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Conditional Fourier–Feynman transform and conditional convolution product given infinite-dimensional vector-valued conditioning function

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Abstract. Let $C_0[0,T]$ denote Wiener space. Define an infinite-dimensional random vector $X_{\mathcal{E},\infty}: C_0[0,T] \to \mathbb{R}^\infty$ by $X_{\mathcal{E},\infty}(x) = (\langle e_1, x\rangle, \langle e_2, x\rangle, \ldots)$ where $\mathcal{E} = \{e_n\}_{n=1}^\infty$ is an orthonormal sequence in $L_2[0,T]$ and $\langle e, x\rangle$ denotes the Paley–Wiener–Zygmund (PWZ) stochastic integral. In this paper, we study a conditional Fourier–Feynman transform (CFFT) and a conditional convolution product (CCP) for scale-invariant measurable (SIM) functionals on $C_0[0,T]$ with the very general conditioning function $X_{\mathcal{E},\infty}$. In particular, we show that the CFFT of the CCP is a product of CFFTs.

1. Introduction

The concepts of the CFFT and the CCP for functionals on the Wiener space $C_0[0,T]$ were introduced by Park and Skoug in [20]. Many authors have provided various results between conditional transforms and conditional convolutions for functionals on $C_0[0,T]$. For instance, see [2, 4, 6, 10, 11, 14]. However the conditioning functions defining the conditional transforms and the conditional convolutions studied in [2, 4, 6, 10, 11, 14] are finite-dimensional vector-valued functions. In [19], Park and Skoug derived an evaluation formula for the conditional Wiener integral given an infinite-dimensional vector-valued conditioning function and established useful formulas to calculate their conditional Wiener integrals.

In this paper, using the evaluation formula [19] for the conditional Wiener integral given an infinite-dimensional conditioning function, we study a relationship between a CFFT and a CCP associated with infinite-dimensional conditioning functions on the Wiener space $C_0[0,T]$. The conditioning function $X_{\mathcal{E},\infty}: C_0[0,T] \to \mathbb{R}^\infty$ used in this paper is given by $X_{\mathcal{E},\infty}(x) = (\langle e_1, x\rangle, \langle e_2, x\rangle, \ldots)$ where $\mathcal{E} = \{e_n\}$ is an orthonormal sequence of functions in $L_2[0,T]$ and $\langle e, x\rangle$ denotes the PWZ stochastic integral $\int_0^T e(t)dx(t)$ [15, 16]. Both the CFFT and the CCP in this paper are defined in terms of a conditional analytic Feynman integral.

In Section 4, we establish the relationship under existence conditions of the CFFT and the CCP for functionals on $C_0[0,T]$. We then in Section 5 provide a class of bounded cylinder functionals whose CFFT and CCP exist. In Section 6, we confirm the relationship for the specific bounded cylinder functionals. In Section 7, we provide a concluding remark for the topic related with the result in this paper.

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2. Preliminaries

In Section 3 below, we introduce the concepts of the CFFT and the CCP for functionals on the complete Wiener measure space $(C_0[0,T], \mathcal{W}(C_0[0,T]), m_w)$, where $\mathcal{W}(C_0[0,T])$ denotes the σ-field of all Wiener measurable subsets. The definitions are based on the concept of the conditional Wiener integral associated with an infinite-dimensional vector-valued conditioning function.

2.1. Conditional Wiener integral

We denote the Wiener integral of a Wiener integrable functional F by

$$E[F] \equiv E_x[F(x)] = \int_{C_0[0,T]} F(x)dm_w(x),$$

and for $u \in L_2[0,T]$ and $x \in C_0[0,T]$, we let $\langle u, x\rangle = \int_0^T u(t)dx(t)$ denote the PWZ stochastic integral. It is known that

$$E_x[\langle u, x\rangle\langle v, x\rangle] = (u, v)_2 \tag{2.1}$$

where $(\cdot,\cdot)_2$ denotes the inner product on $L_2[0,T]$.

Let $\mathcal{H}$ be an infinite-dimensional subspace of $L_2[0,T]$ with a countable orthonormal basis $\mathcal{E} = \{e_n\}$. For notational conveniences, we let

$$\gamma_n(x) \equiv \langle e_n, x\rangle$$

and

$$\beta_n(t) = \int_0^t e_n(s)ds, \ \ t \in [0,T],$$

respectively, for each $n \in \mathbb{N}$.

Let F be a Wiener integrable functional. Then we have the conditional Wiener integral from a well-known probability theory: Let $\mathbb{V}$ be a real linear space with norm $|\cdot|$. Clearly, the normed space $\mathbb{V}$ is a topological vector space with respect to the uniform topology induced by $|\cdot|$. Let $\mathcal{B}(\mathbb{V})$ be the σ-field generated by the class of all open subsets of $\mathbb{V}$. Then $\mathcal{B}(\mathbb{V})$ is known as the Borel σ-field on $\mathbb{V}$. Let X be a $\mathbb{V}$-valued measurable function and Y a $\mathbb{C}$-valued integrable functional on $C_0[0,T]$. Let $\mathcal{F}(X)$ denote the σ-field generated by X. Then by the definition, the conditional expectation of Y given $\mathcal{F}(X)$, written $E(Y|X)$, is any $\mathcal{F}(X)$-measurable function on $C_0[0,T]$ such that

$$\int_A Y(x)dm_w(x) = \int_A E(Y|X)(x)dm_w(x) \quad \text{for} \ \ A \in \mathcal{F}(X).$$

It is well-known that there exists a Borel measurable and P_X-integrable function ψ on $(\mathbb{V}, \mathcal{B}(\mathbb{V}), P_X)$ such that $E(Y|X) = \psi \circ X$, where P_X is the probability distribution of X defined by $P_X(U) = m_w(X^{-1}(U))$ for $U \in \mathcal{B}(\mathbb{V})$. The function $\psi(\eta)$, $\eta \in \mathbb{V}$, is unique up to Borel null sets in $\mathbb{V}$. Following Tucker [21] and Yeh [22], the function $\psi(\eta)$, written $E(Y|X = \eta)$, is called the conditional Wiener integral of Y given X.

Let $X_{\mathcal{E},\infty} : C_0[0,T] \to \mathbb{R}^\infty$ be the function defined by

$$X_{\mathcal{E},\infty}(x) = (\gamma_1(x), \gamma_2(x), \ldots). \tag{2.2}$$

We note that the PWZ stochastic integrals $\gamma_n(x)$, $n \in \mathbb{N}$, form a set of independent standard Gaussian random variables on $C_0[0,T]$. Consider the projection map $\mathcal{P}_{\mathcal{H}} : L_2[0,T] \to \mathcal{H}$ given by

$$\mathcal{P}_{\mathcal{H}}h(t) = \sum_{n=1}^{\infty}(h, e_n)_2 e_n(t). \tag{2.3}$$

Then it follows that $\|\mathcal{P}_{\mathcal{H}}h\|_2 \lneq \|h\|_2$ if $\mathcal{H} = \text{Span}\{e_1, e_2, \ldots\} \subsetneq L_2[0,T]$. For $x \in C_0[0,T]$ and $\vec{\xi} = (\xi_1, \xi_2, \ldots) \in \mathbb{R}^\infty$, let

$$x_\infty(t) = \langle \mathcal{P}_{\mathcal{H}} I_{[0,t]}, x\rangle = \sum_{n=1}^{\infty} \gamma_n(x)(I_{[0,t]}, e_n)_2 = \sum_{n=1}^{\infty} \gamma_n(x)\beta_n(t)$$

and

$$\vec{\xi}_\infty(t) = \sum_{n=1}^{\infty} \xi_n (I_{[0,t]}, e_n)_2 = \sum_{n=1}^{\infty} \xi_n \beta_n(t)$$

where $I_{[0,t]}$ denotes the indicator function of the interval $[0,t]$.

