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Approximation properties of a new type of Gamma operator by two parameter Gamma function

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Abstract. In this work, with the aid of the two parameter Gamma function, we define a new type of the Gamma operator. We have proved Voronovskaya type theorem and rate of convergence. We establish uniform convergence of a sequence of the new type Gamma operator via power series method. The power series method is also used to analyze the rate of convergence of operators. Finally, numerical examples are given to show the approximate properties of the new type Gamma operator.

1. Introduction

The use of positive linear operators to approximate functions is one of the fundamental topics of approximation theory. This theory plays a central role in many fields of mathematics, such as measure theory, harmonic analysis, functional analysis, partial differential equations and probability theory. The Gamma operator is one of the most used operators within approximation theory. The sequence of Gamma operator defined by Lupas and Müller [12] is the following manner:

$$G_n(g;x) = \frac{x^{n+1}}{\Gamma(n+1)} \int_0^\infty e^{-xv} v^n g\left(\frac{n}{v}\right) dv, \ n \in \mathbb{N}$$

for all $x \in (0,\infty)$. The function g for which the integral is absolutely convergent and for all $x \in (0,\infty)$. A lot of researchers studied Gamma-type operators in the literature see e.g. [1, 4, 6, 10, 13, 15]. When assessing Feynman integrals, one of the studies on Gamma operators is the k-Gamma function defined by Diaz and Pariguan [5]. k-Gamma function is described by

$$\Gamma_k(x) = \int_0^\infty e^{\frac{-t^k}{k}} t^{x-1} dt$$

for $k \in \mathbb{C}$, $Re(x) > 0$. By using k-Gamma function, in 2022, İçöz and Demir [8] defined the following new Gamma operator, for all $x \in (0,\infty)$, $k > 0$, $n \in \mathbb{N}$,

$$\tau_n(g;x) = \frac{x^{n+1+1/k}}{\Gamma_k(nk+k+1)} \int_0^\infty e^{-xv} (vk)^{n+1/k} g\left(\frac{n}{v}\right) dv.$$

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In 2017, Gehlot defined two parameter Gamma function as follows:

$$_p\Gamma_k(x) = \int_0^\infty e^{\frac{-t^k}{p}} t^{x-1} dt,$$

$k, p \in \mathbb{R}^+$ and $Re(x) > 0$. Some properties of the classical Gamma function and k-Gamma function can be generalized to the two parameter Gamma function such that $_p\Gamma_k(x) \Rightarrow_k \Gamma_k(x) = \Gamma_k(x)$ as $p = k$ and $_p\Gamma_k(x) \Rightarrow 1, \ \Gamma_1(x) = \Gamma(x)$ as $p, k \to 1$. The properties of fundamental satisfied by the two parameter Gamma function, $_p\Gamma_k(x)$ are

$$\begin{aligned} _p\Gamma_k(1) &= \frac{p^{\frac{1}{k}}}{k}\Gamma\left(\frac{1}{k}\right), \\ _p\Gamma_k(k) &= \frac{p}{k}, \\ _p(x)_{p,k} &= \frac{_p\Gamma_k(x+pk)}{_p\Gamma_k(x)}, \\ _p\Gamma_k(x+k) &= \frac{xp}{k}\,_p\Gamma_k(x), \ p \in \mathbb{N}, \end{aligned} \tag{1}$$

see for detail [9].

In the remainder of this paper is structured as follows. The new type Gamma operators is presented, using two parameter Gamma function and this operator is satisfied conditions of Korovkin theorem in Section 2. The modified Gamma operator is shown Voronovskaya type theorem and rate of convergence in Section 3. In Section 4, with help of the power series method, we prove approximation properties and the rate of convergence of the new type Gamma operator. In addition, numerical results related to Gamma operators are given.

2. A New Type of Gamma Operators

In this part, we shall introduce a new type of Gamma operators and some findings that we shall get. Throughout this work, we get the statement $a_z(h) = h^z$ and $\phi_z(h, x) = (h - x)^z$ for $x \in (0, \infty)$ as polynomial functions. This changed version of the classical Gamma operator, we define by

$$\tau_n^*(g; x) = \frac{(xp)^{n+1+\frac{p}{k}}}{k_p\Gamma_k(nk+k+p)} \int_0^\infty e^{-xv} v^{n+\frac{p}{k}} g\left(\frac{n+p}{v}\right) dv, \tag{2}$$

where for all $x \in (0, \infty)$, $k, p \in \mathbb{R}^+$, $n \in \mathbb{N}$, and $v > 0$, $g \in C_\gamma(0, \infty) = \{g \in C(0, \infty) : g(u) = O(u^\gamma), \text{ as } u \to \infty\}$ for $n > \gamma$, where $C(0, \infty)$ is the set of continuous functions on $(0, \infty)$.

Next, we will give the following lemma and will use in the main theorem.

Definition 2.1. $C^*(0, \infty) := \left\{g \in C(0, \infty) : \lim\limits_{x\to\infty} g(x) \textit{ is finite and exists}\right\}$.

In this paper, we take the following norm

$$\|g\| = \sup\{|g(x)| : x \in (0, \infty)\} \text{ for } g \in C(0, \infty).$$

Lemma 2.2. *With $x \in (0,\infty)$, the following moment values are:*

$$\begin{aligned}
\tau_n^*(a_0(h);x) &= a_0(x),\\
\tau_n^*(a_1(h);x) &= \frac{(n+p)k}{(nk+p)}a_1(x),\\
\tau_n^*(a_2(h);x) &= \frac{[(n+p)k]^2}{(nk+p)(nk+p-k)}a_2(x),\\
\tau_n^*(a_3(h);x) &= \frac{[(n+p)k]^3}{(nk+p)(nk+p-k)(nk+p-2k)}a_3(x),\\
\tau_n^*(a_4(h);x) &= \frac{[(n+p)k]^4}{(nk+p)(nk+p-k)(nk+p-2k)(nk+p-3k)}a_4(x).
\end{aligned}$$

In this case, this modified operator is clearly linear and positive. When the above moment values are generalized, the following lemma is obtained.

