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Derivation of conticrete Hermite-Hadamard-Jensen-Mercer inequalities through k-Caputo fractional derivatives and majorization

Muhammad Adil Khan[a,*], Shah Faisal[a]

[a]*Department of Mathematics, University of Peshawar, Peshawar 25000, Pakistan*

Abstract. The Hermite-Hadamard inequality stands out as one of the highly valuable inequalities due to its exceptional role in research. Many mathematicians are working hard right now to create various improvements, generalizations and extensions of this inequality. This study develops the Hermite-Hadamard-Jensen-Mercer inequalities in a conticrete framework through the application of k-Caputo fractional derivatives and the concept of majorization. These inequalities are also presented in their weighted forms by assuming certain monotonic tuples. Furthermore, the setting of bounds for the discrepancy of the terms of our important inequalities is also accomplished on the basis of the two newly deduced identities. Also, as particular cases, the previous inequalities are obtained by fixing some parameters in the main inequalities.

1. Introduction

Fractional calculus has become a prominent subject of study for researchers due to its broad uses in several scientific domains. Its usefulness can be shown in a variety of fields, including economics [5], biology [6], geophysics [23] and medicine [13]. Additionally, the references [21, 22, 37] offer some further applications of this field. The fundamental principles in this discipline focus upon fractional derivatives and integration. Notably, the idea of Riemann-Liouville fractional integrals has been crucial in the creation of a large number of extended and generalized fractional operators. The operators Caputo [36], Atangana-Baleanu [1], k-Caputo [17], Hadamard [40] and Katugampola [24] are well-known in this context.

It is crucial to provide some fundamental definitions of the field of fractional calculus because the main goal of our study is to derive unified inequalities within its framework.

Definition 1.1. [36]*[**Riemann-Liouville Fractional Integral Operators**] The Riemann-Liouville fractional integrals of the left- and right-sides for a function $\Psi \in L[\vartheta_1, \vartheta_2]$ and order $\varpi > 0$, are respectively defined as:*

$$J^{\varpi}_{\vartheta_1^+}\Psi(r) = \frac{1}{\Gamma(\varpi)} \int_{\vartheta_1}^{r} (r-u)^{\varpi-1}\Psi(u)du, \quad r > \vartheta_1$$

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* Corresponding author: Muhammad Adil Khan
Email addresses: madilkhan@uop.edu.pk (Muhammad Adil Khan), sfaisaluop@gmail.com (Shah Faisal)

and

$$J^{\varpi}_{\vartheta_2^-}\Psi(r) = \frac{1}{\Gamma(\varpi)}\int_r^{\vartheta_2}(u-r)^{\varpi-1}\Psi(u)du, \quad r < \vartheta_2,$$

where $\Gamma(\cdot)$ *denotes the gamma function and* $\Psi(r) = J^{0}_{\vartheta_2^-}\Psi(r) = J^{0}_{\vartheta_1^+}\Psi(r)$.

Definition 1.2. [36]*[**Caputo Fractional Derivative Operators**] The Caputo fractional derivative operators of the left- and right-sides for a function* Ψ *defined on* $[\vartheta_1, \vartheta_2]$ *such that* $\Psi \in C^n[\vartheta_1, \vartheta_2]$ *and having order* $\varpi > 0$, *are respectively defined as:*

$${}^{c}D^{\varpi}_{\vartheta_1^+}\Psi(r) = \frac{1}{\Gamma(n-\varpi)}\int_{\vartheta_1}^{r}\frac{\Psi^{(n)}(u)}{(r-u)^{\varpi-n+1}}du, \quad r > \vartheta_1$$

and

$${}^{c}D^{\varpi}_{\vartheta_2^-}\Psi(r) = \frac{(-1)^n}{\Gamma(n-\varpi)}\int_{r}^{\vartheta_2}\frac{\Psi^{(n)}(u)}{(u-r)^{\varpi-n+1}}du, \quad r < \vartheta_2.$$

It is essential to note that the following is true for $\varpi = 0$ and $n = 1$,

$$\Psi(r) = \left({}^{c}D^{0}_{\vartheta_2^-}\Psi\right)(r) = \left({}^{c}D^{0}_{\vartheta_1^+}\Psi\right)(r).$$

Definition 1.3. [17, 47]*[**k-Caputo Fractional Derivative Operators**] The k-Caputo fractional derivative operators of the left- and right-sides for a function* Ψ *defined on* $[\vartheta_1, \vartheta_2]$ *such that* $\Psi \in C^n[\vartheta_1, \vartheta_2]$, $k \geq 1$, *and having order* $\varpi > 0$, *are respectively defined as :*

$$({}^{c}D^{\varpi,k}_{\vartheta_1^+}\Psi)(r) = \frac{1}{k\Gamma_k(n-\frac{\varpi}{k})}\int_{\vartheta_1}^{r}\frac{\Psi^{(n)}(u)}{(r-u)^{\frac{\varpi}{k}-n+1}}du, \quad r > \vartheta_1$$

and

$$({}^{c}D^{\varpi,k}_{\vartheta_2^-}\Psi)(r) = \frac{(-1)^n}{k\Gamma_k(n-\frac{\varpi}{k})}\int_{r}^{\vartheta_2}\frac{\Psi^{(n)}(u)}{(u-r)^{\frac{\varpi}{k}-n+1}}du, \quad r < \vartheta_2,$$

where $\Gamma_k(\cdot)$ *refers to k-Gamma function satisfying* $\Gamma_k(\varpi + k) = \varpi\Gamma_k(\varpi)$.