In [19], Park and Skoug proved the facts that the process $\{x(t)-x_\infty(t), 0 \le t \le T\}$ and the Gaussian random variable $\gamma_n(x)$ are stochastically independent for each $n \in \mathbb{N}$, and that the processes $\{x(t) - x_\infty(t), 0 \le t \le T\}$ and $\{x_\infty(t), 0 \le t \le T\}$ are also stochastically independent. Using these basic results, Park and Skoug established the following evaluation formula to express conditional Wiener integrals in terms of ordinary Wiener integrals.

Theorem 2.1 ([19]). *Let $F \in L_1(C_0[0,T])$. Then it follows that*

$$E(F|X_{\mathcal{E},\infty} = \vec{\xi}) = E_x\Big[F\Big(x - x_\infty + \vec{\xi}_\infty\Big)\Big] = E_x\Big[F\Big(x - \sum_{n=1}^{\infty}\gamma_n(x)\beta_n + \sum_{n=1}^{\infty}\xi_n\beta_n\Big)\Big] \tag{2.4}$$

for a.e. $\vec{\xi} \in \mathbb{R}^\infty$.

2.2. Cylinder functionals

A functional F on $C_0[0,T]$ is called a cylinder functional, if the functional F is represented by

$$F(x) = f(\langle v_1, x\rangle, \ldots, \langle v_n, x\rangle), \tag{2.5}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a Lebesgue measurable function, $\{v_1, \ldots, v_n\}$ is a linearly independent set of functions in $L_2[0,T]$. The functional F given by (2.5) is Wiener measurable if and only if f is Lebesgue measurable [7].

In order to simplify many expressions in this paper, we use the following conventions: for $\vec{u} = (u_1, \ldots, u_n) \in \mathbb{R}^n$ and a set $\{g_1, \ldots, g_n\}$ of functions in $L_2[0,T]$, let

$$f(\vec{u}) \equiv f(u_1, \ldots, u_n), \quad \langle \vec{g}, x\rangle \equiv (\langle g_1, x\rangle, \ldots, \langle g_n, x\rangle), \quad \text{and} \quad f(\langle \vec{g}, x\rangle) \equiv f(\langle g_1, x\rangle, \ldots, \langle g_n, x\rangle).$$

Equation (2.6) below can be easily obtained by the change of variables theorem.

Lemma 2.2. *Let $\mathcal{G} = \{g_1, \ldots, g_m\}$ be an orthonormal set of functions in $L_2[0,T]$ and let $f : \mathbb{R}^m \to \mathbb{C}$ be a Lebesgue measurable function. Then*

$$E_x[f(\langle \vec{g}, x\rangle)] \overset{*}{=} (2\pi)^{-m/2} \int_{\mathbb{R}^m} f(\vec{u}) \exp\Big\{-\sum_{l=1}^{m} \frac{u_l^2}{2}\Big\} d\vec{u}, \tag{2.6}$$

where by $\overset{*}{=}$ we mean that if either side exists, both sides exist and equality holds.

The following integration formula also used in this paper:

$$\int_{\mathbb{R}} \exp\{-av^2 + bv\} dv = \sqrt{\frac{\pi}{a}} \exp\Big\{\frac{b^2}{4a}\Big\} \tag{2.7}$$

for $a, b \in \mathbb{C}$ with $\text{Re}(a) > 0$.

3. Definitions

In order to define a CFFT and a CCP, we need the concept of the scale-invariant measurability on Wiener space $C_0[0,T]$. A subset B of $C_0[0,T]$ is called an SIM set if $\rho B \in \mathcal{W}(C_0[0,T])$ for all $\rho > 0$, and an SIM set N is called a scale-invariant null set if $m_w(\rho N) = 0$ for all $\rho > 0$. A property which holds except on a scale-invariant null set is said to hold scale-invariant almost everywhere (SI-a.e.). A functional F is said to be SIM provided F is defined on an SIM set and $F(\rho\,\cdot)$ is $\mathcal{W}(C_0[0,T])$-measurable for every $\rho > 0$. For more detailed studies of the scale-invariant measurability, see [7, 12].

The definitions of the CFFT and the CCP are based on the conditional analytic Wiener integral [2, 6, 20]. In this paper, we shall use exclusively the conditioning function $X_{\mathcal{E},\infty}$ given by (2.2) to define our CFFT and CCP on $C_0[0,T]$.

Let $\mathbb{C}_+ = \{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$ and let $\widetilde{\mathbb{C}}_+ = \{\lambda \in \mathbb{C} \setminus \{0\} : \mathrm{Re}(\lambda) \geq 0\}$. Let $X_{\mathcal{E},\infty} : C_0[0,T] \to \mathbb{R}^\infty$ be given by (2.2) and let F be a $\mathbb{C}$-valued SIM functional such that the Wiener integral $E_x[F(\lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. For $\lambda > 0$ and $\vec{\xi}$ in $\mathbb{R}^\infty$, let

$$J_F(\lambda;\vec{\xi}) = E\Big(F(\lambda^{-1/2}\,\cdot)\Big|X_{\mathcal{E},\infty}(\lambda^{-1/2}\,\cdot) = \vec{\xi}\Big)$$

denote the conditional Wiener integral of $F(\lambda^{-1/2}\,\cdot)$ given $X_{\mathcal{E},\infty}(\lambda^{-1/2}\,\cdot)$. If for a.e. $\vec{\xi} \in \mathbb{R}^\infty$, there exists a function $J_F^*(\lambda;\vec{\xi})$, analytic in λ on $\mathbb{C}_+$ such that $J_F^*(\lambda;\vec{\xi}) = J_F(\lambda;\vec{\xi})$ for all $\lambda > 0$, then $J_F^*(\lambda;\,\cdot)$ is defined to be the conditional analytic Wiener integral of F given $X_{\mathcal{E},\infty}$ with parameter λ. For $\lambda \in \mathbb{C}_+$, we write

$$E^{\mathrm{an}w_\lambda}(F|X_{\mathcal{E},\infty} = \vec{\xi}) = J_F^*(\lambda;\vec{\xi}).$$

If for fixed real $q \in \mathbb{R} \setminus \{0\}$, the limit

$$\lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} E^{\mathrm{an}w_\lambda}(F|X_{\mathcal{E},\infty} = \vec{\xi})$$

exists for a.e. $\vec{\xi} \in \mathbb{R}^\infty$, then we denote the value of this limit by $E^{\mathrm{an}f_q}(F|X_{\mathcal{E},\infty} = \vec{\xi})$, and we call it the conditional analytic Feynman integral of F given $X_{\mathcal{E},\infty}$ with parameter q on $C_0[0,T]$.