Lemma 2.3. *For $x \in (0,\infty)$ and $z \in \mathbb{N}$, $\tau_n^*(a_0(h);x) = a_0(x)$, we have*

$$\tau_n^*(a_z(h);x) = \frac{((n+p)k)^z}{\prod_{i=0}^{z-1}(nk+p-ki)}a_z(x), \quad z = 1,2,\dots .$$

Lemma 2.4. *For any $x \in (0,\infty)$, by Lemma 2.2, we get*

$$\begin{aligned}
\tau_n^*(\phi_0(x,h);x) &= 1\\
\tau_n^*(\phi_1(x,h);x) &= \frac{(k-1)p}{nk+p}a_1(x)\\
\tau_n^*(\phi_2(x,h);x) &= \frac{(n+p^2+2p)k^2-(2p^2+p)k+p^2}{(nk+p)(nk+p-k)}a_2(x)\\
\tau_n^*(\phi_3(x,h);x) &= \left(\frac{(n+p)^3k^3}{(nk+p)(nk+p-k)(nk+p-2k)} - \frac{3(n+p)^2k^2}{(nk+p)(nk+p-k)} + \frac{3(n+p)k}{nk+p} - 1\right)a_3(x)\\
\tau_n^*(\phi_4(x,h);x) &= \left(\frac{(n+p)^4k^4}{(nk+p)(nk+p-k)(nk+p-2k)(nk+p-3k)} - 4\frac{(n+p)^3k^3}{(nk+p)(nk+p-k)(nk+p-2k)}\right.\\
&\quad \left. + 6\frac{(n+p)^2k^2}{(nk+p)(nk+p-k)} - 4\frac{(n+p)k}{nk+p} + 1\right)a_4(x).
\end{aligned}$$

From definition of $\tau_n^*(g;x)$, the following lemma is obtained.

Lemma 2.5. *Let $h \in C_B(0,\infty)$. Therefore, we get*

$$\|\tau_n^*(g)\| \le \|g\|.$$

Proof. From Lemma 2.2, we can obtain

$$\begin{aligned}
\|\tau_n^*(g)\| &\le \frac{(yp)^{n+1+\frac{p}{k}}}{{}_p\Gamma_k(nk+k+p)}\int_0^\infty e^{-xv}(vk)^{n+\frac{p}{k}}\left|g\left(\frac{n+p}{v}\right)\right|dv\\
&\le \|g\|\frac{(yp)^{n+1+\frac{p}{k}}}{{}_p\Gamma_k(nk+k+p)}\int_0^\infty e^{-xv}(vk)^{n+\frac{p}{k}}\,dv
\end{aligned}$$

$$= \|g\|\tau_n^*(a_0(g);x) = \|g\|.$$

The proof is done. □

Theorem 2.6. *Let $g \in C^*(0,\infty)$. In uniformly each compact subset of $(0,\infty)$, we have*

$$\lim_{n\to\infty} \tau_n^*(g;x) = g(x).$$

Proof. From Lemma 2.2 and for $z = 0, 1, 2$, we get

$$\lim_{n\to\infty} \tau_n^*(a_z(h);x) = a_z(x)$$

for uniformly each compact subset of $(0,\infty)$. Therefore, by Korovkin theorem in [2], we conclude that $\lim\limits_{n\to\infty} \tau_n^*(g;x) = g(x)$. □

3. Rate of convergence of a New Type Gamma Operator

In this part, we shall offer the approximation properties of $\tau_n^*(g;x)$. Next, we shall give Voronovskaya type theorem for $(\tau_n^*)_{n\geq 1}$.

Theorem 3.1. *Let $g \in C^*(0,\infty)$ such that $g^{'}, g^{''} \in C^*(0,\infty)$. The following limit is valid*

$$\lim_{n\to\infty} n[\tau_n^*(g;x) - g(x)] = \left(\frac{pk-1}{k}\right) x g^{'}(x) + \frac{1}{2}x^2 g^{''}(x).$$

Proof. Using Taylor's formula for the function g

$$g(h) = g(x) + g^{'}(x)(h-x) + \frac{g^{''}(x)(h-x)^2}{2} + \varphi(h,x)(h-x)^2, \tag{3}$$

where $\varphi(h,x) = \frac{g^{''}(\delta)-g^{''}(x)}{2}$, δ between x and h and, $\lim\limits_{h\to x} \varphi(h,x) = 0$. When $(\tau_n^*)_{n\geq 1}$ is applied to (3), it results in that

$$\tau_n^*(g;x) = g(x) + g^{'}(x)\tau_n^*((h-x);x) + \frac{g^{''}(x)}{2}\tau_n^*((h-x)^2;x) + \tau_n^*(\varphi(h,x)(h-x)^2;x).$$

By multiplying both sides of the last inequality by n, we obtain the following formula

$$n[\tau_n^*(g;x) - g(x)] = g^{'}(x) n\tau_n^*((h-x);x) + \frac{g^{''}(x)}{2} n\tau_n^*((h-x)^2;x) + n\tau_n^*(\varphi(h,x)(h-x)^2;x).$$

In the limit case, this equation is

$$\lim_{n\to\infty} n[\tau_n^*(g;x) - g(x)] = g^{'}(x)\lim_{n\to\infty} n\tau_n^*((h-x);x) + \frac{g^{''}(x)}{2}\lim_{n\to\infty} n\tau_n^*((h-x)^2;x) + \lim_{n\to\infty} n\tau_n^*(\varphi(h,x)(h-x)^2;x).$$