According to studies outlined in [8, 27, 29, 32, 42, 44], there has been a lot of research done on fractional calculus and inequalities. A fractional form of the Hermite-Hadamard inequality was developed by Sarikaya et al. [41] using Definition 1.1. This was accomplished by considering the convexity of the function Ψ over the interval $[\vartheta_1, \vartheta_2]$. It is given below:

$$\Psi\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{\Gamma(\varpi+1)}{2(\vartheta_2-\vartheta_1)^{\varpi}}\left\{J^{\varpi}_{\vartheta_1^+}\Psi(\vartheta_2) + J^{\varpi}_{\vartheta_2^-}\Psi(\vartheta_1)\right\} \leq \frac{1}{2}\left\{\Psi(\vartheta_1)+\Psi(\vartheta_2)\right\}. \tag{1}$$

By taking into account Definition 1.2, a function Ψ such that $\Psi \in C^n[\vartheta_1, \vartheta_2]$ and convexity of $\Psi^{(n)}$ over $[\xi_1, \xi_2] \subset [\vartheta_1, \vartheta_2]$, Zhao et al. [48] developed the following inequality:

$$\Psi^{(n)}\left(\vartheta_1+\vartheta_2-\frac{\xi_1+\xi_2}{2}\right) \leq \frac{2^{n-\varpi-1}\Gamma(n-\varpi+1)}{(\xi_2-\xi_1)^{n-\varpi}}\Big[\left({}^{c}D^{\varpi}_{(\vartheta_1+\vartheta_2-(\xi_1+\xi_2/2))^+}\Psi\right)(\vartheta_1+\vartheta_2-\xi_1)$$
$$+(-1)^n\left({}^{c}D^{\varpi}_{(\vartheta_1+\vartheta_2-(\xi_1+\xi_2/2))^-}\Psi\right)(\vartheta_1+\vartheta_2-\xi_2)\Big]$$

$$\leq \Psi^{(n)}(\vartheta_1) + \Psi^{(n)}(\vartheta_2) - \frac{1}{2}\left(\Psi^{(n)}(\xi_1) + \Psi^{(n)}(\xi_2)\right). \tag{2}$$

Zhao et al. [47] continued this approach and established another inequality using Definition 1.3. This inequality is expressed as follows:

$$\begin{aligned}\Psi^{(n)}\left(\vartheta_1 + \vartheta_2 - \frac{\xi_1 + \xi_2}{2}\right) &\leq \frac{2^{n-\frac{\varpi}{k}-1}\Gamma_k(n - \frac{\varpi}{k} + k)}{(\xi_2 - \xi_1)^{n-\frac{\varpi}{k}}}\left[\left({}^cD^{\varpi,k}_{(\vartheta_1+\vartheta_2-(\xi_1+\xi_2/2))^+}\Psi\right)(\vartheta_1 + \vartheta_2 - \xi_1)\right.\\ &\quad \left.+(-1)^n\left({}^cD^{\varpi,k}_{(\vartheta_1+\vartheta_2-(\xi_1+\xi_2/2))^-}\Psi\right)(\vartheta_1 + \vartheta_2 - \xi_2)\right]\\ &\leq \Psi^{(n)}(\vartheta_1) + \Psi^{(n)}(\vartheta_2) - \frac{1}{2}\left(\Psi^{(n)}(\xi_1) + \Psi^{(n)}(\xi_2)\right).\end{aligned} \tag{3}$$

The result given in (3) reduces to result given in (2) for $k = 1$.

Mathematical inequalities allow us to compare and establish relationship between two quantities and to state that whether one is less than, greater than, less than and equal to or greater than and equal to the other [49, 53, 54]. Due to their wide number of applications, they are now considered fundamental part of the fields like economics [30], engineering [7], information theory [43, 46] and mathematical statistics [31]. They produce much important inequalities when studied in connection to the convex functions, for instance, Fejér inequality [39], Hermite-Hadamard inequality [11], Jensen's inequality [2], Ostrowski inequality [25] and the Jensen-Mercer inequality [33].