Let F be a $\mathbb{C}$-valued SIM functional on $C_0[0,T]$ such that the Wiener integral $E[F(y + \lambda^{-1/2}\,\cdot)] \equiv E_x[F(y + \lambda^{-1/2}x)]$ exists as a finite number for all $\lambda > 0$. Then one can easily see from (2.4) that for all $\lambda > 0$,

$$\begin{aligned} E(F(\lambda^{-1/2}\,\cdot)|X_{\mathcal{E},\infty}(\lambda^{-1/2}\,\cdot) = \vec{\xi}) &\equiv E(F(\lambda^{-1/2}\,\cdot)|\gamma_n(\lambda^{-1/2}\,\cdot) = \xi_n,\ n = 1,2,\ldots) \\ &= E_x\bigg[F\bigg(\lambda^{-1/2}x - \lambda^{-1/2}\sum_{n=1}^{\infty}\gamma_n(x)\beta_n + \sum_{n=1}^{\infty}\xi_n\beta_n\bigg)\bigg]. \end{aligned} \tag{3.1}$$

Thus we have that

$$E^{\mathrm{an}w_\lambda}(F|X_{\mathcal{E},\infty} = \vec{\xi}) = E_x^{\mathrm{an}w_\lambda}\bigg[F\bigg(x - \sum_{n=1}^{\infty}\gamma_n(x)\beta_n + \sum_{n=1}^{\infty}\xi_n\beta_n\bigg)\bigg]$$

and

$$E^{\mathrm{an}f_q}(F|X_{\mathcal{E},\infty} = \vec{\xi}) = E_x^{\mathrm{an}f_q}\bigg[F\bigg(x - \sum_{n=1}^{\infty}\gamma_n(x)\beta_n + \sum_{n=1}^{\infty}\xi_n\beta_n\bigg)\bigg], \tag{3.2}$$

where $E_x^{\mathrm{an}w_\lambda}[F(x)]$ and $E_x^{\mathrm{an}f_q}[F(x)]$ denote the analytic Wiener and the analytic Feynman integrals of functionals F on $C_0[0,T]$, see [1, 12].

We are now ready to state the definitions of the CFFT and the CCP of functionals on $C_0[0,T]$.

Definition 3.1. *Let $F : C_0[0,T] \to \mathbb{C}$ be an SIM functional such that the Wiener integral $E[F(y+\lambda^{-1/2}\cdot)]$ exists as a finite number for all $\lambda > 0$. Let $X_{\mathcal{E},\infty} : C_0[0,T] \to \mathbb{R}^\infty$ be given by (2.2). For $\lambda \in \mathbb{C}_+$ and $y \in C_0[0,T]$, let $T_\lambda(F|X_{\mathcal{E},\infty})(y,\vec{\xi})$ denote the conditional analytic Wiener integral of $F(y+\cdot)$ given $X_{\mathcal{E},\infty}$, that is to say,*

$$T_\lambda(F|X_{\mathcal{E},\infty})(y,\vec{\xi}) = E^{\mathrm{anw}_\lambda}(F(y+\cdot)|X_{\mathcal{E},\infty} = \vec{\xi}) = E_x^{\mathrm{anw}_\lambda}\Big[F\Big(y + x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\Big)\Big]. \tag{3.3}$$

We define the L_1 analytic CFFT $T_q^{(1)}(F|X_{\mathcal{E},\infty})(y,\vec{\xi})$ of F given $X_{\mathcal{E},\infty}$ by the formula

$$T_q^{(1)}(F|X_{\mathcal{E},\infty})(y,\vec{\xi}) = \lim_{\substack{\lambda\to -iq\\ \lambda\in\mathbb{C}_+}} T_\lambda(F|X_{\mathcal{E},\infty})(y,\vec{\xi}) = E_x^{\mathrm{anf}_q}\Big[F\Big(y + x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\Big)\Big]. \tag{3.4}$$

We also define the CCP of SIM functionals F and G given $X_{\mathcal{E},\infty}$ by the formula

$$\begin{aligned}
&[(F*G)_\lambda|X_{\mathcal{E},\infty}](y,\vec{\xi})\\
&= \begin{cases} E^{\mathrm{anw}_\lambda}\Big(F\Big(\frac{y+\cdot}{\sqrt{2}}\Big)G\Big(\frac{y-\cdot}{\sqrt{2}}\Big)\Big|X_{\mathcal{E},\infty} = \vec{\xi}\Big), & \lambda\in\mathbb{C}_+\\ E^{\mathrm{anf}_q}\Big(F\Big(\frac{y+\cdot}{\sqrt{2}}\Big)G\Big(\frac{y-\cdot}{\sqrt{2}}\Big)\Big|X_{\mathcal{E},\infty} = \vec{\xi}\Big), & \lambda = -iq,\ q\in\mathbb{R}\setminus\{0\}\end{cases}\\
&= \begin{cases} E_x^{\mathrm{anw}_\lambda}\Big[F\Big(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Big(x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\Big)\Big)\\ \qquad\times G\Big(\frac{y}{\sqrt{2}} - \frac{1}{\sqrt{2}}\Big(x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\Big)\Big)\Big], & \lambda\in\mathbb{C}_+\\ E_x^{\mathrm{anf}_q}\Big[F\Big(\frac{y}{\sqrt{2}} + \frac{1}{\sqrt{2}}\Big(x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\Big)\Big)\\ \qquad\times G\Big(\frac{y}{\sqrt{2}} - \frac{1}{\sqrt{2}}\Big(x - \sum_{n=1}^\infty \gamma_n(x)\beta_n + \sum_{n=1}^\infty \xi_n\beta_n\Big)\Big)\Big], & \lambda = -iq,\ q\in\mathbb{R}\setminus\{0\}.\end{cases}
\end{aligned}\tag{3.5}$$

4. Relationship between conditional Fourier–Feynman transform and conditional convolution product given $\mathbb{R}^\infty$-valued conditioning function

In this section, we establish a relationship between the CFFT and the CCP of general SIM functionals on $C_0[0,T]$. Theorem 4.2 below tells us that the CFFT of the CCP of SIM functionals F and G is a product of the CFFTs of each functional. To ensure this result, we need the following lemma.

Lemma 4.1. *Given an orthonormal sequence $\{e_n\}_{n=1}^\infty$ of functions in $L_2[0,T]$, let $\mathcal{Z}_1, \mathcal{Z}_2 : [0,T]\times C_0[0,T]\times C_0[0,T] \to \mathbb{R}$ be given by*

$$\mathcal{Z}_1(t;x_1,x_2) = x_1(t) - \sum_{n=1}^\infty \gamma_n(x_1)\beta_n(t) + x_2(t) - \sum_{n=1}^\infty \gamma_n(x_2)\beta_n(t)$$

and

$$\mathcal{Z}_2(t;x_1,x_2) = x_1(t) - \sum_{n=1}^\infty \gamma_n(x_1)\beta_n(t) - x_2(t) + \sum_{n=1}^\infty \gamma_n(x_2)\beta_n(t).$$

Then the processes $\{\mathcal{Z}_1(t;\cdot,\cdot) : t\in[0,T]\}$ and $\{\mathcal{Z}_2(t;\cdot,\cdot) : t\in[0,T]\}$ are stochastically independent.

Proof. We first note that for each $t\in[0,T]$, $x(t) = \int_0^t dx(s) = \langle I_{[0,t]}, x\rangle$. Using this and equation (2.1), we also see that $E[x(t)\gamma_n(x)] = \beta_n(t)$ for each $t\in[0,T]$. Using this fact and the facts that $E_x[x(t)] = 0$ and $E_x[x(s)x(t)] = \min\{s,t\}$ for all $s,t\in[0,T]$, it follows that

$$E_{x_1}[E_{x_2}[\mathcal{Z}_1(s;x_1,x_2)\mathcal{Z}_2(t;x_1,x_2)]] = 0 = E_{x_1}[E_{x_2}[\mathcal{Z}_1(s;x_1,x_2)]]E_{x_1}[E_{x_2}[\mathcal{Z}_2(t;x_1,x_2)]]$$

for $s,t\in[0,T]$. □

In Theorem 4.2 below, we show that the CFFT of the CCP is a product of CFFTs.

