methods, while Mourad E. H. Ismail has made significant contributions to the field of q-calculus, which is a generalization of classical calculus. These inequalities have been extensively studied in classical calculus and have proven to be powerful tools in analyzing functions. The aim of this study, is to explore and analyze some Milne's rule type inequalities for convex functions within the framework of (q-calculus). The main focus is to determining the optimal values, defining bounds, and understanding the properties of these inequalities in q-calculus. This analysis will not only deepen our understanding of how convex functions behave in q-calculus but also provide practical tools for solving problems that involve convex functions within q-calculus. Milne's inequality, which is comparable to the Simpson's inequality in terms of its applicability is the only inequality that provides estimates of error bounds for Milne's formula. Assume that $f : [a,b] \to \mathbb{R}$ is four times differentiable function on (a,b), and $\left\|f^4\right\|_\infty$, then

$$\left|\frac{1}{(b-a)^2}\int_a^b f(x)\,dx - \frac{(b-a)}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right| \leq \frac{7(b-a)^5}{23040}\sup_{x\in(a,b)}\left|f^{(4)}(x)\right|. \tag{2}$$

In recent years, researchers have paid close attention to Milne's inequality. In [4], authors introduced fractional analogs of this inequality by utilizing the properties of convex functions, bounded functions, bounded variation, and Lipschitz conditions. In [5], certain integral inequalities for the Milne's formula using local fractal integrals via generalized convexity property of functions were examined. In [6], the authors investigated the fractional error bounds for Milne's inequality associated with convex functions.

The q-Hermite-Hadamard type inequality is an extension of the classical Hermite-Hadamard inequality in mathematical analysis. This inequality provides bounds on the convex functions based on their values at the endpoints of an interval. The q-Hermite-Hadamard type inequality was derived from the notion of q-calculus, which is the extensions of traditional calculus that deal with q-derivatives. The q-Hermite-Hadamard type inequality is expressed as follows:

$$f\left(\frac{qa+b}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ {}_ad_qx \leq \frac{qf(a)+f(b)}{1+q}, \tag{3}$$

for $q \in (0,1)$. The fundamental reason to established this inequality (3), was discussed in [7].

The Hermite-Hadamard inequality can be shown in q-calculus in a number of ways, but Burmudo et al. [8] provided the new version of Hermite-Hadamard inequality for $q \in (0,1)$, which is as follows:

$$f\left(\frac{a+qb}{1+q}\right) \leq \frac{1}{b-a}\int_a^b f(x)\ {}^b d_q x \leq \frac{f(a)+qf(b)}{1+q}, \tag{4}$$

and

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{2(b-a)}\left[\int_a^b f(x)\ {}_a d_q x + \int_a^b f(x)\ {}^b d_q x\right] \leq \frac{f(a)+f(b)}{2}. \tag{5}$$

These inequalities has gained significant attention in recent years due to its various applications in fields such as quantum physics, economics, and information theory. It provides a valuable tool for establishing bounds on certain mathematical quantities and analyzing convex functions in a q-calculus. In [7, 9], authors using the q-differentiable convexity throughout, to determine the bounds of the midpoint and trapezoidal formulas in q-calculus from the inequality (3), respectively. The trapezoidal formula in q-calculus was constructed using the inequality (4), and Budak used the same methods as in [10] to get the midpoint bounds. Utilizing the inequality (5), some further bounds for midpoint and trapezoidal formulas in q-calculus were established in [11]. In [12, 13], the authors proved new Hermite-Hadamard inequalities in q-calculus for piecewise continous convex function, respectively. We demonstrate certain dual Simpson's type inequalities in q-calculus in response to modern research. We establish required inequalities for q-differentiable convex functions within the framework of q-calculus by first proving a novel quantum integral identity. The error bounds for the dual Simpson's formula in quantum and classical calculus can be determined with the use of these inequalities. One can consult to [14, 15], and sources referenced therein for more contemporary inequalities of Simpson's and Newton's type in q-calculus. For more detials, one can consult [16–18] for more interesting inequalities. However, in the results demonstrated here, only the first differentiability of the function is necessary, but previously calssical Milne's type inequality (2) has been proven, and we need a function that is four times differentiable. We only need the first differentiability of functions to find the error bounds for Milne's rule.

Inspired by the continuing studies, we establish Milne's type inequalities using the convexity property of the function in terms of q-calculus. We also show that the inequalities given here are an extension of some existing ones and we give some numerical examples to show the validity of newly established inequalities.

The organization of the paper is as follows: The second Section provides an overview of convex functions and q-calculus. The third Section presents our main results on Milne's inequality with the help of q-identity. In the fourth Section, we provide some numerical examples and graphical illustrations to validate our results. Finally, we give some concluding remarks about this work and some future directions in Section 5.

2. Preliminaries of q-calculus

q-calculus, which focuses on a meaningful modification of integration and differentiation techniques from a theoretical perspective, is a fundamental research topic in the field of mathematical analysis. The q-derivative and integral operations were first carefully studied by Jackson. Jackson's q-operators played a pivotal role in the development of q-theory, which has tremendous applications in special functions, modern mathematical analysis, physics, number theory, combinatorics, cryptography, etc. For more detials see [20, 21]. To proceed further, let us recall the essentials of q-calculus.

Tariboon and Ntouyas [19] introduced the left or q_a-derivative and integral concept in 2013. They also discuss their properties, here we recall the definitions from their work. Throughout this Section, the functions f and h are continuous $f, h : [a,b] \to \mathbb{R}$.

Definition 2.1. *[19] The q_a-derivative of f at $x \in (a,b]$ is defined by*

$${}_a D_q f(x) = \frac{f(x) - f(a+(x-a)q)}{(x-a)(1-q)}.$$

If $x = a$, *we define* ${}_aD_q f(a) = \lim_{x\to a} {}_aD_q f(x)$ *if it exists and it is finite.*

Definition 2.2. *[19] The* q_a*-integral is defined by*

$$\int_a^x f(t)\ {}_ad_q t = (x-a)(1-q)\sum_{n=0}^{\infty} q^n f(a+(x-a)q^n),$$

where $x \in [a,b]$.

In 2020, some new definitions of quantum derivative and integral using a different approach were introduced by Bermudo et al. [8], namely right or q^b-derivative and integral. They also discussed some basic properties of the given operators, here we recall the following definitions from their work.