The present-day research in the field of inequalities is being conducted in two subcategories of inequalities, i.e., discrete inequalities and continuous inequalities [38, 52]. Researchers, in both cases, try to develop new generalized inequalities by means of applying generalized convexity or integral operators, or sometimes they use both to fulfill the task [9, 50, 51]. However, there is a need to develop such an idea that will enable the researchers to conduct research in continuous and discrete version simultaneously. The concept of majorization is one such concept that meets this requirement and can be used to obtain a unified form of discrete and continuous inequalities. Majorization defines a partial order relationship between two tuples, demonstrating the degree of similarity or dispersion between elements in one tuple and those in the other.With the help of majorization, one can convert difficult problems related to optimization into simple ones, which are then easily solved [3, 20]. Recently, Faisal et al. [14] used the concept of majorization along with the convex function and established a generalized inequality of the Hermite-Hadamard-Mercer type, which simultaneously serves as both continuous and discrete inequality. This new inequality is named conticrete inequality. The word "conticerete" stands for both continuous and discrete and has been adopted using English-Language rules. This inequality is given below:

Theorem 1.4. [14] *Let Ψ be a function that exhibits convexity on the interval $\mathcal{I}$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\Omega), \boldsymbol{\tau} = (\tau_1, \ldots, \tau_\Omega), \boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\Omega)$ be three tuples with $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in \mathcal{I}$ for all $\varsigma' = 1, \ldots, \Omega$. If $\boldsymbol{\tau} \prec \boldsymbol{\xi}$ and $\boldsymbol{\varepsilon} \prec \boldsymbol{\xi}$, then*

$$\begin{aligned}\Psi\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'} + \varepsilon_{\varsigma'}}{2}\right)\right) &\leq \sum_{\varsigma'=1}^{\Omega}\Psi(\xi_{\varsigma'}) - \sum_{\varsigma'=1}^{\Omega-1}\frac{1}{\varepsilon_{\varsigma'} - \tau_{\varsigma'}}\int_{\tau_{\varsigma'}}^{\varepsilon_{\varsigma'}}\Psi(u)du\\ &\leq \sum_{\varsigma'=1}^{\Omega}\Psi(\xi_{\varsigma'}) - \sum_{\varsigma'=1}^{\Omega-1}\Psi\left(\frac{\tau_{\varsigma'} + \varepsilon_{\varsigma'}}{2}\right).\end{aligned} \tag{4}$$

To study more about the fractional form of the combined Hermite-Hadamard-Jensen-Mercer inequalities using majorization, see [4, 15, 16].

The subsequent sections of this paper are structured as follows: Section 2 consists of some basic results and definitions which will be used to obtain our desired inequalities. Section 3 contains the derivation of the new type of Hermite-Hadamard-Jensen-Mercer inequalities for majorized tuples using k-Caputo fractional operators. These new results have been expressed as Theorem 3.1 and Theorem 3.3. This section also consists of two more results established by considering the lemmas stated in Section 2. Also, this section contains some remarks presented at the end of each theorem, showing that the results obtained in this paper give rise to the existing results in the literature for some particular values. Section 4 presents the investigation of two new identities, which have been further used to obtain bounds for the main results. This section also possesses some remarks and corollaries which show that these results cover the previous results in the literature. The last section presents the conclusion of the whole work.

2. Preliminaries

This section provides essential results and definitions which are directly needed in order to obtain the main contents of this paper.

Hermite-Hadamard Inequality ([11]):
Let Ψ be a function that exhibits convexity on $[\vartheta_1, \vartheta_2]$. Then

$$\Psi\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \leq \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Psi(u)du \leq \frac{\Psi(\vartheta_1)+\Psi(\vartheta_2)}{2}. \tag{5}$$

The inequality given above, can be obtained in reverse direction when the applied function is a concave function. Researchers, have also obtained this inequality by using various classes of convex functions such as η-convex [19], coordinate convex [10], strongly convex [34] and s-convex function [12].

Jensen-Mercer Inequality ([33]):
Let Ψ be a function that exhibits convexity on $[\vartheta_1, \vartheta_2]$ and $\beta_{\varsigma'} \in [\vartheta_1, \vartheta_2]$, $\rho_{\varsigma'} \geq 0$ for all $\varsigma' = 1, 2, \ldots, \Omega$ with $\sum_{\varsigma'=1}^{\Omega} \rho_{\varsigma'} = 1$. Then

$$\Psi\left(\vartheta_1+\vartheta_2-\sum_{\varsigma'=1}^{\Omega}\rho_{\varsigma'}\beta_{\varsigma'}\right) \leq \Psi(\vartheta_1)+\Psi(\vartheta_2)-\sum_{\varsigma'=1}^{\Omega}\rho_{\varsigma'}\Psi(\beta_{\varsigma'}). \tag{6}$$