Theorem 4.2. *Let $X_{\mathcal{E},\infty}$ be given by equation* (2.2). *Let F and G be SIM functionals on $C_0[0,T]$. Assume that $T_q^{(1)}\Big([(F*G)_q|X_{\mathcal{E},\infty}](\,\cdot\,,\vec{\eta}_1)\Big|X_{\mathcal{E},\infty}\Big)(\cdot,\vec{\eta}_2)$, $T_q^{(1)}(F|X_{\mathcal{E},\infty})(\cdot,\vec{\eta})$, and $T_q^{(1)}(G|X_{\mathcal{E},\infty})(\cdot,\vec{\eta})$, all exist at $q\in\mathbb{R}\setminus\{0\}$. Then*

$$T_q^{(1)}\Big([(F*G)_q|X_{\mathcal{E},\infty}](\,\cdot\,,\vec{\eta}_1)\Big|X_{\mathcal{E},\infty}\Big)(y,\vec{\eta}_2)=T_q^{(1)}(F|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2+\vec{\eta}_1}{\sqrt{2}}\Big)T_q^{(1)}(G|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2-\vec{\eta}_1}{\sqrt{2}}\Big)\tag{4.1}$$

for SI-a.e. $y\in C_0[0,T]$.

Proof. In view of (3.4), (3.3), and (3.1), it will suffice to show that

$$T_\lambda\Big([(F*G)_\lambda|X_{\mathcal{E},\infty}](\,\cdot\,,\vec{\eta}_1)\Big|X_{\mathcal{E},\infty}\Big)(y,\vec{\eta}_2)=T_\lambda(F|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2+\vec{\eta}_1}{\sqrt{2}}\Big)T_\lambda(G|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2-\vec{\eta}_1}{\sqrt{2}}\Big)$$

for $\lambda>0$. But using equations (3.1), (3.3), and (3.5), we observe that for all $\lambda>0$,

$$\begin{aligned}
&T_\lambda\Big([(F*G)_\lambda|X_{\mathcal{E},\infty}](\,\cdot\,,\vec{\eta}_1)\Big|X\Big)(y,\vec{\eta}_2)\\
&=E_{x_1}\Big[[(F*G)_\lambda|X_{\mathcal{E},\infty}]\Big(y+\frac{1}{\sqrt{\lambda}}\Big\{x_1-\sum_{n=1}^{\infty}\gamma_n(x_1)\beta_n\Big\}+\sum_{n=1}^{\infty}\eta_{2n}\beta_n,\vec{\eta}_1\Big)\Big]\\
&=E_{x_1}\Big[E_{x_2}\Big[F\Big(\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{2\lambda}}\Big\{x_1-\sum_{n=1}^{\infty}\gamma_n(x_1)\beta_n+x_2-\sum_{n=1}^{\infty}\gamma_n(x_2)\beta_n\Big\}+\sum_{n=1}^{\infty}\frac{\eta_{2n}+\eta_{1n}}{\sqrt{2}}\beta_n\Big)\\
&\qquad\times G\Big(\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{2\lambda}}\Big\{x_1-\sum_{n=1}^{\infty}\gamma_n(x_1)\beta_n-x_2+\sum_{n=1}^{\infty}\gamma_n(x_2)\beta_n\Big\}+\sum_{n=1}^{\infty}\frac{\eta_{2n}-\eta_{1n}}{\sqrt{2}}\beta_n\Big)\Big]\Big].
\end{aligned}$$

By Lemma 4.1 above,

$$(2\lambda)^{-1/2}\Big(x_1-\sum_{n=1}^{\infty}\gamma_n(x_1)\beta_n+x_2-\sum_{n=1}^{\infty}\gamma_n(x_2)\beta_n\Big)$$

and

$$(2\lambda)^{-1/2}\Big(x_1-\sum_{n=1}^{\infty}\gamma_n(x_1)\beta_n-x_2+\sum_{n=1}^{\infty}\gamma_n(x_2)\beta_n\Big)$$

are independent processes. Hence the expectation of FG equals the product of the expectations and so we see that

$$\begin{aligned}
&T_\lambda\Big([(F*G)_\lambda|X_{\mathcal{E},\infty}](\,\cdot\,,\vec{\eta}_1)\Big|X_{\mathcal{E},\infty}\Big)(y,\vec{\eta}_2)\\
&=E_{x_1}\Big[E_{x_2}\Big[F\Big(\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{\lambda}}\Big\{\frac{x_1+x_2}{\sqrt{2}}-\sum_{n=1}^{\infty}\gamma_n\Big(\frac{x_1+x_2}{\sqrt{2}}\Big)\beta_n\Big\}+\sum_{n=1}^{\infty}\frac{\eta_{2n}+\eta_{1n}}{\sqrt{2}}\beta_n\Big)\Big]\Big]\\
&\quad\times E_{x_1}\Big[E_{x_2}\Big[G\Big(\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{\lambda}}\Big\{\frac{x_1-x_2}{\sqrt{2}}-\sum_{n=1}^{\infty}\gamma_n\Big(\frac{x_1-x_2}{\sqrt{2}}\Big)\beta_n\Big\}+\sum_{n=1}^{\infty}\frac{\eta_{2n}-\eta_{1n}}{\sqrt{2}}\beta_n\Big)\Big]\Big].
\end{aligned}$$

Now the processes $(x_1+x_2)/\sqrt{2}$ and $(x_1-x_2)/\sqrt{2}$ on $C_0[0,T]\times C_0[0,T]$ are equivalent to the Wiener process,

x, on $C_0[0,T]$ by the rotation-invariant of the Wiener measure m_w. Thus for $\lambda > 0$,

$$\begin{aligned}&T_\lambda\Big([(F*G)_\lambda|X_{\mathcal{E},\infty}](\,\cdot\,,\vec{\eta}_1)\Big|X_{\mathcal{E},\infty}\Big)(y,\vec{\eta}_2)\\&=E_x\bigg[F\bigg(\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{\lambda}}\bigg\{x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n\bigg\}+\sum_{n=1}^{\infty}\frac{\eta_{2n}+\eta_{1n}}{\sqrt{2}}\beta_n\bigg)\bigg]\\&\quad\times E_x\bigg[G\bigg(\frac{y}{\sqrt{2}}+\frac{1}{\sqrt{\lambda}}\bigg\{x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n\bigg\}+\sum_{n=1}^{\infty}\frac{\eta_{2n}-\eta_{1n}}{\sqrt{2}}\beta_n\bigg)\bigg]\\&=T_\lambda(F|X_{\mathcal{E},\infty})\bigg(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2+\vec{\eta}_1}{\sqrt{2}}\bigg)T_\lambda(G|X_{\mathcal{E},\infty})\bigg(\frac{y}{\sqrt{2}},\frac{\vec{\eta}_2-\vec{\eta}_1}{\sqrt{2}}\bigg)\end{aligned}$$

and the theorem is verified. □

5. Conditional Fourier–Feynman transform and conditional convolution product of bounded cylinder functionals

In order to establish equation (4.1) above, we assumed that the transforms appearing (4.1) all exists. In this section, we present specific bounded cylinder functionals on $C_0[0,T]$ whose CFFT and CCP exist.

We introduce the class of bounded cylinder functionals on $C_0[0,T]$. Let $\mathcal{M}(\mathbb{R}^\nu)$ be the space of $\mathbb{C}$-valued Borel measures on $\mathcal{B}(\mathbb{R}^\nu)$, the Borel class on $\mathbb{R}^\nu$. It is known that a $\mathbb{C}$-valued Borel measure μ has a finite total variation $\|\mu\|$, and the class $\mathcal{M}(\mathbb{R}^\nu)$ is a Banach algebra under the norm $\|\cdot\|$ and with convolution as multiplication.