It is known that the values are

$$\lim_{n\to\infty} n\tau_n^*((h-x);x) = \lim_{n\to\infty} n\left[\frac{p(k-1)}{nk+p}\right]x = \frac{p(k-1)}{k}x,$$

and

$$\lim_{n\to\infty} n\tau_n^*((h-x)^2;x) = \lim_{n\to\infty} n\left[\frac{(n+p^2+2p)k^2-(2p^2+p)k+p^2}{(nk+p)(nk+p-k)}\right]x^2 = x^2$$

from Lemma 2.4. So, we have

$$\lim_{n\to\infty} n[\tau_n^*(g;x) - g(x)] = \left(\frac{p(k-1)}{k}\right) x g^{'}(x) + \frac{g^{''}(x)}{2} x^2 + \lim_{n\to\infty} n\tau_n^*(\varphi(h,x)\phi_{x,2}(h);x). \tag{4}$$

We get, by the Cauchy-Schwarz inequality,

$$n\tau_n^*(\varphi(h,x)\phi_{x,2}(h);x)) \le \sqrt{\tau_n^*(\varphi^2(h,x);x)}\sqrt{n^2\tau_n^*(\phi_{x,4}(h);x)}. \tag{5}$$

Then, by Theorem 2.6, we can deduce that

$$\lim_{n\to\infty} \tau_n^*(\varphi^2(h,x);x) = \varphi^2(x,x) = 0 \tag{6}$$

since $\varphi^2(x,x) = 0$, $\varphi(\cdot,x) \in C^*(0,\infty)$ and in view of the fact that $\tau_n^*(\phi_{x,4}(h);x) = O(n^{-2})$. When equations (5) and (6) are writen in equation (4), the proof is done. □

Next, we obtain the rates of convergence for $(\tau_n^*)_{n\ge1}$. The modulus of continuity of g indicated by $\omega_{x_0}(g,\delta)$, for interval $(0,x_0]$, $x_0 \ge 0$, is defined as follows

$$\omega_{x_0}(g,\delta) = \sup_{\substack{|h-x|\le\delta \\ x,h\in(0,x_0]}} |g(h) - g(x)|$$

It is clear that the modulus of continuity $\omega_{x_0}(g,\delta) \to 0$ as $\delta \to 0$ for $g \in C_B(0,\infty)$. $C_B(0,\infty)$ denotes the space of all continous functions bounded on $(0,\infty)$.

Theorem 3.2. *Let $x_0 > 0$ and $g \in C_B(0,\infty)$. The following expression exists for the modulus of continuity $\omega_{x_0+1}(g,\delta)$ in the finite interval $(0,x_0+1] \subset (0,\infty)$.*

$$\begin{aligned} |\tau_n^*(g;x) - g(x)| \le{}& 3N_g\left(\frac{\left(n+p^2+2p\right)k^2 - \left(2p^2+p\right)+p^2}{(nk+p)(nk+p-k)} a_2(x)\right) x_0^2(1+x_0)^2 \\ &+ 2\omega_{x_0+1}\left(g, \sqrt{\left(\frac{(n+p^2+2p)k^2 - (2p^2+p)k + p^2}{(nk+p)(nk+p-k)}\right) x_0^2}\right) \end{aligned} \tag{7}$$

where N_g is a constant associated with g.

Proof. Let $g \in C_B(0,\infty)$, $h > x_0 + 1$ and $0 < x \le x_0$. Therefore, we can conclude

$$\begin{aligned} \left|g(h) - g(x)\right| &\le |g(h)| + |g(x)| \\ &\le 3N_g(h-x)^2(1+x_0)^2 \end{aligned}$$

for $h - x > 1$. Also, the expression holds

$$\left|g(h) - g(x)\right| \le \omega_{x_0+1}(g,|h-x|) \le \omega_{x_0+1}(g,\delta)\left(1 + \frac{1}{\delta}|h-x|\right)$$

for $h \le x_0 + 1$. Consequently, from last inequality we deduce from

$$\left|g(h) - g(x)\right| \le 3N_g(h-x)^2(1+x_0)^2 + \omega_{x_0+1}(g,\delta)\left(1 + \frac{1}{\delta}|h-x|\right) \tag{8}$$

for $0 < x \le x_0$ and $0 < h < \infty$. Applying the operator to (8) $(\tau_n^*)_{n\ge1}$ and Cauchy-Schwarz inequality, we get

$$\left|\tau_n^*(g;x)-g(x)\right| \leq 3N_g\tau_n^*((h-x)^2;x)(1+x_0)^2+\omega_{x_0+1}(g,\delta)\left(1+\frac{1}{\delta}\sqrt{\tau_n^*((h-x)^2;x)}\right)$$
$$\leq 3N_g\left(\frac{(n+p^2+2p)k^2-(2p^2+p)k+p^2}{(nk+p)(nk+p-k)}\right)(1+x_0)^2+2\omega_{x_0+1}\left(g,\sqrt{\left(\frac{(n+p^2+2p)k^2-(2p^2+p)k+p^2}{(nk+p)(nk+p-k)}\right)x_0^2}\right),$$

by choosing $\delta=\sqrt{\left(\frac{(n+p^2+2p)k^2-(2p^2+p)k+p^2}{(nk+p)(nk+p-k)}\right)x_0^2}$. Then, the proof is done. □

Definition 3.3. *Let*

$$C_B^2(0,\infty)=\left\{g\in C_B(0,\infty):g',g''\in C_B(0,\infty)\right\}, \tag{9}$$

with the norm

$$\left\|g\right\|_{C_B{}^2(0,\infty)}=\left\|g\right\|_{C_B(0,\infty)}+\|g'\|_{C_B(0,\infty)}+\|g''\|_{C_B(0,\infty)} \tag{10}$$

also

$$\left\|g\right\|_{C_B(0,\infty)}=\sup_{x\in(0,\infty)}\left|g(x)\right| \tag{11}$$

in [7].