Majorization ([45]):
Given that two real tuples $\mathbf{a} = (a_1, \ldots, a_\Omega)$ and $\mathbf{b} = (b_1, \ldots, b_\Omega)$ such that their components are arranged in descending order, i.e., $a_{[\Omega]} \leq a_{[\Omega-1]} \leq \cdots \leq a_{[1]}, b_{[\Omega]} \leq b_{[\Omega-1]} \leq \cdots \leq b_{[1]}$, we say that the tuple $\mathbf{a}$ majorizes tuple $\mathbf{b}$ (in symbols $\mathbf{b} \prec \mathbf{a}$), if the conditions listed below are true:

$$\sum_{\varsigma'=1}^{s} a_{[\varsigma']} \geq \sum_{\varsigma'=1}^{s} b_{[\varsigma']} \tag{7}$$

for $s = 1, 2, \ldots, \Omega - 1$ and

$$\sum_{\varsigma'=1}^{\Omega} a_{\varsigma'} = \sum_{\varsigma'=1}^{\Omega} b_{\varsigma'}. \tag{8}$$

Based on the concept of majoriation, the extended form of the Jensen-Mercer inequality is given as follows.

Theorem 2.1. [35] [**Discrete Majorized Jensen-Mercer Inequality**] *Let Ψ be a function that exhibits convexity on the interval $\mathcal{I}$ and $(x_{i\varsigma'})$ be a matrix of order $n \times \Omega$ with $x_{i\varsigma'} \in \mathcal{I}$ for all $i = 1, 2, \ldots, n$, $\varsigma' = 1, 2, \ldots, \Omega$. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\Omega)$ be a tuple with $\omega_{\varsigma'} \in \mathcal{I}$ for $\varsigma' = 1, 2, \ldots, \Omega$ and $0 \leq \rho_i$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \rho_i = 1$. Also, assume that $\boldsymbol{\omega}$ majorizes every row of $(x_{i\varsigma'})$, then*

$$\Psi\left(\sum_{\varsigma'=1}^{\Omega}\omega_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\sum_{i=1}^{n}\rho_i x_{i\varsigma'}\right) \leq \sum_{\varsigma'=1}^{\Omega}\Psi(\omega_{\varsigma'})-\sum_{\varsigma'=1}^{\Omega-1}\sum_{i=1}^{n}\rho_i\Psi(x_{i\varsigma'}). \tag{9}$$

We also need lemmas mentioned below to get our new results [14].

Lemma 2.2. *Let Ψ be a function that exhibits convexity on the interval $\mathcal{I}$ and $(x_{i\varsigma'})$ be a matrix of order $n \times \Omega$ with $x_{i\varsigma'} \in \mathcal{I}$ for all $i = 1, 2, \ldots, n, \varsigma' = 1, 2, \ldots, \Omega$. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\Omega)$ and $\mathbf{p} = (p_1, \ldots, p_\Omega)$ be two tuples such that $\omega_{\varsigma'} \in I, p_{\varsigma'} \geq 0$ with $p_\Omega \neq 0, \eta = \frac{1}{p_\Omega}$ for all $\varsigma' = 1, \ldots, \Omega$ and $0 \leq \rho_i$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \rho_i = 1$. Also, assume that a decreasing tuple $(x_{i1}, \ldots, x_{i\varsigma'})$ satisfies $\sum_{\varsigma'=1}^{s} p_{\varsigma'} x_{i\varsigma'} \leq \sum_{\varsigma'=1}^{s} p_{\varsigma'} \omega_{\varsigma'}$ (for $s = 1, 2, \ldots, \Omega - 1$) and $\sum_{\varsigma'=1}^{\Omega} p_{\varsigma'} \omega_{\varsigma'} = \sum_{\varsigma'=1}^{\Omega} p_{\varsigma'} x_{i\varsigma'}$ for each $i = 1, 2, \ldots, n$, then*

$$\Psi\left(\sum_{\varsigma'=1}^{\Omega} \eta p_{\varsigma'} \omega_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \sum_{i=1}^{n} \eta \rho_i p_{\varsigma'} x_{i\varsigma'}\right) \leq \sum_{\varsigma'=1}^{\Omega} \eta p_{\varsigma'} \Psi(\omega_{\varsigma'}) - \sum_{\varsigma'=1}^{\Omega-1} \sum_{i=1}^{n} \eta \rho_i p_{\varsigma'} \Psi(x_{i\varsigma'}). \tag{10}$$