Given a complex measure μ in $\mathcal{M}(\mathbb{R}^\nu)$, the Fourier–Stieltjes transform $\widehat{\mu}$ of μ is a $\mathbb{C}$-valued function on $\mathbb{R}^\nu$ defined by

$$\widehat{\mu}(\vec{u})=\int_{\mathbb{R}^\nu}\exp\Big\{i\sum_{j=1}^{\nu}u_jv_j\Big\}d\mu(\vec{v}). \tag{5.1}$$

Given an orthonormal set $\mathcal{A}=\{\alpha_1,\ldots,\alpha_\nu\}$ of functions in $L_2[0,T]$, let $\widehat{\mathfrak{T}}_{\mathcal{A}}$ be the class of functionals F_μ on $C_0[0,T]$ defined by

$$F_\mu(x)=\widehat{\mu}(\langle\vec{\alpha},x\rangle)=\int_{\mathbb{R}^\nu}\exp\Big\{i\sum_{j=1}^{\nu}\langle\alpha_j,x\rangle v_j\Big\}d\mu(\vec{v}) \tag{5.2}$$

for SI-a.e. $x\in C_0[0,T]$. Given any $\mu\in\mathcal{M}(\mathbb{R}^\nu)$, the function $\widehat{\mu}$ corresponding to μ by (5.1) is bounded (and so is F_μ), because $|\widehat{\mu}(\vec{u})|\le\|\mu\|<+\infty$ for every $\vec{u}\in\mathbb{R}^\nu$. Note that the functional F_μ having the form (5.2) is SIM on $C_0[0,T]$.

Given an infinite-dimensional subspace $\mathcal{H}$ of $L_2[0,T]$, let $\mathcal{E}=\{e_n\}$ be a countable orthonormal basis of $\mathcal{H}$ and let the projection map $\mathcal{P}_{\mathcal{H}}:L_2[0,T]\to\mathcal{H}$ be given by (2.3). Even if $\mathcal{A}=\{\alpha_1,\ldots,\alpha_\nu\}$ is an orthonormal set of functions in $L_2[0,T]$, the set $\{\alpha_1-\mathcal{P}_{\mathcal{H}}\alpha_1,\ldots,\alpha_\nu-\mathcal{P}_{\mathcal{H}}\alpha_\nu\}$ may not orthonormal. Let $\{g_1,\ldots,g_m\}$ be an orthonormal basis of the subspace $\mathrm{Span}\{\alpha_1-\mathcal{P}_{\mathcal{H}}\alpha_1,\ldots,\alpha_\nu-\mathcal{P}_{\mathcal{H}}\alpha_\nu\}$. Then we see that for each $j\in\{1,\ldots,\nu\}$,

$$\alpha_j-\mathcal{P}_{\mathcal{H}}\alpha_j=\sum_{l=1}^{m}(\alpha_j-\mathcal{P}_{\mathcal{H}}\alpha_j,g_l)_2g_l.$$

Lemma 5.1. *Given a linearly independent subset $\{\alpha_1,\ldots,\alpha_\nu\}$ of $L_2[0,T]$, let $\mathcal{G}=\{g_1,\ldots,g_m\}$ be an orthonormal basis of the subspace* $\mathrm{Span}\{\alpha_1-\mathcal{P}_{\mathcal{H}}\alpha_1,\ldots,\alpha_\nu-\mathcal{P}_{\mathcal{H}}\alpha_\nu\}$ *of $L_2[0,T]$. Then for any $\rho\in\mathbb{R}\setminus\{0\}$, it follows that*

$$E_x\bigg[\exp\Big\{i\rho\sum_{j=1}^{\nu}\Big\langle\alpha_j,x-\sum_{n=1}^{\infty}\gamma_n(x)\beta_n\Big\rangle v_j\Big\}\bigg]=\exp\bigg\{-\frac{\rho^2}{2}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j-\mathcal{P}_{\mathcal{H}}\alpha_j,g_l)_2v_j\Big)^2\bigg\} \tag{5.3}$$

where the projection map $\mathcal{P}_{\mathcal{H}}:L_2[0,T]\to\mathcal{H}$ is given by (2.3).

Proof. Notice that for each $j \in \{1, \ldots, \nu\}$,

$$\begin{aligned}\Big\langle \alpha_j, x - \sum_{n=1}^{\infty} \gamma_n(x)\beta_n \Big\rangle &= \langle \alpha_j, x\rangle - \sum_{n=1}^{\infty} \gamma_n(x)\langle \alpha_j, \beta_n\rangle = \Big\langle \alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, x\Big\rangle \\ &= \Big\langle \sum_{l=1}^{m} (\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 g_l, x\Big\rangle \\ &= \sum_{l=1}^{m} (\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 \langle g_l, x\rangle. \end{aligned} \tag{5.4}$$

Using (5.4), (2.6), the Fubini theorem, and (2.7), it follows that

$$\begin{aligned}&E_x\Big[\exp\Big\{i\rho \sum_{j=1}^{\nu} \Big\langle \alpha_j, x - \sum_{n=1}^{\infty}\gamma_n(x)\beta_n\Big\rangle v_j\Big\}\Big] \\ &= E_x\Big[\exp\Big\{i\rho\sum_{j=1}^{\nu}\Big(\sum_{l=1}^{m}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2\langle g_l, x\rangle\Big)v_j\Big\}\Big] \\ &= E_x\Big[\exp\Big\{i\rho\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)\langle g_l, x\rangle\Big\}\Big] \\ &= \prod_{l=1}^{m}\Big((2\pi)^{-1/2}\int_{\mathbb{R}}\exp\Big\{i\rho\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)u_l - \sum_{l=1}^{m}\frac{u_l^2}{2}\Big\}du_l\Big) \\ &= \prod_{l=1}^{m}\Big(\exp\Big\{-\frac{\rho^2}{2}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2\Big\}\Big).\end{aligned}$$

From this we obtain equation (5.3). □

In our first theorem of this section, we establish the existences of the CFFT $T_q^{(1)}(F_\mu|X_{\mathcal{E},\infty})$ of functionals F_μ in the class $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

Theorem 5.2. *Let $F_\mu \in \widehat{\mathfrak{T}}_{\mathcal{A}}$ be given by equation* (5.2), *and let $X_{\mathcal{E},\infty}$ be given by equation* (2.2). *Then for a.e. $\vec{\xi} \in \mathbb{R}^\infty$, it follows that*

$$\begin{aligned}&T_q^{(1)}(F|X_{\mathcal{E},\infty})(y, \vec{\xi}) \\ &= \int_{\mathbb{R}^\nu}\exp\Big\{i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j - \frac{i}{2q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2 + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j\Big\}d\mu(\vec{v})\end{aligned} \tag{5.5}$$

for all $q \in \mathbb{R}\setminus\{0\}$ and SI-a.e. $y \in C_0[0, T]$.