Theorem 3.4. *Let τ_n^* be the operator described in (2). Then, we have for any $g\in C_B^2(0,\infty)$,*

$$\left|\tau_n^*(g;x)-g(x)\right|\leq\frac{1}{2}\sqrt{\chi}(2+\sqrt{\chi})\left\|g\right\|_{C_B^2(0,\infty)},$$

where χ is $\tau_n^*(\phi_2(x,h);x)$ in Lemma 2.4.

Proof. Let $g\in C_B^2(0,\infty)$. Using formula of Taylor, we have the equality

$$g(h)=g(x)+g'(x)(h-x)+\frac{1}{2}g''(\xi)(h-x)^2,$$

where ξ between x and h, from which it follows:

$$\left|g(h)-g(x)\right|\leq N_1|h-x|+\frac{1}{2}N_2(h-x)^2$$

where

$$N_1=\sup_{x\in(0,\infty)}|g'(x)|=\|g'\|_{C_B(0,\infty)}\leq\|g\|_{C_B^2(0,\infty)},$$

$$N_2=\sup_{x\in(0,\infty)}|g''(x)|=\|g''\|_{C_B(0,\infty)}\leq\|g\|_{C_B^2(0,\infty)},$$

because of (10). So, we get

$$\left|g(h)-g(x)\right|\leq\left(|h-x|+\frac{1}{2}(h-x)^2\right)\|g\|_{C_B^2(0,\infty)}.$$

Since

$$\left|\tau_n^*(g;x)-g(x)\right|=\left|\tau_n^*(g(h)-g(x);x)\right|\leq\tau_n^*(|g(h)-g(x)|;x),$$

and $\tau_n^*(|h-x|;x)\leq\tau_n^*((h-x)^2;x)^{\frac{1}{2}}=\sqrt{\chi}$, we obtain

$$\left|\tau_n^*(g;x)-g(x)\right|\leq\left(\tau_n^*(|h-x|;x)+\frac{1}{2}\tau_n^*((h-x)^2;x)\right)\|g\|_{C_B^2(0,\infty)}$$
$$\leq\frac{1}{2}\sqrt{\chi}(2+\sqrt{\chi})\|g\|_{C_B^2;(0,\infty)}.$$

The desired result is achieved. □

4. Rate of convergence by Power Series Method

In this part, we will study the convergence via the power series method for the a new type Gamma operator $(\tau_n^*)_{n\geq 1}$.

Let (q_k) be a real sequence with $q_0 > 0$, $q_k \geq 0$ $(k \in \mathbb{N})$, and such that the corresponding power series $q(\xi) = \sum\limits_{k=0}^{\infty} q_k \xi^k$ has radius of convergence R with $0 < R \leq \infty$. If the limit

$$\lim_{\xi \to R^-} \frac{1}{q(\xi)} \sum_{k=0}^{\infty} x_k q_k \xi^k = L$$

exists then $x = (x_k)$ is convergent in the sense of power series method see [11, 14]. It is well-known that the power series method is regular if and only if for each $k \in \mathbb{N}$

$$\lim_{\xi \to R^-} \frac{q_k \xi^k}{q(\xi)} = 0, \tag{12}$$

holds see [3].

Theorem 4.1. *Let (τ_n^*) be a sequence of positive linear operator acting from $C^*(0,\infty)$ into itself such that*

$$\lim_{\xi \to R^-} \frac{1}{q(\xi)} \left\| \sum_{n=0}^{\infty} (\tau_n^*(e_i) - e_i)\, q_n (\cdot)^n \right\| = 0, \tag{13}$$

for every $i \in \{1,2,3\}$ where $e_i(x) = x^i$ for any $g \in C^*(0,\infty)$ and $x \in [1,c] \subset (0,\infty)$

$$\lim_{\xi \to R^-} \frac{1}{q(\xi)} \left\| \sum_{n=0}^{\infty} (\tau_n^*(g) - g)\, q_n (.)^n \right\| = 0. \tag{14}$$

Proof. It is clear that (14) follows the expression (13). Conversely, let $g \in C^*(0,\infty)$. Therefore, there exists a constant $L > 0$ such that $\left|g(\xi)\right| \leq L$ for all $\xi \in (0,\infty)$. So, it follows that

$$\left|g(\xi) - g(x)\right| \leq 2L. \tag{15}$$

Also, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\left|g(\xi) - g(x)\right| \leq \varepsilon, \tag{16}$$

whenever $|\xi - x| < \delta$ for all $\xi \in (0,\infty)$. Set $\psi(\xi,x) = (\xi - x)^2$. If $|\xi - x| \geq \delta$ therefore, we get

$$\left|g(\xi) - g(x)\right| \leq \frac{2K}{\delta^2}\psi(\xi,x). \tag{17}$$

Now, from the expressions (15) and (17), we can write

$$\left|g(\xi) - g(x)\right| \leq \varepsilon + \frac{2K}{\delta^2}\psi(\xi,x). \tag{18}$$

Since $\tau_n^*(1;x)$ is linear and monotone operator, we have

$$\tau_n^*(1;x)\left(-\varepsilon - \frac{2K}{\delta^2}\psi(\xi,x)\right) < \tau_n^*(1;x)\,(g(\xi) - g(x)) < \tau_n^*(1;x)\left(\varepsilon + \frac{2K}{\delta^2}\psi(\xi,x)\right)$$

which implies

$$-\varepsilon\tau_n^*(1;x) - \frac{2K}{\delta^2}\psi(\xi,x)\,\tau_n^*(1;x) < \tau_n^*(g;x) - g(x)\,\tau_n^*(1;x) < \varepsilon\tau_n^*(1;x) + \frac{2K}{\delta^2}\tau_n^*(\psi(\xi);x). \tag{19}$$