Lemma 2.3. *Let Ψ be a function that exhibits convexity on the interval $\mathcal{I}$ and $(x_{i\varsigma'})$ be a matrix of order $n \times \Omega$ with $x_{i\varsigma'} \in \mathcal{I}$ for all $i = 1, 2, \ldots, n, \varsigma' = 1, 2, \ldots, \Omega$. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_\Omega)$ and $\mathbf{p} = (p_1, \ldots, p_\Omega)$ be two tuples such that $\omega_{\varsigma'} \in I, p_{\varsigma'} \geq 0$ with $p_\Omega \neq 0, \eta = \frac{1}{p_\Omega}$ for all $\varsigma' = 1, \ldots, \Omega$ and $0 \leq \rho_i$ for $i = 1, 2, \ldots, n$ with $\sum_{i=1}^{n} \rho_i = 1$. Also, assume that for every $i = 1, 2, \ldots, n$, both $x_{i\varsigma'}$ and $(\omega_{\varsigma'} - x_{i\varsigma'})$ behave the same monotonicity property and $\sum_{\varsigma'=1}^{\Omega} p_{\varsigma'} \omega_{\varsigma'} = \sum_{\varsigma'=1}^{\Omega} p_{\varsigma'} x_{i\varsigma'}$ holds, then*

$$\Psi\left(\sum_{\varsigma'=1}^{\Omega} \eta p_{\varsigma'} \omega_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \sum_{i=1}^{n} \eta \rho_i p_{\varsigma'} x_{i\varsigma'}\right) \leq \sum_{\varsigma'=1}^{\Omega} \eta p_{\varsigma'} \Psi(\omega_{\varsigma'}) - \sum_{\varsigma'=1}^{\Omega-1} \sum_{i=1}^{n} \eta \rho_i p_{\varsigma'} \Psi(x_{i\varsigma'}). \tag{11}$$

3. Main Results

The novel findings of our work are discussed below within the context of k-Caputo fractional operators.

Theorem 3.1. *Let Ψ be a function provided that $\Psi \in C^n(\mathcal{I})$ and $\Psi^{(n)}$ exhibits convexity on the interval $\mathcal{I}$. Additionally, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_\Omega)$ and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\Omega)$ be three tuples with $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in \mathcal{I}$ for all $\varsigma' = 1, \ldots, \Omega$ and $\varpi > 0, \tau_\Omega > \varepsilon_\Omega$. If $\boldsymbol{\tau} \prec \boldsymbol{\xi}$ and $\boldsymbol{\varepsilon} \prec \boldsymbol{\xi}$, then*

$$\begin{aligned}
&\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'} + \varepsilon_{\varsigma'}}{2}\right)\right) \\
&\leq \frac{k\Gamma_k(n - \frac{\varpi}{k} + k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'} - \tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \tau_{\varsigma'}\right)\right. \\
&\quad \left. +(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \varepsilon_{\varsigma'}\right)\right] \\
&\leq \frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \varepsilon_{\varsigma'}\right) + \Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \tau_{\varsigma'}\right)\right] \\
&\leq \sum_{\varsigma'=1}^{\Omega} \Psi^{(n)}(\xi_{\varsigma'}) - \frac{1}{2}\left(\sum_{\varsigma'=1}^{\Omega-1} \Psi^{(n)}(\tau_{\varsigma'}) + \sum_{\varsigma'=1}^{\Omega-1} \Psi^{(n)}(\varepsilon_{\varsigma'})\right).
\end{aligned} \tag{12}$$

Proof. By using the fact that $\varrho \in [0, 1]$ and $\Psi^{(n)}$ is convex, we have

$$\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'} + \varepsilon_{\varsigma'}}{2}\right)\right)$$

$$\leq \frac{1}{2}\Bigg[\Psi^{(n)}\Bigg(\varrho\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\Bigg)+(1-\varrho)\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\Bigg)\Bigg)$$

$$+\Psi^{(n)}\Bigg(\varrho\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\Bigg)+(1-\varrho)\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\Bigg)\Bigg)\Bigg]. \tag{13}$$

The aforementioned inequality is multiplied by $\varrho^{n-\frac{\varpi}{k}-1}$, and then integrated with respect to ϱ to get

$$\frac{1}{n-\frac{\varpi}{k}}\Psi^{(n)}\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\Big(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\Big)\Bigg)$$

$$\leq \frac{1}{2}\Bigg[\int_0^1 \varrho^{n-\frac{\varpi}{k}-1}\Psi^{(n)}\Bigg(\varrho\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\Bigg)+(1-\varrho)\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\Bigg)\Bigg)d\varrho$$

$$+\int_0^1 \varrho^{n-\frac{\varpi}{k}-1}\Psi^{(n)}\Bigg(\varrho\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\Bigg)+(1-\varrho)\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\Bigg)\Bigg)d\varrho\Bigg]$$

$$=\frac{1}{2\Big(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\Big)^{n-\varpi}}\Bigg[\int_{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}}^{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}}\frac{\Psi^{(n)}(u)}{\Big(u-\Big(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\Big)\Big)^{\frac{\varpi}{k}-n+1}}du$$

$$+\int_{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}}^{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}}\frac{\Psi^{(n)}(u)}{\Big(\Big(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\Big)-u\Big)^{\frac{\varpi}{k}-n+1}}du\Bigg]. \tag{14}$$