Proof. Using (5.2), (3.1) with F replaced with $F_\mu(y + \cdot)$, the Fubini theorem, and (5.3) with ρ replaced with $\lambda^{-1/2}$, it follows that for $(\lambda, \vec{\xi}) \in (0, +\infty) \times \mathbb{R}^\infty$,

$$\begin{aligned}&J_{F_\mu(y+\cdot)}(\lambda; \vec{\xi}) \equiv E\Big(F_\mu(y + \lambda^{-1/2}\cdot)\Big|X_{\mathcal{E},\infty}(\lambda^{-1/2}\cdot) = \vec{\xi}\Big) \\ &= E_x\Big[F_\mu\Big(y + \lambda^{-1/2}x - \lambda^{-1/2}\sum_{n=1}^{\infty}\gamma_n(x)\beta_n + \sum_{n=1}^{\infty}\xi_n\beta_n\Big)\Big]\end{aligned}$$

$$= \int_{\mathbb{R}^\nu} \exp\Big\{ i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j + i\sum_{j=1}^{\nu}\Big\langle\alpha_j, \sum_{n=1}^{\infty}\xi_n\beta_n\Big\rangle v_j\Big\} E_x\Big[\exp\Big\{ i\lambda^{-1/2}\sum_{j=1}^{\nu}\Big\langle\alpha_j, x - \sum_{n=1}^{\infty}\gamma_n(x)\beta_n\Big\rangle v_j\Big\}\Big] d\mu(\vec{v})$$

$$= \int_{\mathbb{R}^\nu} \exp\Big\{ i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j - \frac{1}{2\lambda}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2\Big\} d\mu(\vec{v}).$$

Let

$$J^*_{F_\mu(y+\cdot)}(\lambda;\vec{\xi}) = \int_{\mathbb{R}^\nu} \exp\Big\{ i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j - \frac{1}{2\lambda}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2\Big\} d\mu(\vec{v}) \tag{5.6}$$

for $\lambda \in \mathbb{C}_+$. Since $\mathrm{Re}(\lambda) > 0$ for all $\lambda \in \mathbb{C}_+$, it follows that

$$\begin{aligned} \big|J^*_{F_\mu(y+\cdot)}(\lambda;\vec{\xi})\big| &\le \int_{\mathbb{R}^\nu} \Big|\exp\Big\{ i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j - \frac{1}{2\lambda}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2\Big\}\Big| d|\mu|(\vec{v}) \\ &\le \int_{\mathbb{R}^\nu} d|\mu|(\vec{v}) = \|\mu\| < +\infty. \end{aligned} \tag{5.7}$$

Hence, applying the dominated convergence theorem, we see that $J^*_{F(y+\cdot)}(\lambda;\vec{\xi})$ is a continuous function of $\lambda \in \widetilde{\mathbb{C}}_+$. Since

$$K(\lambda) \equiv \exp\Big\{ -\frac{1}{2\lambda}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2\Big\}$$

is analytic on $\mathbb{C}_+$, using the Fubini theorem, it follows that

$$\int_\Gamma J^*_{F_\mu(y+\cdot)}(\lambda;\vec{\xi})d\lambda = \int_{\mathbb{R}^\nu} \exp\Big\{ i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j\Big\}\Big(\int_\Gamma K(\lambda)d\lambda\Big) d\mu(\vec{v}) = 0$$

for all rectifiable closed curves Γ lying in $\mathbb{C}_+$. Thus by the Morera theorem, $J^*_{F_\mu(y+\cdot)}(\lambda;\vec{\xi})$ is analytic on $\mathbb{C}_+$. Therefore, the conditional analytic Wiener integral

$$T_\lambda(F_\mu|X_{\mathcal{E},\infty})(y,\vec{\xi}) = E^{\mathrm{an}w_\lambda}(F_\mu(y+\cdot)|X_{\mathcal{E},\infty} = \vec{\xi}) = J^*_{F_\mu(y+\cdot)}(\lambda;\vec{\xi})$$

exists and is given by the right-hand side of (5.6). Finally, by the dominated convergence theorem (the use of which is justified by (5.7)), the L_1 analytic CFFT $T_q^{(1)}(F_\mu|X_{\mathcal{E},\infty} = \vec{\xi})$ of F_μ exists and is given by (5.5). □

From the definition of the conditional Feynman integral and the L_1 analytic CFFT, it follows that $T_q^{(1)}(F|X_{\mathcal{E},\infty})(0,\vec{\xi}) = E^{\mathrm{an}f_q}(F|X_{\mathcal{E},\infty} = \vec{\xi})$. We thus have the following corollary.

Corollary 5.3. *Let F_μ and $X_{\mathcal{E},\infty}$ be as in Theorem 5.2. Then the conditional analytic Feynman integral $E^{\mathrm{an}f_q}(F_\mu|X_{\mathcal{E},\infty} = \vec{\xi})$ of F_μ exists for all $q \in \mathbb{R}\setminus\{0\}$ and a.e. $\vec{\xi} \in \mathbb{R}^\infty$, and is given by the formula*

$$E^{\mathrm{an}f_q}(F_\mu|X_{\mathcal{E},\infty} = \vec{\xi}) = \int_{\mathbb{R}^\nu} \exp\Big\{ -\frac{i}{2q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2 + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j\Big\} d\mu(\vec{v}).$$

Remark 5.4. *Given a functional F_μ in $\widehat{\mathfrak{T}}_{\mathcal{A}}$ with the corresponding measure $\mu \in \mathcal{M}(\mathbb{R}^\nu)$, and given a non-zero real number q and a vector $\vec{\xi} \in \mathbb{R}^\infty$, define a set function $\mu_{q,\vec{\xi}} : \mathcal{B}(\mathbb{R}^\nu) \to \mathbb{C}$ by the formula*

$$\mu_{q,\vec{\xi}}(G) = \int_G \exp\Big\{ -\frac{i}{2q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2 + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2 v_j\Big\} d\mu(\vec{v}) \tag{5.8}$$

for each G in $\mathcal{B}(\mathbb{R}^\nu)$, the Borel σ-field on $\mathbb{R}^\nu$. Then $\mu_{q,\vec{\xi}}$ is obviously a complex measure in $\mathcal{M}(\mathbb{R}^\nu)$ and $\|\mu_{q,\vec{\xi}}\| = \|\mu\|$ for any $q \in \mathbb{R} \setminus \{0\}$ and $\vec{\xi} \in \mathbb{R}^\infty$. Then equation (5.5) can be rewritten by

$$T_q^{(1)}(F_\mu|X_{\mathcal{E},\infty})(y,\vec{\xi}) = \int_{\mathbb{R}^\nu} \exp\Big\{i\sum_{j=1}^{\nu}\langle\alpha_j, y\rangle v_j\Big\} d\mu_{q,\vec{\xi}}(\vec{v}) = \widehat{\mu_{q,\vec{\xi}}}(\langle\vec{\alpha}, y\rangle) \tag{5.9}$$

for SI-a.e. $y \in C_0[0,T]$, and so the L_1 analytic CFFT $T_q^{(1)}(F_\mu|X_{\mathcal{E},\infty})(\,\cdot\,,\vec{\xi})$ of F_μ with parameter q is an element of $\widehat{\mathfrak{T}_{\mathcal{A}}}$ for each $\vec{\xi} \in \mathbb{R}^\infty$.

In view of Theorem 5.2 and Remark 5.4, we easily obtain the following corollary.

Corollary 5.5. *Let F_μ and $X_{\mathcal{E},\infty}$ be as in Theorem 5.2. Then, for any finite sequence $\{q_1,\dots,q_N\}$ in $\mathbb{R} \setminus \{0\}$ which satisfies the condition*

$$\frac{1}{q_1} + \cdots + \frac{1}{q_k} \neq 0 \text{ for each } k \in \{1,\dots,N\}, \tag{5.10}$$

it follows that

$$T_{q_N}^{(1)}\Big(T_{q_{N-1}}^{(1)}\Big(\cdots T_{q_1}^{(1)}(F_\mu|X_{\mathcal{E},\infty})(\,\cdot\,,\vec{\xi}^{(1)})\cdots\Big|X_{\mathcal{E},\infty}\Big)(\,\cdot\,,\vec{\xi}^{(N-1)})\Big|X_{\mathcal{E},\infty}\Big)(y,\vec{\xi}^{(N)}) = T_{\alpha_N}^{(1)}(F_\mu|X_{\mathcal{E},\infty})\Big(y,\sum_{k=1}^{N}\vec{\xi}^{(k)}\Big)$$

for SI-a.e. $y \in C_0[0,T]$ and a.e. $(\vec{\xi}^{(1)},\dots,\vec{\xi}^{(N)})$ in $(\mathbb{R}^\infty)^N$, the product of N copies of $\mathbb{R}^\infty$, where

$$\alpha_N = \Big(\frac{1}{q_1} + \cdots + \frac{1}{q_N}\Big)^{-1}. \tag{5.11}$$

In our next theorem, we also establish the existence of the CCP of functionals F_{μ_1} and F_{μ_2} in the class $\widehat{\mathfrak{T}_{\mathcal{A}}}$.