Definition 2.3. *[8] The* q^b*-derivative of* f *at* $x \in (a,b]$ *is defined by*

$${}^bD_q f(x) = \frac{f(b+(x-b)q) - f(x)}{(b-x)(1-q)}.$$

If $x = b$, *we define* ${}^bD_q f(b) = \lim_{x\to b} {}^bD_q f(x)$ *if it exists and it is finite.*

Definition 2.4. *[8] The* q^b*-integral is defined by*

$$\int_x^b f(t)\ {}^bd_q t = (b-x)(1-q)\sum_{n=0}^{\infty} q^n f(b+(x-b)q^n),$$

where $x \in [a,b]$.

Lemma 2.5. *[19] We have the following equality for the functions* f *and* h*:*

$$\int_0^c h(t)\ {}_aD_q f(ta+(1-t)b)\ {}_0d_q t = \left.\frac{h(t)f(ta+(1-t)b)}{b-a}\right|_0^c - \frac{1}{b-a}\int_0^c f(ta+(1-t)b)\ {}_0D_q h(t)\ {}_0d_q t. \quad (6)$$

Lemma 2.6. *[8] We have the following equality for the functions* f *and* h*:*

$$\int_0^c h(t)\ {}^bD_q f(ta+(1-t)b)\ {}_0d_q t = \frac{1}{b-a}\int_0^c f(ta+(1-t)b)\ {}_0D_q h(t)\ {}_0d_q t - \left.\frac{h(t)f(ta+(1-t)b)}{b-a}\right|_0^c. \quad (7)$$

3. Milne's Type Inequalities

Here, we use quantum differentiable convex functions to demonstrate the main inequalities. For this, first, we give the following quantum integral identity involving a quantum differentiable function. For the sake of brevity, we use the following quantum number notation:

$$[n]_q = \frac{1-q^n}{1-q} = \sum_{k=0}^{n-1} q^k,\ \ q \in (0,1).$$

Lemma 3.1. *If* $f : [a,b] \to \mathbb{R}$ *be a q-differentiable function. If* ${}^{\frac{3a+b}{4}}D_q f(t)$, ${}^{\frac{a+b}{2}}D_q f(t)$, ${}^{\frac{a+3b}{4}}D_q f(t)$ *and* ${}^bD_q f(t)$ *are q-integrable on* $[a,b]$, *and* $t \in [0,1]$, *then the following equality holds:*

$$\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_q x + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_q x + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_q x + \int_{\frac{a+3b}{4}}^{b} f(x)\ {}^bd_q x\right)$$

$$
\begin{aligned}
&-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\\
&=\frac{b-a}{16}\left[I_1+I_2+I_3+I_4\right],
\end{aligned}
\tag{8}
$$

where,

$$
\begin{aligned}
I_1 &= \int_0^1 (qt-1)\ {}^{\frac{3a+b}{4}}D_q f\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)d_qt\\
I_2 &= \int_0^1 \left(qt+\frac{2}{3}\right)\ {}^{\frac{a+b}{2}}D_q f\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)d_qt\\
I_3 &= \int_0^1 \left(qt-\frac{5}{3}\right)\ {}^{\frac{a+3b}{4}}D_q f\left(t\left(\frac{a+b}{2}\right)+(1-t)\left(\frac{a+3b}{4}\right)\right)d_qt\\
I_4 &= \int_0^1 qt\ {}^{b}D_q f\left(t\left(\frac{a+3b}{4}\right)+(1-t)\,b\right)d_qt.
\end{aligned}
$$

Proof. By the definition of q-integral and using Lemma 2.6, we have

$$
\begin{aligned}
I_1 &= \int_0^1 (qt-1)\ {}^{\frac{3a+b}{4}}D_q f\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)d_qt\\
&= -4qt\frac{f\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)}{b-a}\Bigg|_0^1+\frac{4q}{b-a}\int_0^1 f\left(qta+(1-qt)\left(\frac{3a+b}{4}\right)\right)d_qt\\
&+\frac{4}{b-a}\left[f(a)-f\left(\frac{3a+b}{4}\right)\right]\\
&=-\frac{4q}{b-a}f(a)+\frac{4q}{b-a}\left[\frac{1-q}{q}\sum_{n=0}^{\infty}q^{n+1}f\left(q^{n+1}a+\left(1-q^{n+1}\right)\left(\frac{3a+b}{4}\right)\right)\right]\\
&+\frac{4}{b-a}\left[f(a)-f\left(\frac{3a+b}{4}\right)\right]\\
&=\frac{4}{b-a}\left[(1-q)\sum_{n=0}^{\infty}q^{n}f\left(q^{n}a+(1-q^{n})\left(\frac{3a+b}{4}\right)\right)\right]-\frac{4v}{b-a}f\left(\frac{3a+b}{4}\right)\\
&=\frac{16}{(b-a)^2}\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_qx-\frac{4}{b-a}f\left(\frac{3a+b}{4}\right),
\end{aligned}
\tag{9}
$$

similarly,

$$
\begin{aligned}
I_2 &= \int_0^1 \left(qt+\frac{2}{3}\right)\ {}^{\frac{a+b}{2}}D_q f\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)d_qt\\
&= -4qt\frac{f\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)}{b-a}\Bigg|_0^1+\frac{4q}{b-a}\int_0^1 f\left(qt\left(\frac{3a+b}{4}\right)+(1-qt)\left(\frac{a+b}{2}\right)\right)d_qt\\
&-\frac{8}{3(b-a)}\left[f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)\right]\\
&=-\frac{4q}{b-a}f\left(\frac{3a+b}{4}\right)+\frac{4q}{b-a}\left[\frac{1-q}{q}\sum_{n=0}^{\infty}q^{n+1}f\left(q^{n+1}\left(\frac{3a+b}{4}\right)+\left(1-q^{n+1}\right)\left(\frac{a+b}{2}\right)\right)\right]
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{8}{3(b-a)}\left[f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)\right] \\
&=\frac{16}{(b-a)^2}\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx-\frac{20}{3(b-a)}f\left(\frac{3a+b}{4}\right)+\frac{8}{3(b-a)}f\left(\frac{a+b}{2}\right), \qquad (10)
\end{aligned}
$$

$$
\begin{aligned}
I_3 &= \int_0^1\left(qt-\frac{5}{3}\right)\ {}^{\frac{a+3b}{4}}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\left(\frac{a+3b}{4}\right)\right)d_qt \\
&=\frac{16}{(b-a)^2}\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_qx+\frac{8}{3(b-a)}f\left(\frac{a+b}{2}\right)-\frac{20}{3(b-a)}f\left(\frac{a+3b}{4}\right), \qquad (11)
\end{aligned}
$$

and

$$
\begin{aligned}
I_4 &= \int_0^1 qt\ {}^bD_qf\left(t\left(\frac{a+3b}{4}\right)+(1-t)\,b\right)d_qt \\
&=\frac{16}{(b-a)^2}\int_{\frac{a+3b}{4}}^{b} f(x)\ {}^bd_qx-\frac{4}{b-a}f\left(\frac{a+3b}{4}\right). \qquad (12)
\end{aligned}
$$