On the other hand, we can write

$$\tau_n^*(g;x)-g(x)=\tau_n^*(g;x)-g(x)\,\tau_n^*(1;x)+g(x)\left[\tau_n^*(1;x)-1\right]. \tag{20}$$

From the expressions (19) and (20), we get

$$\tau_n^*(g;x)-g(x)<\varepsilon\tau_n^*(1;x)+\frac{2K}{\delta^2}\tau_n^*(\psi(\xi);x)+g(x)\left[\tau_n^*(1;x)-1\right]. \tag{21}$$

Also, we get

$$\begin{aligned}\tau_n^*(\psi(\xi);x) &= \tau_n^*\left((\xi-x)^2;x\right)\\ &= \tau_n^*\left(\left(\xi^2-2\xi x+x^2\right);x\right)\\ &= \xi^2\tau_n^*(1;x)-2x\tau_n^*(\xi;x)+\tau_n^*\left(\xi^2;x\right).\end{aligned}$$

From (21) and the last equality, we attain

$$\begin{aligned}\tau_n^*(g;x)-g(x) &< \varepsilon\tau_n^*(1;x)+\frac{2K}{\delta^2}\left\{\xi^2\left[\tau_n^*(1;x)-1\right]-2x\left[\tau_n^*(\xi;x)-x\right]+\left[\tau_n^*\left(\xi^2;x\right)-\xi^2\right]\right\}+g(x)\left[\tau_n^*(1;x)-1\right]\\ &= \varepsilon+\varepsilon\left[\tau_n^*(1;x)-1\right]+g(x)\left[\tau_n^*(1;x)-1\right]+\frac{2K}{\delta^2}\left\{\xi^2\left[\tau_n^*(1;x)-1\right]-2x\left[\tau_n^*(\xi;x)-x\right]+\left[\tau_n^*\left(\xi^2;x\right)-\xi^2\right]\right\}.\end{aligned}$$

Therefore, by making some calculations we obtain

$$\left|\tau_n^*(g;x)-g(x)\right| \leq \varepsilon+\left(\varepsilon+K+\frac{2K}{\delta^2}\right)\left|\tau_n^*(1;x)-1\right|+\frac{4K}{\delta^2}\left|\tau_n^*(\xi;x)-x\right|+\frac{2K}{\delta^2}\left|\tau_n^*\left(\xi^2;x\right)-\xi^2\right|$$

and

$$\begin{aligned}\frac{1}{q(\xi)}\left\|\sum_{n=0}^{\infty}\left(\tau_n^*(g)-g\right)q_n(\cdot)^n\right\| &\leq \varepsilon+\left(\varepsilon+K+\frac{2K}{\delta^2}\right)\frac{1}{q(\xi)}\left\|\sum_{n=0}^{\infty}\left(\tau_n^*(e_0)-e_0\right)q_n(\cdot)^n\right\|+\frac{4K}{\delta^2}\frac{1}{q(\xi)}\left\|\sum_{n=0}^{\infty}\left(\tau_n^*(e_1)-e_1\right)q_n(\cdot)^n\right\|\\ &\quad+\frac{2K}{\delta^2}\frac{1}{q(\xi)}\left\|\sum_{n=0}^{\infty}\left(\tau_n^*(e_1)-e_1\right)q_n(\cdot)^n\right\|.\end{aligned}$$

From the last inequality and (13) we conclude the expression (14). □

Now, we can study the rate of convergence of the power series method for the new type Gamma Operator $(\tau_n^*)_{n\geq 1}$.

Theorem 4.2. *Let $\omega_{x_0+1}(g,\delta)$ be the modulus of continuity on the finite interval $(0,x_0+1]\subset(0,\infty)$ for $g\in C_B(0,\infty)$. Let ϕ be a positive real function described on $(0,R)$. If $\omega_{x_0+1}(g,\delta)=O(\phi)$, then we have*

$$\frac{1}{q(\xi)}\left\|\sum_{n=0}^{\infty}\left(\tau_n^*(e_i)-e_i\right)q_n(\cdot)^n\right\|=O(\phi),\ \text{as } \xi\to R^-$$

where $\psi:(0,R)\to\mathbb{R}$ is described as

$$\psi=\sup_x\left\{\tau_n^*\left(\left(\frac{k}{n}-x\right)^2\right)\right\}.$$

Proof. Let $g \in C_B(0,\infty)$, $0 < x \leq x_0$, and $t > y_0 + 1$. For any $\xi \in (0,R)$ and $\delta > 0$, we have the following

$$\begin{aligned}
\left|\sum_{n=0}^{\infty}\left(\tau_n^*(g;x) - g(x)\right) q_n(\cdot)^n\right| &\leq \sum_{n=0}^{\infty}\tau_n^*\left(\left|g(\xi) - g(x)\right|;x\right) q_n^n \xi^n \\
&\leq \sum_{n=0}^{\infty}\tau_n^*\left(\omega_{x_0+1}\left(g, \frac{\left|\frac{k}{n} - x\right|}{\delta}\delta\right);x\right) q_n \xi^n \\
&\leq \sum_{n=0}^{\infty}\tau_n^*\left(1 + \left[\left|\frac{\left|\frac{k}{n} - x\right|}{\delta}\right|\right]\omega_{x_0+1}(g,\delta);x\right) q_n \xi^n \\
&\leq \omega_{x_0+1}(g,\delta)\sum_{n=0}^{\infty}\tau_n^*\left(1 + \frac{\left(\frac{k}{n} - x\right)^2}{\delta^2};x\right) q_n \xi^n \\
&\leq \omega_{x_0+1}(g,\delta)\sum_{n=0}^{\infty}\tau_n^*\left(e_0(\xi);x\right) q_n \xi^n + \frac{\omega_{x_0+1}(g,\delta)}{\delta^2}\sum_{n=0}^{\infty}\tau_n^*\left(\frac{\left(\frac{k}{n} - y\right)^2}{\delta^2};x\right) q_n \xi^n \\
&= q(\xi)\,\omega_{x_0+1}(g,\delta) + \frac{\omega_{x_0+1}(g,\delta)}{\delta^2}\sup_x\left\{\tau_n^*\left(\left(\frac{k}{n} - x\right)^2\right)\right\}\sum_{n=0}^{\infty} q_n \xi^n \\
&= q(\xi)\,\omega_{x_0+1}(g,\delta) + \frac{\omega_{x_0+1}(g,\delta)}{\delta^2}\sup_x\left\{\tau_n^*\left(\left(\frac{k}{n} - x\right)^2\right)\right\} q(\xi).
\end{aligned}$$