We see that the required condition $\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}>\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}$ for the applicability of fractional operator can be easily established by taking into consideration $\tau_\Omega>\varepsilon_\Omega$. After this, the k-Caputo fractional operators are utilized in (14) to get

$$\frac{1}{n-\frac{\varpi}{k}}\Psi^{(n)}\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\Big(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\Big)\Bigg)$$

$$\leq\frac{k\Gamma_k(n-\frac{\varpi}{k})}{2\Big(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\Big)^{n-\frac{\varpi}{k}}}\Bigg[\Bigg({}^{c}D^{\varpi,k}_{\big(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\big)^{+}}\Psi\Bigg)\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\Bigg)$$

$$+(-1)^n\Bigg({}^{c}D^{\varpi,k}_{\big(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\big)^{-}}\Psi\Bigg)\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\Bigg)\Bigg], \tag{15}$$

which gives

$$\Psi^{(n)}\Bigg(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\Big(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\Big)\Bigg)$$

$$\leq \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right.$$
$$\left.+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]. \tag{16}$$

Consequently, we have derived the first portion of (12). Now, the the convexity of $\Psi^{(n)}$ is used to derive the second portion of the inequality (12), as shown below:

$$\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)$$
$$\leq \varrho\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+(1-\varrho)\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right) \tag{17}$$

and

$$\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)$$
$$\leq \varrho\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+(1-\varrho)\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right). \tag{18}$$

By combining (17) and (18) and then utilizing Theorem 2.1 for $n = 1$ and $\rho_1 = 1$, we have

$$\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)$$
$$+\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)$$
$$\leq \Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)$$
$$\leq 2\sum_{\varsigma'=1}^{\Omega}\Psi^{(n)}(\xi_{\varsigma'})-\left(\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\tau_{\varsigma'})+\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\varepsilon_{\varsigma'})\right). \tag{19}$$

By multiplying the aforementioned inequality with $\varrho^{n-\frac{\varpi}{k}-1}$ and then integrating the acquired inequality with respect to ϱ, we achieve the remaining portion of (12). □

Remark 3.2. *(i) By setting $k = 1$ in* (12)*, we arrive at the inequality* (11) *proved in [16].*

(ii) By inserting $n = 1, k = 1, \Omega = 2$ and $\varpi = 0$ in (12)*, we obtain the results mentioned in [28].*

(iii) By inserting $k = 1, \Omega = 2, \tau_1 = \vartheta_1$ and $\varepsilon_1 = \vartheta_2$, we attain the inequality (2.2) *constructed in [18].*

We provide another form of our finding in the following manner, following identical steps to that used in the aforementioned theorem.

Theorem 3.3. *If the conditions stated in Theorem* 3.1 *hold true, then*

$$\begin{aligned}
&\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\\
&\leq \frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^-}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right.\\
&\quad\left.+\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^+}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right]\\
&\leq \sum_{\varsigma'=1}^{\Omega}\Psi^{(n)}(\xi_{\varsigma'})-\frac{1}{2}\left[\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\tau_{\varsigma'})+\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\varepsilon_{\varsigma'})\right].
\end{aligned}\tag{20}$$

Proof. The convexity of $\Psi^{(n)}$ and $\varrho\in[0,1]$, allows us to write that

$$\begin{aligned}
\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)&\leq\frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)\right.\\
&\quad\left.+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)\right].
\end{aligned}\tag{21}$$

The preceding inequality is multiplied by $\varrho^{n-\frac{\varpi}{k}-1}$, and then integrated with respect to ϱ to get

$$\begin{aligned}
&\frac{1}{n-\frac{\varpi}{k}}\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\\
&\leq\frac{1}{2}\left[\int_0^1\varrho^{n-\frac{\varpi}{k}-1}\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)d\varrho\right.\\
&\quad\left.+\int_0^1\varrho^{n-\frac{\varpi}{k}-1}\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)d\varrho\right]\\
&=\frac{1}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\varepsilon_{\varsigma'}-\tau_{\varsigma'}}{2}\right)\right)^{n-\frac{\varpi}{k}}}\left[\int_{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}}^{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)}\frac{\Psi^{(n)}(u)}{\left(u-\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)^{\frac{\varpi}{k}-n+1}}du\right.\\
&\quad\left.+\int_{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)}^{\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}}\frac{\Psi^{(n)}(u)}{\left(\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)-u\right)^{\frac{\varpi}{k}-n+1}}du\right].
\end{aligned}\tag{22}$$

For applying the k-Caputo fractional operators in (22), the required necessary condition that

$$\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)>\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}$$

and

$$\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)<\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'},$$

can be constructed by taking $\tau_\Omega > \varepsilon_\Omega$.
Now, (22) implies

$$\begin{aligned}&\frac{1}{n-\frac{\varpi}{k}}\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\\&=\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k})}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[(-1)^n\left({}^cD^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^-}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right.\\&\left.\quad+\left({}^cD^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^+}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right],\end{aligned}$$

which gives

$$\begin{aligned}&\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\\&\leq\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[(-1)^n\left({}^cD^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^-}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right.\\&\left.\quad+\left({}^cD^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^+}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right].\end{aligned}\tag{23}$$