Adding (9)-(12) and multiplying by $\frac{b-a}{16}$ on both side then we obtain the required result. The proof is completed. □

Now, we calculate the integrals that will be used in our next results .

$$E_{11}=\int_0^1(1-qt)\,d_qt=\frac{1}{[2]_q}, \qquad (13)$$

$$E_{12}=\int_0^1\left(qt+\frac{2}{3}\right)d_qt=\frac{q}{[2]_q}+\frac{2}{3}, \qquad (14)$$

$$E_{13}=\int_0^1\left(\frac{5}{3}-qt\right)d_qt=\frac{5}{3}-\frac{q}{[2]_q}, \qquad (15)$$

$$E_{14}=\int_0^1 qt d_qt=\frac{q}{[2]_q}, \qquad (16)$$

$$
\begin{aligned}
&E_1=\int_0^1 t(1-qt)\,d_qt=\frac{1}{[2]_q[3]_q},\ E_2=\int_0^1(1-t)(1-qt)\,d_qt=\frac{q}{[3]_q},\\
&E_3=\int_0^1 t\left(qt+\frac{2}{3}\right)d_qt=\frac{q}{[3]_q}+\frac{2}{3[2]_q},\ E_4=\int_0^1(1-t)\left(qt+\frac{2}{3}\right)d_qt=\frac{q}{[2]_q}+\frac{2}{3}-\frac{q}{[3]_q}-\frac{2}{3[2]_q},\\
&E_5=\int_0^1 t\left(\frac{5}{3}-qt\right)d_qt=\frac{5}{3[2]_q}-\frac{q}{[3]_q},\ E_6=\int_0^1(1-t)\left(\frac{5}{3}-qt\right)d_qt=\frac{5}{3}-\frac{q}{[2]_q}-\frac{5}{3[2]_q}+\frac{q}{[3]_q},\\
&E_7=\int_0^1\left(qt^2\right)d_qt=\frac{q}{[3]_q},\ E_8=\int_0^1(1-t)(qt)\,d_qt=\frac{q}{[2]_q}-\frac{q}{[3]_q}.
\end{aligned}
$$

Theorem 3.2. *Assume that the assumptions of Lemma 3.1 hold. If* $\left|{}^{\frac{3a+b}{4}}D_qf\right|$, $\left|{}^{\frac{a+b}{2}}D_qf\right|$, $\left|{}^{\frac{a+3b}{4}}D_qf\right|$ *and* $\left|{}^{b}D_qf\right|$ *are convex on* $[a,b]$ *, then the following inequality holds:*

$$
\left|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_qx+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_qx+\int_{\frac{a+3b}{4}}^{b} f(x)\ {}^bd_qx\right)\right.
$$

$$-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\Bigg|$$
$$\leq \frac{b-a}{16}\left[(E_1)\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|+(E_2)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|+(E_3)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|\right.$$
$$+(E_4)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|+(E_5)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|+(E_6)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|$$
$$\left.+(E_7)\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|+(E_8)\left|{}^{b}D_qf(b)\right|\right]. \tag{17}$$

Proof. By Lemma 3.1, we have

$$\Bigg|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}}f(x)\ {}^{\frac{3a+b}{4}}d_qx+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}f(x)\ {}^{\frac{a+3b}{4}}d_qx+\int_{\frac{a+3b}{4}}^{b}f(x)\ {}^{b}d_qx\right)$$
$$-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\Bigg|$$
$$\leq \frac{b-a}{16}\left[\int_0^1|qt-1|\left|{}^{\frac{3a+b}{4}}D_qf\left(at+(1-t)\left(\frac{3a+b}{4}\right)\right)\right|d_qt\right.$$
$$+\int_0^1\left|qt+\frac{2}{3}\right|\left|{}^{\frac{a+b}{2}}D_qf\left(\left(\frac{3a+b}{4}\right)t+(1-t)\left(\frac{a+b}{2}\right)\right)\right|d_qt$$
$$\left.+\int_0^1\left|\frac{5}{3}-qt\right|\left|{}^{\frac{a+3b}{4}}D_qf\left(\left(\frac{a+b}{2}\right)t+(1-t)\left(\frac{a+3b}{4}\right)\right)\right|d_qt+\int_0^1|qt|\left|{}^{b}D_qf\left(\left(\frac{a+3b}{4}\right)t+(1-t)b\right)\right|d_qt\right]. \tag{18}$$

Since $\left|{}^{\frac{3a+b}{4}}D_qf\right|, \left|{}^{\frac{a+b}{2}}D_qf\right|, \left|{}^{\frac{a+3b}{4}}D_qf\right|$ *and* $\left|{}^{b}D_qf\right|$ *are convex on* $[a,b]$, *therefore*

$$\left|{}^{b}D_qf(at+(1-t)b)\right|\leq t\left|{}^{b}D_qf(a)\right|+(1-t)\left|{}^{b}D_qf(b)\right|. \tag{19}$$