By taking $\delta = \psi$ and from the last expression we conclude that

$$0 \leq \frac{1}{q(\xi)}\left\|\sum_{n=0}^{\infty}\left(\tau_n^*(g) - g\right) q_n(\cdot)^n\right\| \leq 2\omega_{x_0+1}(g,\delta).$$

Therefore, the proof is done. □

5. Numerical Examples

In this part, we shall establish some numerical examples to confirm the rates of convergence of $\tau_n^*(g;x)$ in two dimensions ($k = 1/3,\ p = 1/30$ is fixed for all of Figures). In our experiments, we compare a new type of the Gamma operator with by using k-Gamma function İçöz and Demir [8] defined a new Gamma operator.

In our first example, $\tau_n^*(g;x)$ and $\tau_n(g;x)$ are applied to the test functions $g : [0, 10] \to [0, \infty)$ such that $g(x) = x^2$.

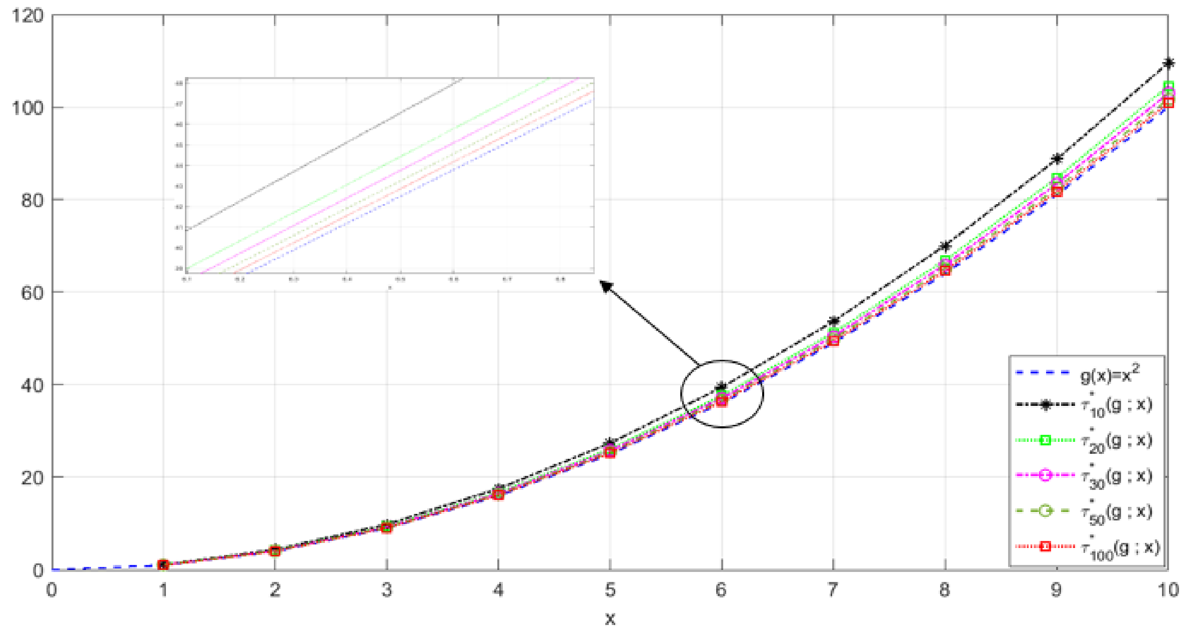
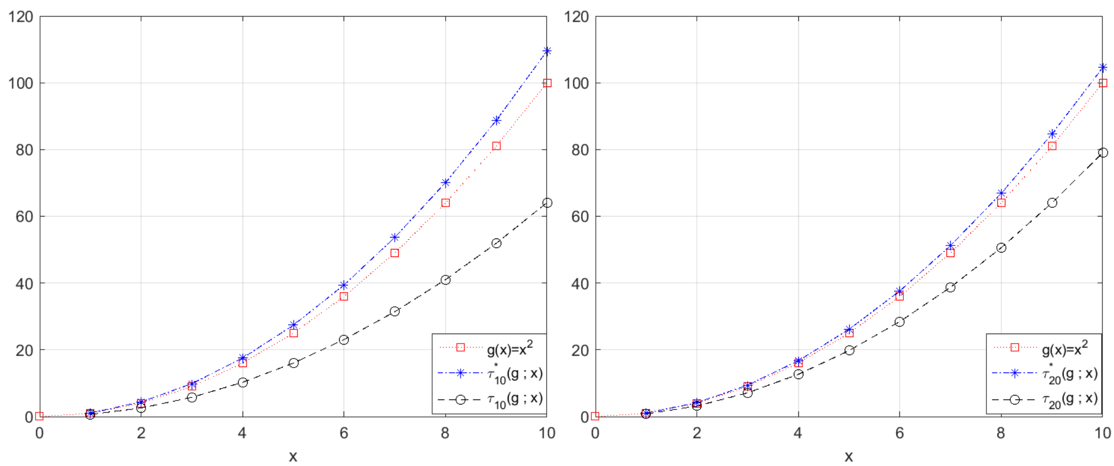


Figure 1: Comparasion of $\tau_n^*(g;x)$ with $g(x) = x^2$ (blue) for the value of $n = 10,\ 20,\ 30,\ 50$ and 100. $\tau_{10}^*(g;x)$ (black), $\tau_{20}^*(g;x)$ (light green), $\tau_{30}^*(g;x)$ (magenta), $\tau_{50}^*(g;x)$ (green), $\tau_{100}^*(g;x)$ (red).