Thus, the first portion of (20) has been accomplished. We continue to demonstrate the second portion of (20) by using Theorem 2.1 for $n=2$, $\rho_1=\frac{\varrho}{2}$ and $\rho_2=\frac{2-\varrho}{2}$, in the subsequent way:

$$\begin{aligned}&\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)\\&\leq\sum_{\varsigma'=1}^{\Omega}\Psi^{(n)}(\xi_{\varsigma'})-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\tau_{\varsigma'})+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\varepsilon_{\varsigma'})\right)\end{aligned}\tag{24}$$

and

$$\begin{aligned}&\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)\\&\leq\sum_{\varsigma'=1}^{\Omega}\Psi^{(n)}(\xi_{\varsigma'})-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\varepsilon_{\varsigma'})+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\tau_{\varsigma'})\right).\end{aligned}\tag{25}$$

Adding (24) and (25), we get

$$\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)$$
$$\leq 2\sum_{\varsigma'=1}^{\Omega}\Psi^{(n)}(\xi_{\varsigma'})-\left(\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\tau_{\varsigma'})+\sum_{\varsigma'=1}^{\Omega-1}\Psi^{(n)}(\varepsilon_{\varsigma'})\right). \tag{26}$$

By multiplying the aforementioned inequality with $\varrho^{n-\frac{\varpi}{k}-1}$ and then integrating the acquired inequality with respect to ϱ, we attain the remaining portion of (20). □

Remark 3.4. *(i) By fixing $k = 1$ in (20), we gain the inequality (25) established in [16].*

(ii) By fixing $\Omega = 2$ in (20), we acquire the inequality (8) derived in [47].

We derive the following additional result with the assistance of Lemma 2.2.

Theorem 3.5. *Let Ψ be a function provided that $\Psi \in C^n(\mathcal{I})$ and $\Psi^{(n)}$ exhibits convexity on the interval $\mathcal{I}$. Additionally, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_\Omega)$, $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\Omega)$ and $\mathbf{p} = (p_1, \ldots, p_\Omega)$ be four tuples where $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in I$, $p_{\varsigma'} \geq 0$ with $p_\Omega \neq 0$, $\eta = \frac{1}{p_\Omega}$ for all $\varsigma' = 1, \ldots, \Omega$, and $\varpi > 0$, $\tau_\Omega > \varepsilon_\Omega$. If $\boldsymbol{\tau}$ and $\boldsymbol{\varepsilon}$ are decreasing tuples with*

$$\sum_{\varsigma'=1}^{s}p_{\varsigma'}\tau_{\varsigma'}\leq\sum_{\varsigma'=1}^{s}p_{\varsigma'}\xi_{\varsigma'},\ \sum_{\varsigma'=1}^{s}p_{\varsigma'}\varepsilon_{\varsigma'}\leq\sum_{\varsigma'=1}^{s}p_{\varsigma'}\xi_{\varsigma'}$$

for $s = 1, \ldots, \Omega - 1$ and

$$\sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\xi_{\varsigma'}=\sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\tau_{\varsigma'},\ \sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\xi_{\varsigma'}=\sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\varepsilon_{\varsigma'},$$

then

$$\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\eta\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{p_{\varsigma'}\tau_{\varsigma'}+p_{\varsigma'}\varepsilon_{\varsigma'}}{2}\right)\right)$$
$$\leq\frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\eta p_{\varsigma'}\varepsilon_{\varsigma'}-\eta p_{\varsigma'}\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right.$$
$$\left.+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right]$$
$$\leq\frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right]$$
$$\leq\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\Psi^{(n)}(\xi_{\varsigma'})-\frac{1}{2}\left[\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\Psi^{(n)}(\tau_{\varsigma'})+\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\Psi^{(n)}(\varepsilon_{\varsigma'})\right]. \tag{27}$$

Proof. By using the fact that $\varrho \in [0, 1]$ and $\Psi^{(n)}$ is convex, we have

$$\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\eta\sum_{\varsigma'=1}^{\Omega-1}p_{\varsigma'}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)$$

$$\leq \frac{1}{2}\left[\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right)\right.$$
$$\left.+\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right)\right]. \tag{28}$$