Using notion (19) in (18), we have

$$\Bigg|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}}f(x)\ {}^{\frac{3a+b}{4}}d_qx+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}f(x)\ {}^{\frac{a+3b}{4}}d_qx+\int_{\frac{a+3b}{4}}^{b}f(x)\ {}^{b}d_qx\right)$$
$$-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\Bigg|$$
$$\leq \left[\int_0^1|qt-1|\left[t\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|+(1-t)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|\right]d_qt\right.$$
$$+\int_0^1\left|qt+\frac{2}{3}\right|\left[t\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|+(1-t)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|\right]d_qt$$
$$+\int_0^1\left|\frac{5}{3}-qt\right|\left[t\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|+(1-t)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|\right]d_qt$$
$$\left.+\int_0^1|qt|\left[t\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|+(1-t)\left|{}^{b}D_qf(b)\right|\right]d_qt\right]$$
$$\leq \left[\left(\int_0^1 t|qt-1|d_qt\right)\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|+\left(\int_0^1(1-t)|qt-1|d_qt\right)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|\right.$$
$$+\left(\int_0^1 t\left|qt+\frac{2}{3}\right|d_qt\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|+\left(\int_0^1(1-t)\left|qt+\frac{2}{3}\right|d_qt\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|$$

$$+\left(\int_0^1 t\left|\frac{5}{3}-qt\right|d_qt\right)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|+\left(\int_0^1 (1-t)\left|\frac{5}{3}-qt\right|d_qt\right)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|$$

$$+\left(\int_0^1 t\left|qt\right|d_qt\right)\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|+\left(\int_0^1 (1-t)\left|qt\right|d_qt\right)\left|{}^{b}D_qf(b)\right|\Bigg]$$

$$= \frac{b-a}{16}\Bigg[(E_1)\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|+(E_2)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|+(E_3)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|$$

$$+(E_4)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|+(E_5)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|+(E_6)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|$$

$$+(E_7)\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|+(E_8)\left|{}^{b}D_qf(b)\right|\Bigg].$$

Which is the desired conclusion. □

Corollary 3.3. *If we take the $q \to 1^-$ in Theorem 3.2, then we obtain the following inequality:*

$$\left|\frac{1}{(b-a)}\int_a^b f(x)\,dx-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right|$$

$$\leq \frac{b-a}{96}\Bigg[\left|f'(a)\right|+2\left|f'\left(\frac{3a+b}{4}\right)\right|+4\left|f'\left(\frac{3a+b}{4}\right)\right|$$

$$+3\left|f'\left(\frac{a+b}{2}\right)\right|+3\left|f'\left(\frac{a+b}{2}\right)\right|+4\left|f'\left(\frac{a+3b}{4}\right)\right|$$

$$+2\left|f'\left(\frac{a+3b}{4}\right)\right|+\left|f'(b)\right|\Bigg].$$

Theorem 3.4. *We assume that the given conditions of Lemma 3.1 hold. If the mappings $\left|{}^{\frac{3a+b}{4}}D_qf\right|^r$, $\left|{}^{\frac{a+b}{2}}D_qf\right|^r$, $\left|{}^{\frac{a+3b}{4}}D_qf\right|^r$ and $\left|{}^{b}D_qf\right|^r$ are convex on $[a,b]$ and $\frac{1}{p}+\frac{1}{r}=1$ with $p,r>1$, then the following Milne's formula type inequality holds:*

$$\Bigg|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_qx+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_qx+\int_{\frac{a+3b}{4}}^{b} f(x)\ {}^{b}d_qx\right)$$

$$-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\Bigg|$$

$$\leq \frac{b-a}{16}\Bigg[\left(\int_0^1 \left|1-qt\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r+q\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r}{[2]_q}\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1 \left|qt+\frac{2}{3}\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r+q\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r}{[2]_q}\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1 \left|\frac{5}{3}-qt\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^r}{[2]_q}\right)^{\frac{1}{r}}$$

$$+\left(\frac{q^p}{[p+1]_q}\right)^{\frac{1}{p}}\left(\frac{\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|^r+q\left|{}^{b}D_qf(b)\right|^r}{[2]_q}\right)^{\frac{1}{r}}\Bigg]. \quad (20)$$

Proof. By using q-Hölder inequality in (18), we have

$$\begin{aligned}
&\left|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_qx + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_qx + \int_{\frac{a+3b}{4}}^{b} f(x)\ {}^{b}d_qx\right)\right.\\
&\left.-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\right|\\
&\le \frac{b-a}{16}\left[\left(\int_0^1 \left|1-qt\right|^p d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}^{\frac{3a+b}{4}}D_qf\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)\right|^r d_qt\right)^{\frac{1}{r}}\right.\\
&+\left(\int_0^1\left|qt+\frac{2}{3}\right|^p d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}^{\frac{a+b}{2}}D_qf\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)\right|^r d_qt\right)^{\frac{1}{r}}\\
&+\left(\int_0^1\left|\frac{5}{3}-qt\right|^p d_qt\right)^{\frac{1}{p}}\left(\int_0^1\left|{}^{\frac{a+3b}{4}}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\left(\frac{a+3b}{4}\right)\right)\right|^r d_qt\right)^{\frac{1}{r}}\\
&\left.+\left(\frac{q^p}{[p+1]_q}\right)^{\frac{1}{p}}\left(\int_0^1\left|{}^{b}D_qf\left(t\left(\frac{a+3b}{4}\right)+(1-t)\,b\right)\right|^r d_qt\right)^{\frac{1}{r}}\right]. \qquad (21)
\end{aligned}$$

Since $\left|{}^{\frac{3a+b}{4}}D_qf\right|^r$, $\left|{}^{\frac{a+b}{2}}D_qf\right|^r$, $\left|{}^{\frac{a+3b}{4}}D_qf\right|^r$ and $\left|{}^{b}D_qf\right|^r$ are convex on $[a,b]$, we have

$$\begin{aligned}
&\int_0^1\left|{}^{\frac{3a+b}{4}}D_qf\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)\right|^r d_qt\\
&\le \int_0^1\left[t\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r+(1-t)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r\right]d_qt\\
&=\frac{1}{[2]_q}\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r+\frac{q}{[2]_q}\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r. \qquad (22)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1\left|{}^{\frac{a+b}{2}}D_qf\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)\right|^r d_qt\\
&\le \int_0^1\left[t\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r+(1-t)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r\right]d_qt\\
&=\frac{1}{[2]_q}\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r+\frac{q}{[2]_q}\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r, \qquad (23)\\
&\int_0^1\left|{}^{\frac{a+3b}{4}}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\left(\frac{a+3b}{4}\right)\right)\right|^r d_qt\\
&\le \int_0^1\left[t\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^r+(1-t)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^r\right]d_qt\\
&=\frac{1}{[2]_q}\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^r+\frac{q}{[2]_q}\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^r, \qquad (24)
\end{aligned}$$

and

$$\int_0^1\left|{}^{b}D_qf\left(t\left(\frac{a+3b}{4}\right)+(1-t)\,b\right)\right|^r d_qt$$

$$\leq \int_0^1 \left[t\left|{}^bD_qf\left(\frac{a+3b}{4}\right)\right|^r + (1-t)\left|{}^bD_qf(b)\right|^r\right]d_qt$$
$$= \frac{1}{[2]_q}\left|{}^bD_qf\left(\frac{a+3b}{4}\right)\right|^r + \frac{q}{[2]_q}\left|{}^bD_qf(b)\right|^r. \tag{25}$$