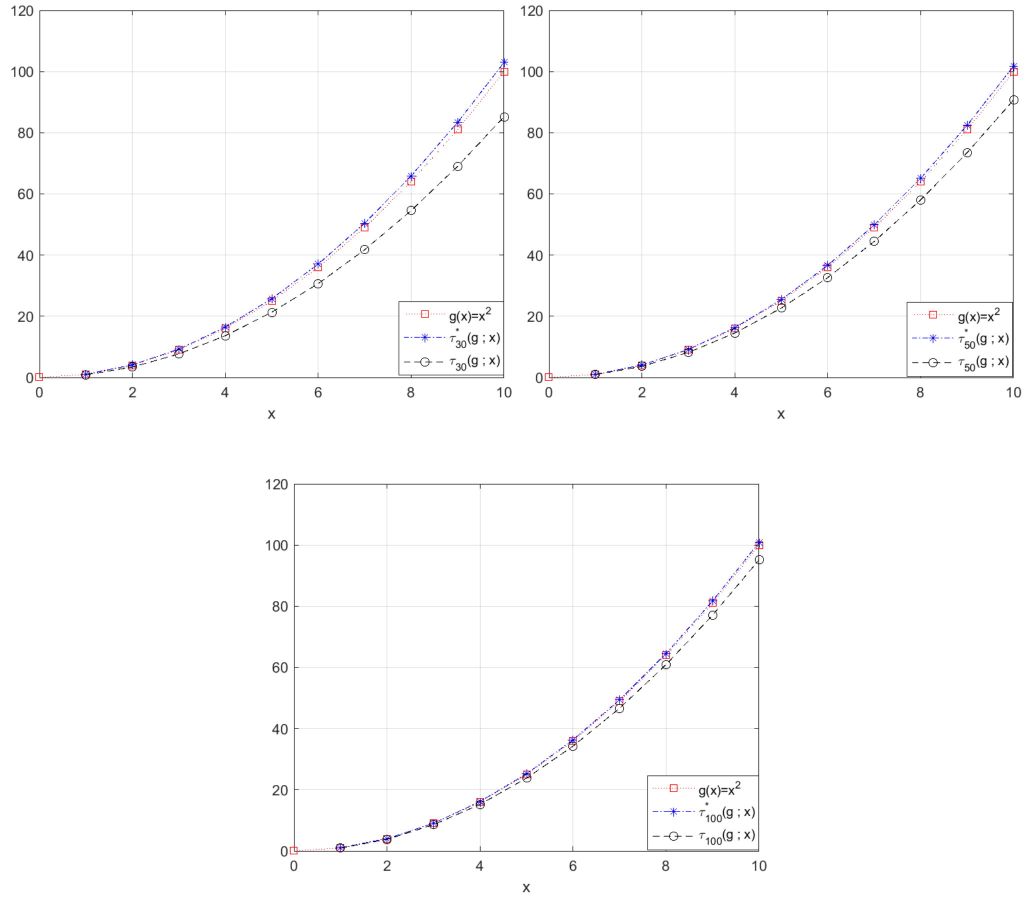


Figure 2: Comporasion of $\tau_n^*(g;x)$ and $\tau_n(g;x)$ with $g(x) = x^2$ for the value of $n = 10, 20, 30, 50, 100$; test function (red), $\tau_n^*(g;x)$ (blue), $\tau_n(g;x)$ (black) .

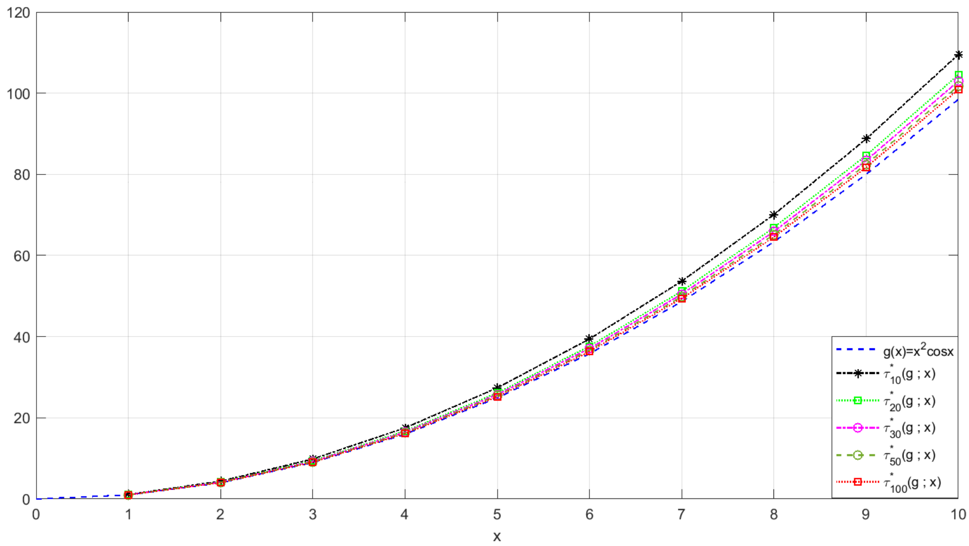


Figure 3: Comparasion of $\tau_n^*(g;x)$ with $g(x) = x^2cos(x)$ (blue) for the value of $n = 10, 20, 30, 50$ and 100. $\tau_{10}^*(g;x)$ (black), $\tau_{20}^*(g;x)$ (light green), $\tau_{30}^*(g;x)$ (magenta), $\tau_{50}^*(g;x)$ (green), $\tau_{100}^*(g;x)$ (red).