Theorem 5.6. *Let F_{μ_1} and F_{μ_2} be the functionals in $\widehat{\mathfrak{T}_{\mathcal{A}}}$ with corresponding Borel measures μ_1 and μ_2, respectively, in $\mathcal{M}(\mathbb{R}^\nu)$, and let $X_{\mathcal{E},\infty}$ be given by equation (2.2). Then for a.e. $\vec{\xi} \in \mathbb{R}^\infty$, it follows that*

$$\begin{aligned}[(F*G)_q|X_{\mathcal{E},\infty}](y,\vec{\xi}) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \exp\Big\{ i\langle\alpha_j, y\rangle\frac{(u_j+v_j)}{\sqrt{2}} - \frac{i}{4q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2(u_j - v_j)\Big)^2 \\ + \frac{i}{\sqrt{2}}\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2(u_j - v_j)\Big\} d\mu_1(\vec{u})d\mu_2(\vec{v})\end{aligned} \tag{5.12}$$

for all $q \in \mathbb{R} \setminus \{0\}$ and SI-a.e. $y \in C_0[0,T]$.

Proof. Using similar techniques as those in the proof of Theorem 5.2, it follows equation (5.12) immediately by the definition of the CCP. □

Remark 5.7. *Given two functionals F_{μ_1} and F_{μ_2} in the class $\widehat{\mathfrak{T}_{\mathcal{A}}}$ with the corresponding measures μ_1 and μ_2 in $\mathcal{M}(\mathbb{R}^\nu)$, and given a non-zero real q and a vector $\vec{\xi} \in \mathbb{R}^\infty$, define a set function $\varphi_{q,\vec{\xi}} : \mathcal{B}(\mathbb{R}^\nu \times \mathbb{R}^\nu) \to \mathbb{C}$ by the formula*

$$\varphi_{q,\vec{\xi}}(H) = \iint_H \exp\Big\{ - \frac{i}{4q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2(u_j - v_j)\Big)^2 + \frac{i}{\sqrt{2}}\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\xi_n(\alpha_j, e_n)_2(u_j - v_j)\Big\} d\mu_1(\vec{u})d\mu_2(\vec{v}) \tag{5.13}$$

for each H in $\mathcal{B}(\mathbb{R}^\nu \times \mathbb{R}^\nu)$, the Borel σ-field on $\mathbb{R}^\nu \times \mathbb{R}^\nu$. Then $\varphi_{q,\vec{\xi}}$ is a complex measure on $\mathcal{B}(\mathbb{R}^\nu \times \mathbb{R}^\nu)$. Define a function $\phi : \mathbb{R}^\nu \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ by $\phi(\vec{u}, \vec{v}) = (\vec{u} + \vec{v})/\sqrt{2}$. Then ϕ is a continuous function, and so it is $\mathcal{B}(\mathbb{R}^\nu \times \mathbb{R}^\nu)$-measurable. Thus the set function $\varphi_{q,\vec{\xi}} \circ \phi^{-1} : \mathcal{B}(\mathbb{R}^\nu) \to \mathbb{C}$ is in $\mathcal{M}(\mathbb{R}^\nu)$ obviously. Under these setting, equation (5.12) *can be rewritten by*

$$[(F_{\mu_1} * F_{\mu_2})_q | X_{\mathcal{E},\infty}](y, \vec{\xi}) = \int_{\mathbb{R}^\nu} \exp\Big\{ i \sum_{j=1}^{\nu} \langle \alpha_j, y \rangle r_j \Big\} d\varphi_{q,\vec{\xi}} \circ \phi^{-1}(\vec{r}) = \widehat{\varphi_{q,\vec{\xi}} \circ \phi^{-1}}(\langle \vec{\alpha}, y \rangle)$$

for SI-a.e. $y \in C_0[0,T]$. Thus the CCP $[(F_{\mu_1} * F_{\mu_2})_q | X_{\mathcal{E},\infty}](\cdot, \vec{\xi})$ of F_{μ_1} and F_{μ_2} is an element of $\widehat{\mathfrak{T}}_{\mathcal{A}}$ for each $\vec{\xi} \in \mathbb{R}^\infty$.

6. Confirm the relationship

In this section, we confirm the relationship, equation (4.1), between the CFFTs and the CCPs for the functionals F_{μ_1} and F_{μ_2} in the class $\widehat{\mathfrak{T}}_{\mathcal{A}}$, via direct calculations.

Theorem 6.1. *Let F_{μ_1}, F_{μ_2}, and $X_{\mathcal{E},\infty}$ be as in Theorem 5.6. Then for all $q \in \mathbb{R} \setminus \{0\}$, equation* (4.1) *with F and G replaced with F_{μ_1} and F_{μ_2} respectively, holds true.*

Proof. Using (5.9) with F_μ and μ replaced with $[(F_{\mu_1} * F_{\mu_2})_q | X_{\mathcal{E},\infty}]$ and $\varphi_{q,\vec{\xi}^{(1)}} \circ \phi^{-1}$ respectively, (5.8) with μ replaced with $\varphi_{q,\vec{\xi}^{(1)}} \circ \phi^{-1}$, and (5.13), it follows that for SI-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&T_q^{(1)}\Big(\Big[(F_{\mu_1} * F_{\mu_2})_q \Big| X_{\mathcal{E},\infty}\Big](\cdot, \vec{\xi}^{(1)}) \Big| X_{\mathcal{E},\infty}\Big)(y, \vec{\xi}^{(2)}) \\
&= \int_{\mathbb{R}^\nu} \exp\Big\{ i \sum_{j=1}^{\nu} \langle \alpha_j, y \rangle r_j \Big\} d(\varphi_{q,\vec{\xi}^{(1)}} \circ \phi^{-1})_{q,\vec{\xi}^{(2)}}(\vec{r}) \\
&= \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} \exp\Big\{ i \sum_{j=1}^{\nu} \langle \alpha_j, y \rangle \frac{u_j + v_j}{\sqrt{2}} - \frac{i}{2q} \sum_{l=1}^{m} \Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2 \frac{u_j + v_j}{\sqrt{2}} \Big)^2 \\
&\qquad\qquad + i \sum_{j=1}^{\nu} \sum_{n=1}^{\infty} \xi_n^{(2)} (\alpha_j, e_n)_2 \frac{u_j + v_j}{\sqrt{2}} \Big\} d\varphi_{q,\vec{\xi}^{(1)}}(\vec{u}, \vec{v}) \\
&= \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} \exp\Big\{ i \sum_{j=1}^{\nu} \langle \alpha_j, y \rangle \frac{u_j + v_j}{\sqrt{2}} \\
&\qquad\qquad - \frac{i}{4q} \sum_{l=1}^{m} \Big[\Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2 (u_j + v_j) \Big)^2 + \Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2 (u_j - v_j) \Big)^2 \Big] \\
&\qquad\qquad + \frac{i}{\sqrt{2}} \sum_{j=1}^{\nu} \sum_{n=1}^{\infty} \Big(\xi_n^{(2)} (\alpha_j, e_n)_2 (u_j + v_j) + \xi_n^{(1)} (\alpha_j, e_n)_2 (u_j - v_j) \Big) \Big\} d\mu_1(\vec{u}) d\mu_2(\vec{v}).
\end{aligned}$$