On the other way, we also have the equalities

$$\int_0^1 td_qt = \frac{1}{[2]_q}, \int_0^1 (1-t)\,d_qt = \frac{q}{[2]_q},$$

and

$$\int_0^1 (qt)^p\,d_qt = \frac{q^p}{[p+1]_q}. \tag{26}$$

By substituting (22)-(26) in (21), then we obtain the required result. □

Corollary 3.5. *By taking $q \to 1^-$ in Theorem 3.4, then we obtain the following inequality:*

$$\left|\frac{1}{(b-a)}\int_a^b f(x)\,dx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leq \frac{b-a}{16}\left[\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|f'(a)\right|^r + \left|f'\left(\frac{3a+b}{4}\right)\right|^r}{2}\right)^{\frac{1}{r}}\right.$$
$$+\left(\frac{1}{(p+1)}+\frac{2}{3}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{3a+b}{4}\right)\right|^r + \left|f'\left(\frac{a+b}{2}\right)\right|^r}{2}\right)^{\frac{1}{r}}$$
$$+\left(\frac{5}{3}-\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+b}{2}\right)\right|^r + \left|f'\left(\frac{a+3b}{3}\right)\right|^r}{2}\right)^{\frac{1}{r}}$$
$$\left.+\left(\frac{1}{(p+1)}\right)^{\frac{1}{p}}\left(\frac{\left|f'\left(\frac{a+3b}{4}\right)\right|^r + \left|f'(b)\right|^r}{2}\right)^{\frac{1}{r}}\right].$$

Theorem 3.6. *We assume that the given conditions of Lemma 3.1 hold. If the mappings $\left|{}^{\frac{3a+b}{4}}D_qf\right|^r$, $\left|{}^{\frac{a+b}{2}}D_qf\right|^r$, $\left|{}^{\frac{a+3b}{4}}D_qf\right|^r$ and $\left|{}^bD_qf\right|^r$ are convex on $[a,b]$ with $r \geq 1$, then the following Milne's formula type inequality holds:*

$$\left|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_qx + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_qx + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_qx + \int_{\frac{a+3b}{4}}^{b} f(x)\ {}^bd_qx\right)\right.$$
$$\left.-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leq \frac{b-a}{16}\left[\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r + q(1+q)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r}{[2]_q[3]_q}\right)^{\frac{1}{r}}\right.$$

$$+\left(\frac{2+5q}{3\left[2\right]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(2+5q+5q^2\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r+q\left(2+2q+5q^2\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r}{3\left[2\right]_q\left[3\right]_q}\right)^{\frac{1}{r}}$$

$$+\left(\frac{5+2q}{3\left[2\right]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(5+2q+2q^2\right)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^r+q\left(5+5q+2q^2\right)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^r}{3\left[2\right]_q\left[3\right]_q}\right)^{\frac{1}{r}}$$

$$+\left(\frac{q}{\left[2\right]_q}\right)^{1-\frac{1}{r}}\left(\frac{q\left(1+q\right)\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|^r+q^3\left|{}^{b}D_qf\left(b\right)\right|^r}{\left[2\right]_q\left[3\right]_q}\right)^{\frac{1}{r}}\Bigg]. \tag{27}$$

Proof. By using q-power mean inequality in (18), we have

$$\Bigg|\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}}f(x)\ {}^{\frac{3a+b}{4}}d_qx+\int_{\frac{3a+b}{4}}^{\frac{a+b}{2}}f(x)\ {}^{\frac{a+b}{2}}d_qx+\int_{\frac{a+b}{2}}^{\frac{a+3b}{4}}f(x)\ {}^{\frac{a+3b}{4}}d_qx+\int_{\frac{a+3b}{4}}^{b}f(x)\ {}^{b}d_qx\right)$$
$$-\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right)-f\left(\frac{a+b}{2}\right)+2f\left(\frac{a+3b}{4}\right)\right]\Bigg|$$
$$\leq\frac{b-a}{16}\left[\left(\int_0^1\left|1-qt\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1(1-qt)\left|{}^{\frac{3a+b}{4}}D_qf\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)\right|^rd_qt\right)^{\frac{1}{r}}\right.$$
$$+\left(\int_0^1\left|qt+\frac{2}{3}\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1\left|qt+\frac{2}{3}\right|\left|{}^{\frac{a+b}{2}}D_qf\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)\right|^rd_qt\right)^{\frac{1}{r}}$$
$$+\left(\int_0^1\left|\frac{5}{3}-qt\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1\left|\frac{5}{3}-qt\right|\left|{}^{\frac{a+3b}{4}}D_qf\left(t\left(\frac{a+b}{2}\right)+(1-t)\left(\frac{a+3b}{4}\right)\right)\right|^rd_qt\right)^{\frac{1}{r}}$$
$$\left.+\left(\int_0^1\left|qt\right|d_qt\right)^{1-\frac{1}{r}}\left(\int_0^1qt\left|{}^{b}D_qf\left(t\left(\frac{a+3b}{4}\right)+(1-t)\,b\right)\right|^rd_qt\right)^{\frac{1}{r}}\right]. \tag{28}$$