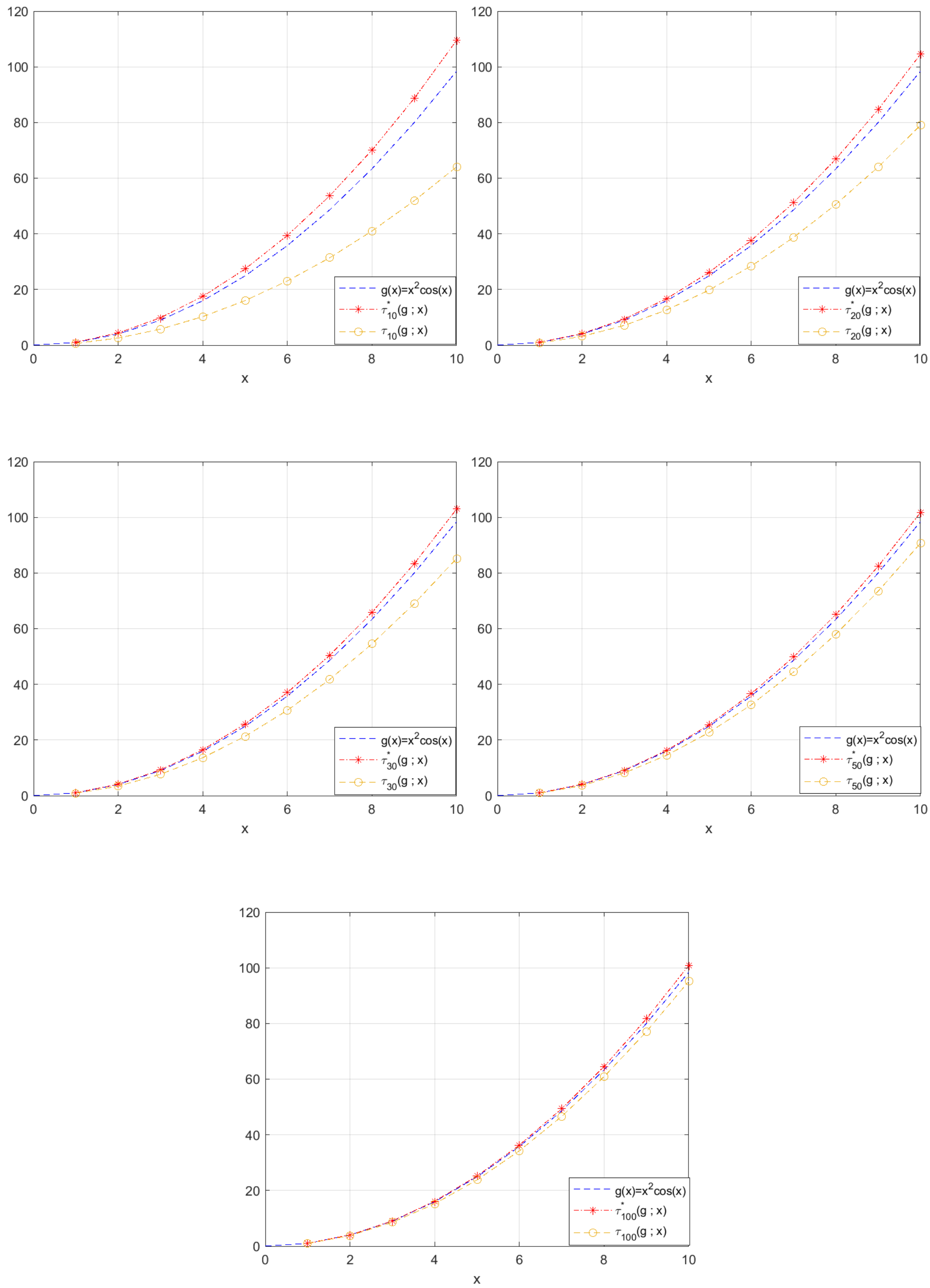


Figure 4: Comporasion of $\tau_n^*(g;x)$ and $\tau_n(g;x)$ with $g(x) = x^2cos(x)$ for the value of $n = 10,\ 20,\ 30,\ 50,\ 100$; test function (blue), $\tau_n^*(g;x)$ (red), $\tau_n(g;x)$ (yellow) .

In Figure 1 and Figure 3 show convergence plots for our operator for $n = 10,\ 20,\ 30,\ 50,\ 100$. It is clear that the observed increase in the convergence rate of approximations of the defined operators can be attributed to increasing n.

In Figure 2, we plot the calculation result of $\tau_n(g;x)$, $\tau_n^*(g;x)$ and $g(x) = x^2$. $\tau_n^*(g;x)$ shows better convergence behaviour than $\tau_n(g;x)$ related to the $g(x) = x^2$.

In Figure 4, we plot the calculation result of $\tau_n(g;x)$, $\tau_n^*(g;x)$ and $g(x) = x^2cos(x)$. It is obvious that $\tau_n^*(g;x)$ shows better convergence behaviour than $\tau_n(g;x)$ related to the $g(x) = x^2cos(x)$.

6. Conclusion

In this work, we described a new type of Gamma Operator by two parameters Gamma function. We present some approximation properties of this operator such as rate of convergence, Voronovksya type theorem. We also research, by using the power series method, Korovkin approximation theorems and give the rate of convergence of these operators. Lastly, we get the numerical examples to confirm its approach properties . All of numerical examples have been written in MATLAB. The numerical results show that the new type of Gamma operator exhibits good approximation behavior compared to the *k*-Gamma operator.

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