We note that

$$\begin{aligned}
&\Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2 (u_j + v_j) \Big)^2 + \Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2 (u_j - v_j) \Big)^2 \\
&= 2 \sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2^2 (u_j^2 + v_j^2) + 2 \sum_{j_1 \neq j_2}^{\nu} (\alpha_{j_1} - \mathcal{P}_{\mathcal{H}} \alpha_{j_1}, g_l)_2 (\alpha_{j_2} - \mathcal{P}_{\mathcal{H}} \alpha_{j_2}, g_l)_2 \Big(u_{j_1} u_{j_2} + v_{j_1} v_{j_2} \Big) \\
&= 2 \Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2 u_j \Big)^2 + 2 \Big(\sum_{j=1}^{\nu} (\alpha_j - \mathcal{P}_{\mathcal{H}} \alpha_j, g_l)_2^2 v_j \Big)^2
\end{aligned}$$

and

$$\xi_n^{(2)}(\alpha_j, e_n)_2(u_j + v_j) + \xi_n^{(1)}(\alpha_j, e_n)_2(u_j - v_j) = (\xi_n^{(2)} + \xi_n^{(1)})(\alpha_j, e_n)_2 u_j + (\xi_n^{(2)} - \xi_n^{(1)})(\alpha_j, e_n)_2 v_j.$$

Using these, the Fubini theorem, and (5.5), we conclude that for SI-a.e. $y \in C_0[0,T]$,

$$\begin{aligned}
&T_q^{(1)}\Big(\Big[(F_{\mu_1} * F_{\mu_2})_q \Big| X_{\mathcal{E},\infty}\Big](\cdot, \vec{\xi}^{(1)})\Big| X_{\mathcal{E},\infty}\Big)(y, \vec{\xi}^{(2)})\\
&= \int_{\mathbb{R}^\nu} \exp\bigg\{ i\sum_{j=1}^{\nu}\Big\langle \alpha_j, \frac{y}{\sqrt{2}}\Big\rangle u_j - \frac{i}{2q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 u_j\Big)^2 + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\frac{\xi_n^{(2)}+\xi_n^{(1)}}{\sqrt{2}}(\alpha_j, e_n)_2 u_j\bigg\} d\mu_1(\vec{u})\\
&\quad\times \int_{\mathbb{R}^\nu} \exp\bigg\{ i\sum_{j=1}^{\nu}\Big\langle \alpha_j, \frac{y}{\sqrt{2}}\Big\rangle v_j - \frac{i}{2q}\sum_{l=1}^{m}\Big(\sum_{j=1}^{\nu}(\alpha_j - \mathcal{P}_{\mathcal{H}}\alpha_j, g_l)_2 v_j\Big)^2 + i\sum_{j=1}^{\nu}\sum_{n=1}^{\infty}\frac{\xi_n^{(2)}-\xi_n^{(1)}}{\sqrt{2}}(\alpha_j, e_n)_2 v_j\bigg\} d\mu_2(\vec{v})\\
&= T_q^{(1)}(F_{\mu_1}|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)}+\vec{\xi}^{(1)}}{\sqrt{2}}\Big) T_q^{(1)}(F_{\mu_2}|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}}, \frac{\vec{\xi}^{(2)}-\vec{\xi}^{(1)}}{\sqrt{2}}\Big)
\end{aligned}$$

as desired. □

We finally provide a more general relationship involving the iterated CFFT for functionals in $\widehat{\mathfrak{T}}_{\mathcal{A}}$.

Corollary 6.2. *Let F_{μ_1}, F_{μ_2}, and $X_{\mathcal{E},\infty}$ be as in Theorem 5.6. Then for any finite sequence $\{q_1, \dots, q_N\}$ in $\mathbb{R}\setminus\{0\}$ which satisfies the condition* (5.10) *above, it follows that*

$$\begin{aligned}
&T_{q_N}^{(1)}\Big(T_{q_{N-1}}^{(1)}\Big(\cdots T_{q_1}^{(1)}\Big([(F_{\mu_1} * F_{\mu_2})_{\alpha_N}|X_{\mathcal{E},\infty}](\cdot, \vec{\eta})\Big| X_{\mathcal{E},\infty}\Big)(\cdot, \vec{\xi}^{(1)})\cdots \Big| X_{\mathcal{E},\infty}\Big)(\cdot, \vec{\xi}^{(N-1)})\Big| X_{\mathcal{E},\infty}\Big)(y, \vec{\xi}^{(N)})\\
&= T_{\alpha_N}^{(1)}\Big([(F * G)_{\alpha_N}|X_{\mathcal{E},\infty}](\cdot, \vec{\eta})\Big| X_{\mathcal{E},\infty}\Big)\Big(y, \sum_{k=1}^{N}\vec{\xi}^{(k)}\Big)\\
&= T_{\alpha_N}^{(1)}(F_{\mu_1}|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sum_{k=1}^{N}\vec{\xi}^{(k)} + \frac{\vec{\eta}}{\sqrt{2}}\Big) T_{\alpha_N}^{(1)}(F_{\mu_2}|X_{\mathcal{E},\infty})\Big(\frac{y}{\sqrt{2}}, \frac{1}{\sqrt{2}}\sum_{k=1}^{N}\vec{\xi}^{(k)} - \frac{\vec{\eta}}{\sqrt{2}}\Big)
\end{aligned}$$

for SI-a.e. $y \in C_0[0,T]$, where α_N is the real number given by (5.11).

7. Concluding remark

The definitions of the CFFT and the CCP based on the conditional Feynman integral. As mentioned above, the conditioning functions defining the conditional transforms and the conditional convolutions studied in [2, 4, 6, 10, 11, 14] are finite-dimensional vector-valued functions. But, Park and Skoug [19] derived a simple formula to calculate conditional Wiener integrals associated with infinite-dimensional vector-valued conditioning functions. In the celebrated papers [17, 18], Park and Skoug established simple formulas in order to evaluate the conditional Wiener integral which can be used in heat and Schrödinger equations.

The Feynman–Kac functionals are given by

$$F(x) = \exp\Big\{-\int_0^t \theta(s, x(s))dt\Big\} \tag{7.1}$$

where θ is a complex-valued potential on $[0,T]\times\mathbb{R}$. The conditional Feynman integrals of the Feynman–Kac functionals are important in a branch of the study of the Schrödinger equation.

Many physical problems concerning the Schrödinger equation can be represented in terms of the conditional Feynman integral $E^{\mathrm{anf}_q}(F|X_t)$ of the Feynman–Kac functional F, where $X_t(x) = x(t)$. The conditional Feynman integral of the Feynman–Kac functionals given by (7.1) thus is important in the study of the

Feynman integration theory [18]. Moreover, the conditional Feynman integral provides solutions of the integral equations which are formally equivalent to the Schrödinger equation [8, 9, 13, 17, 18]. We are obliged to point out that the conditional Feynman integral was defined in terms the conditional Wiener integral. Based on this history, evaluation formulas for conditional Wiener integrals have been established through the papers [17, 19, 22]. For a detailed survey of the conditional Wiener and Feynman integrals, see [5].

As a first application of the evaluation formula (Theorem 2.1 above) for the conditional Wiener integrals associated with infinite dimensional conditioning functions, we in this paper extended the ideas in [19, 20] to the CFFT and the CCP for functionals on $C_0[0, T]$. However, the fundamental concept of the conditional Wiener integral given infinite dimensional conditioning functions would have been very useful to us in establishing many of the results in [3, 8, 9, 17, 18]. We thus feel that the concept of CFFTs and CCPs associated with infinite dimensional conditioning functions, as well as Theorem 2.1, will prove to be very useful in future work by ourselves as well as other researchers in this area.

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