Since $\left|{}^{\frac{3a+b}{4}}D_qf\right|^r$, $\left|{}^{\frac{a+b}{2}}D_qf\right|^r$, $\left|{}^{\frac{a+3b}{4}}D_qf\right|^r$ and $\left|{}^{b}D_qf\right|^r$ are convex on $[a,b]$, then we have

$$\int_0^1\left|(1-qt)\right|\left|{}^{\frac{3a+b}{4}}D_qf\left(ta+(1-t)\left(\frac{3a+b}{4}\right)\right)\right|^rd_qt$$
$$\leq\int_0^1\left|(1-qt)\right|\left[t\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r+(1-t)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r\right]d_qt$$
$$=\frac{\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r+q(1+q)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r}{\left[2\right]_q\left[3\right]_q}, \tag{29}$$

similarly, by convexity of $\left|{}^{\frac{a+b}{2}}D_qf\right|^r$, $\left|{}^{\frac{a+3b}{4}}D_qf\right|^r$ and $\left|{}^{b}D_qf\right|^r$ and the results obtained in the proof of Theorem 3.2, we have

$$\int_0^1\left|qt+\frac{2}{3}\right|\left|{}^{\frac{a+b}{2}}D_qf\left(t\left(\frac{3a+b}{4}\right)+(1-t)\left(\frac{a+b}{2}\right)\right)\right|^rd_qt$$
$$=\frac{\left(2+5q+5q^2\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r+q\left(2+2q+5q^2\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r}{3\left[2\right]_q\left[3\right]_q}, \tag{30}$$

$$\int_0^1 \left|\frac{5}{3} - qt\right| \left|{}^{\frac{a+3b}{4}}D_q f\left(t\left(\frac{a+b}{2}\right) + (1-t)\left(\frac{a+3b}{4}\right)\right)\right|^r d_q t$$
$$= \frac{\left(5+2q+2q^2\right)\left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+b}{2}\right)\right|^r + q\left(5+5q+2q^2\right)\left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+3b}{4}\right)\right|^r}{3\,[2]_q\,[3]_q}, \tag{31}$$

and

$$\int_0^1 |qt| \left|{}^{b}D_q f\left(t\left(\frac{a+3b}{4}\right) + (1-t)\,b\right)\right|^r d_q t$$
$$= \frac{q\,(1+q)\left|{}^{b}D_q f\left(\frac{a+3b}{4}\right)\right|^r + q^3\left|{}^{b}D_q f\,(b)\right|^r}{[2]_q\,[3]_q}. \tag{32}$$

If we substitute (29)-(32) and (13)-(16) in (28) then we obtain the required result. □

Corollary 3.7. *By taking $q \to 1^-$ in Theorem 3.6, then we obtain the following inequality:*

$$\left|\frac{1}{(b-a)}\int_a^b f(x)\,dx - \frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right]\right|$$
$$\leq \frac{b-a}{32}\left[\left(\frac{\left|f'(a)\right|^r + 2\left|f'\left(\frac{3a+b}{4}\right)\right|^r}{3}\right)^{\frac{1}{r}} + \left(\frac{2\left|f'\left(\frac{a+3b}{4}\right)\right|^r + \left|f'(b)\right|^r}{3}\right)^{\frac{1}{r}}\right]$$
$$+\frac{7\,(b-a)}{96}\left[\left(\frac{4\left|f'\left(\frac{3a+b}{4}\right)\right|^r + 3\left|f'\left(\frac{a+b}{2}\right)\right|^r}{7}\right)^{\frac{1}{r}} + \left(\frac{3\left|f'\left(\frac{a+b}{2}\right)\right|^r + 4\left|f'\left(\frac{a+3b}{4}\right)\right|^r}{7}\right)^{\frac{1}{r}}\right].$$

4. Numerical Examples

To prove our theory, we now provide some examples of our main results.

Example 4.1. *Let $f : [a,b] = [0,1] \to \mathbb{R}$ be a function defined by $f(x) = x^3$. Then f is q- differentiable. Moreover, for $q = \frac{3}{4}$*

$$\left|{}^{b}D_q f(x)\right| = \left|{}^{b}D_{\frac{3}{4}} f(x)\right| = \frac{37}{16}x^2 + \frac{5}{8}bx + \frac{b^2}{16}$$

is a convex on $[0,1]$. By applying Theorem 3.2 to the function $f(x) = x^3$, we have

$$\frac{1}{3}\left[2f\left(\frac{3a+b}{4}\right) - f\left(\frac{a+b}{2}\right) + 2f\left(\frac{a+3b}{4}\right)\right] = 0.25.$$

and

$$\frac{1}{(b-a)}\left(\int_a^{\frac{3a+b}{4}} f(x)\ {}^{\frac{3a+b}{4}}d_q x + \int_{\frac{3a+b}{4}}^{\frac{a+b}{2}} f(x)\ {}^{\frac{a+b}{2}}d_q x + \int_{\frac{a+b}{2}}^{\frac{a+3b}{4}} f(x)\ {}^{\frac{a+3b}{4}}d_q x + \int_{\frac{a+3b}{4}}^{b} f(x)\ {}^{b}d_q x\right) = 0.2346.$$

Thus, the left-hand side of (17) is

$$|0.2346 - 0.25| = 0.0153.$$

Now, we consider

$$\begin{aligned}
\left|{}^{\frac{3a+b}{4}}D_q f(a)\right| &= 0.0039,\ \left|{}^{\frac{3a+b}{4}}D_q f\left(\frac{3a+b}{4}\right)\right| = 0.1875,\ \left|{}^{\frac{a+b}{2}}D_q f\left(\frac{3a+b}{4}\right)\right| = 0.2383,\\
\left|{}^{\frac{a+b}{2}}D_q f\left(\frac{a+b}{2}\right)\right| &= 0.75,\ \left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+b}{2}\right)\right| = 0.8476,\ \left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+3b}{4}\right)\right| = 1.6875,\\
\left|{}^{b}D_q f\left(\frac{a+3b}{4}\right)\right| &= 1.8320,\ \left|{}^{b}D_q f(b)\right| = 3,\ E_1\left(\frac{3}{4}\right) = 0.2471,\ E_2\left(\frac{3}{4}\right) = 0.3243,\\
E_3\left(\frac{3}{4}\right) &= 0.7053,\ E_4\left(\frac{3}{4}\right) = 0.3900,\ E_5\left(\frac{3}{4}\right) = 0.6281,\ E_6\left(\frac{3}{4}\right) = 0.6100,\ E_7\left(\frac{3}{4}\right) = 0.3243,
\end{aligned}$$

and

$$E_8\left(\frac{3}{4}\right) = 0.1042.$$

Hence, the right-hand side of (17) is

$$\begin{aligned}
&\frac{b-a}{16}\left[(E_1)\left|{}^{\frac{3a+b}{4}}D_q f(a)\right| + (E_2)\left|{}^{\frac{3a+b}{4}}D_q f\left(\frac{3a+b}{4}\right)\right| + (E_3)\left|{}^{\frac{a+b}{2}}D_q f\left(\frac{3a+b}{4}\right)\right|\right.\\
&+ (E_4)\left|{}^{\frac{a+b}{2}}D_q f\left(\frac{a+b}{2}\right)\right| + (E_5)\left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+b}{2}\right)\right| + (E_6)\left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+3b}{4}\right)\right|\\
&\left.+ (E_7)\left|{}^{b}D_q f\left(\frac{a+3b}{4}\right)\right| + (E_8)\left|{}^{b}D_q f(b)\right|\right] = 0.1869.
\end{aligned}$$

It is clear that

$$0.0153 \leq 0.1869.$$

This demonstrates the result described in Theorem 3.2.

Example 4.2. *Let $f : [a,b] = [0,1] \to \mathbb{R}$ be a function defined by $f(x) = x^3$ and $p = r = 2$. Then f is q- differentiable. Moreover, for $q = \frac{3}{4}$*

$$\left|{}^{b}D_q f(x)\right|^2 = \left|{}^{b}D_{\frac{3}{4}} f(x)\right|^2 = \left(\frac{37}{16}x^2 + \frac{5}{8}bx + \frac{b^2}{16}\right)^2$$

is a convex on $[0,1]$. By applying Theorem 3.4, we have the left-hand side of (20) is 0.0153. Since

$$\begin{aligned}
\left|{}^{\frac{3a+b}{4}}D_q f(a)\right|^2 &= 0.00001521,\ \left|{}^{\frac{3a+b}{4}}D_q f\left(\frac{3a+b}{4}\right)\right|^2 = 0.0352,\\
\left|{}^{\frac{a+b}{2}}D_q f\left(\frac{3a+b}{4}\right)\right|^2 &= 0.0568,\ \left|{}^{\frac{a+b}{2}}D_q f\left(\frac{a+b}{2}\right)\right|^2 = 0.5625,\ \left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+b}{2}\right)\right|^2 = 0.7185,\\
\left|{}^{\frac{a+3b}{4}}D_q f\left(\frac{a+3b}{4}\right)\right|^2 &= 2.8477,\ \left|{}^{b}D_q f\left(\frac{a+3b}{4}\right)\right|^2 = 3.3563,\ \left|{}^{b}D_q f(b)\right|^2 = 9,\\
\int_0^1 (1-qt)^p\, d_qt &= 0.3861,\ \int_0^1 \left|qt + \frac{2}{3}\right|^p d_qt = 1.2591,\ \int_0^1 \left|\frac{5}{3} - qt\right|^p d_qt = 1.5925,
\end{aligned}$$

and

$$\int_0^1 (qt)^p\, d_qt = 0.2432.$$

The right-hand side of (20) is

$$\frac{b-a}{16}\left[\left(\int_0^1 (1-qt)^p\, d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r + q\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r}{[2]_q}\right)^{\frac{1}{r}}\right.$$

$$+\left(\int_0^1 \left|qt+\frac{2}{3}\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r + q\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r}{[2]_q}\right)^{\frac{1}{r}}$$

$$+\left(\int_0^1 \left|\frac{5}{3}-qt\right|^p d_qt\right)^{\frac{1}{p}}\left(\frac{\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^r + q\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^r}{[2]_q}\right)^{\frac{1}{r}}$$

$$\left.+\left(\frac{q^p}{[p+1]_q}\right)^{\frac{1}{p}}\left(\frac{\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|^r + q\left|{}^{b}D_qf(b)\right|^r}{[2]_q}\right)^{\frac{1}{r}}\right] = 0.2449.$$

It is clear that

$$0.0153 \leq 0.2449.$$

This demonstrates the result described in Theorem 3.4.

Example 4.3. *Let $f:[a,b]=[0,1]\to\mathbb{R}$ be a function defined by $f(x)=x^3$ and $p=r=2$. Then f is q- differentiable. Moreover, for $q=\frac{3}{4}$*

$$\left|{}^{b}D_qf(x)\right|^2 = \left|{}^{b}D_{\frac{3}{4}}f(x)\right|^2 = \left(\frac{37}{16}x^2+\frac{5}{8}bx+\frac{b^2}{16}\right)^2$$

is a convex on $[0,1]$. By applying Theorem 3.6, we have the left-hand side of (27) is 0.0153. Since

$$\begin{aligned}
\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^2 &= 0.00001521,\ \left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^2 = 0.0351,\ \left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^2 = 0.0567,\\
\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^2 &= 0.5625,\ \left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^2 = 0.7184,\ \left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^2 = 2.8476,\\
\left|{}^{b}D_qf\left(\frac{a+3b}{4}\right)\right|^2 &= 3.3562,\ \left|{}^{b}D_qf(b)\right|^2 = 9,
\end{aligned}$$

the right-hand side of (27) is

$$\frac{b-a}{16}\left[\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{\left|{}^{\frac{3a+b}{4}}D_qf(a)\right|^r + q(1+q)\left|{}^{\frac{3a+b}{4}}D_qf\left(\frac{3a+b}{4}\right)\right|^r}{[2]_q\,[3]_q}\right)^{\frac{1}{r}}\right.$$

$$+\left(\frac{2+5q}{3\,[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(2+5q+5q^2\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{3a+b}{4}\right)\right|^r + q\left(2+2q+5q^2\right)\left|{}^{\frac{a+b}{2}}D_qf\left(\frac{a+b}{2}\right)\right|^r}{3\,[2]_q\,[3]_q}\right)^{\frac{1}{r}}$$

$$+\left(\frac{5+2q}{3\,[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{\left(5+2q+2q^2\right)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+b}{2}\right)\right|^r + q\left(5+5q+2q^2\right)\left|{}^{\frac{a+3b}{4}}D_qf\left(\frac{a+3b}{4}\right)\right|^r}{3\,[2]_q\,[3]_q}\right)^{\frac{1}{r}}$$

q	Left Term	Right Term
0.2	0.0703	0.1613
0.4	0.0450	0.1724
0.6	0.0265	0.1812
0.8	0.0120	0.1888

Table 1: Comparative analysis between the left and right terms for discretization of ″q″ in Theorem 3.2.

q	Left Term	Right Term
0.2	0.0703	0.2749
0.4	0.0450	0.2632
0.6	0.0265	0.2527
0.8	0.0120	0.2423

Table 2: Comparative analysis between the left and right terms for discretization of ″q″ in Theorem 3.4.

$$+\left(\frac{q}{[2]_q}\right)^{1-\frac{1}{r}}\left(\frac{q\left(1+q\right)\left|{}^{b}D_q f\left(\frac{a+3b}{4}\right)\right|^r + q^3\left|{}^{b}D_q f\left(b\right)\right|^r}{[2]_q\,[3]_q}\right)^{\frac{1}{r}}\Bigg] = 0.2084.$$

It is clear that

$$0.0153 \leq 0.2084.$$

This demonstrates the result described in Theorem 3.6.

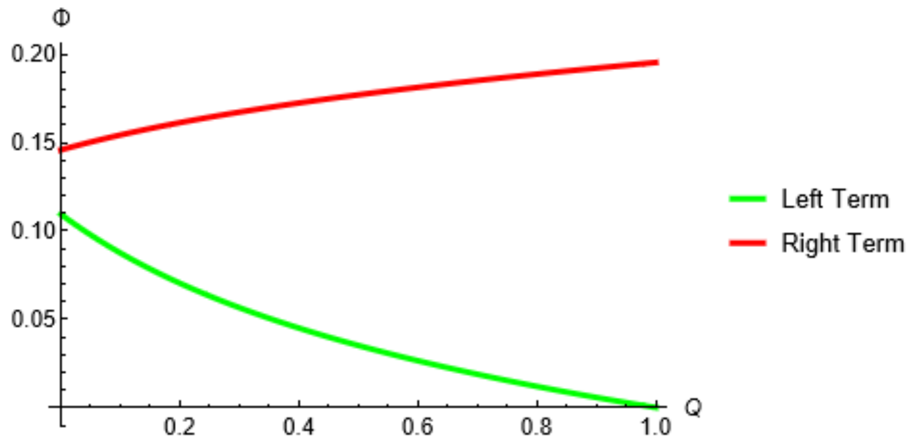

Figure 1: Depicts the comparative analysis between the left and right terms for discretization of ″q″ in Theorem 3.2.

Remark 4.4. *When* $q = \frac{3}{4}$ *in Theorem 3.2, by Example 4.1, then we have* $0.0153 \leq 0.1869$ *which is more efficient bound as compared to existing ones.*

Remark 4.5. *When* $q = \frac{3}{4}$ *in Theorem 3.4, by Example 4.2, then we have* $0.0153 \leq 0.2449$ *which is more efficient bound as compared to existing ones.*

Remark 4.6. *When* $q = \frac{3}{4}$ *in Theorem 3.6, by Example 4.3, then we have* $0.0153 \leq 0.2084$ *which is more efficient bound as compared to existing ones.*

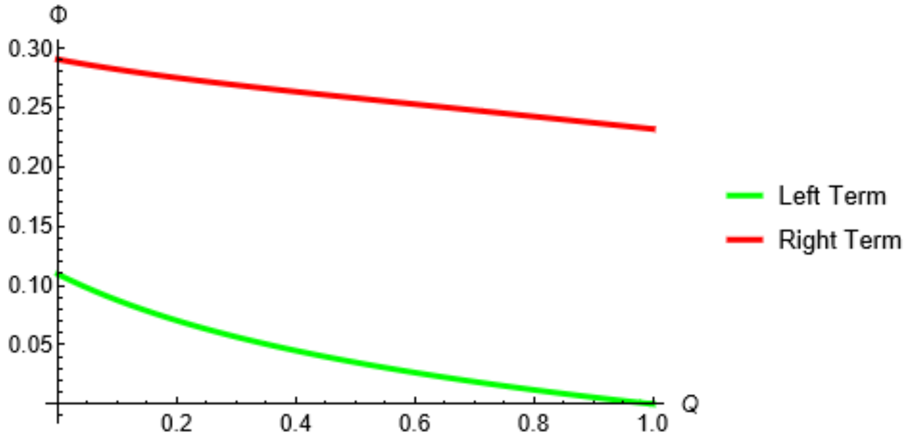

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Figure 2: Depicts the comparative analysis between the left and right terms for discretization of ″q″ in Theorem 3.4.

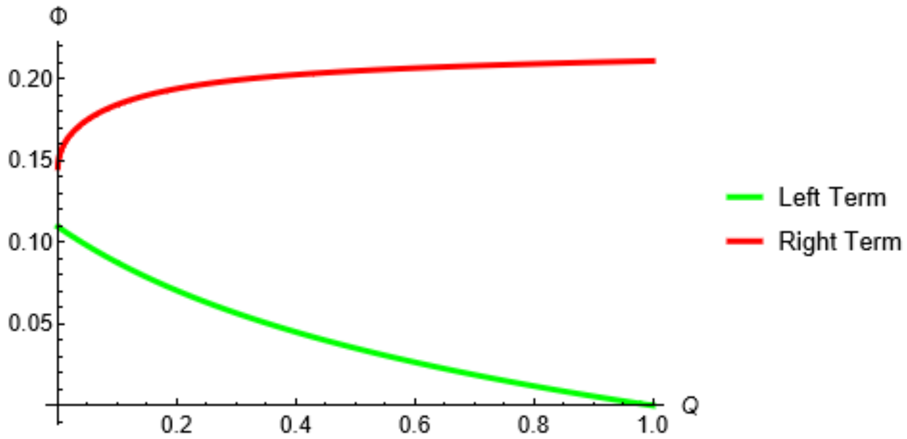

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Figure 3: Depicts the comparative analysis between the left and right terms for discretization of ″q″ in Theorem 3.6.

q	Left Term	Right Term
0.2	0.0703	0.1939
0.4	0.0450	0.2024
0.6	0.0265	0.2065
0.8	0.0120	0.2092

Table 3: Comparative analysis between the left and right terms for discretization of ″q″ in Theorem 3.6.

5. Conclusion

In this work, we developed some new inequalities to obtain error bounds for Milne's rule in the frameworks of classical and q-calculus, by using q-integrals. Moreover, we give the computational analysis of new Milne's rule type inequalities for convex functions. This work also shown that the results presented here are better than existing ones. The method used in this work to prove these inequalities is quite simple and less conditional as compared to existing results. It is an interesting and innovative problem that future researchers may get similar inequalities for convex and coordinate convex functions via different q-integrals operators.

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