The aforementioned inequality is multiplied by $\varrho^{n-\frac{\varpi}{k}-1}$, and then integrated with respect to ϱ to get

$$\frac{1}{n-\frac{\varpi}{k}}\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\eta\sum_{\varsigma'=1}^{\Omega-1}p_{\varsigma'}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)$$
$$\leq \frac{1}{2}\left[\int_0^1\varrho^{n-\frac{\varpi}{k}-1}\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right)d\varrho\right.$$
$$\left.+\int_0^1\varrho^{n-\frac{\varpi}{k}-1}\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right)d\varrho\right]$$
$$=\frac{1}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\eta p_{\varsigma'}\varepsilon_{\varsigma'}-\eta p_{\varsigma'}\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}$$
$$\times\left[\int_{\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}}^{\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}}\frac{\Psi^{(n)}(u)}{\left(u-\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right)^{\frac{\varpi}{k}-n+1}}du\right.$$
$$\left.+\int_{\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}}^{\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}}\frac{\Psi^{(n)}(u)}{\left(\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)-u\right)^{\frac{\varpi}{k}-n+1}}du\right]. \tag{29}$$

By assuming $\tau_\Omega > \varepsilon_\Omega$, we can easily establish that $\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}<\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}$, after which k-Caputo fractional operators are utilized in (29) to obtain

$$\frac{1}{n-\frac{\varpi}{k}}\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\eta\sum_{\varsigma'=1}^{\Omega-1}p_{\varsigma'}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)$$
$$\leq\frac{k\Gamma_k(n-\frac{\varpi}{k})}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\eta p_{\varsigma'}\varepsilon_{\varsigma'}-\eta p_{\varsigma'}\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right.$$
$$\left.+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right],$$

which leads to

$$\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\eta\sum_{\varsigma'=1}^{\Omega-1}p_{\varsigma'}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)$$

$$\leq \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\eta p_{\varsigma'}\varepsilon_{\varsigma'}-\eta p_{\varsigma'}\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right.$$
$$\left.+(-1)^{n}\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right]. \tag{30}$$

Consequently, we have determined the first portion of (27). Further, the convexity of $\Psi^{(n)}$ is employed to determine the remaining portions of (27) as follows:

$$\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right)$$
$$\leq \varrho\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)+(1-\varrho)\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right) \tag{31}$$

and

$$\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right)$$
$$\leq \varrho\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+(1-\varrho)\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right). \tag{32}$$

By combining (31) and (32) and then using Lemma 2.2 for $\rho_1=\varrho$, $\rho_2=1-\varrho$ and $n=2$, we gain

$$\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right)$$
$$+\Psi^{(n)}\left(\varrho\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+(1-\varrho)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right)$$
$$\leq \Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)$$
$$\leq 2\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\Psi^{(n)}(\xi_{\varsigma'})-\left(\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\Psi^{(n)}(\tau_{\varsigma'})+\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\Psi^{(n)}(\varepsilon_{\varsigma'})\right).$$

By multiplying the aforementioned inequality with $\varrho^{n-\frac{\varpi}{k}-1}$ and then integrating the acquired inequality with respect to ϱ, we attain the remaining portions of (27). □

The next result is deduced through the utilization of Lemma 2.3.

Theorem 3.6. *Let Ψ be a function provided that $\Psi \in C^n(\mathcal{I})$ and $\Psi^{(n)}$ exhibits convexity on the interval $\mathcal{I}$. Additionally, let $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_\Omega)$, $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_\Omega)$, $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_\Omega)$ and $\mathbf{p}=(p_1,\ldots,p_\Omega)$ be four tuples where $\xi_{\varsigma'},\tau_{\varsigma'},\varepsilon_{\varsigma'}\in I$, $p_{\varsigma'}\geq 0$ with $p_\Omega\neq 0$, $\eta=\frac{1}{p_\Omega}$ for all $\varsigma'=1,\ldots,\Omega$, and $\varpi>0$, $\tau_\Omega>\varepsilon_\Omega$. If $\boldsymbol{\xi}-\boldsymbol{\tau}$, $\boldsymbol{\tau}$, $\boldsymbol{\xi}-\boldsymbol{\varepsilon}$ and $\boldsymbol{\varepsilon}$ behave the same monotonicity property and*

$$\sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\xi_{\varsigma'}=\sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\tau_{\varsigma'},\ \sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\xi_{\varsigma'}=\sum_{\varsigma'=1}^{\Omega}p_{\varsigma'}\varepsilon_{\varsigma'}$$

hold, then

$$
\begin{aligned}
&\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\eta\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{p_{\varsigma'}\tau_{\varsigma'}+p_{\varsigma'}\varepsilon_{\varsigma'}}{2}\right)\right)\\
&\leq \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\eta p_{\varsigma'}\varepsilon_{\varsigma'}-\eta p_{\varsigma'}\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right.\\
&\quad\left.+(-1)^{n}\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)\right]\\
&\leq \frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\tau_{\varsigma'}\right)\right]\\
&\leq \sum_{\varsigma'=1}^{\Omega}\eta p_{\varsigma'}\Psi^{(n)}(\xi_{\varsigma'})-\frac{1}{2}\left(\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\Psi^{(n)}(\tau_{\varsigma'})+\sum_{\varsigma'=1}^{\Omega-1}\eta p_{\varsigma'}\Psi^{(n)}(\varepsilon_{\varsigma'})\right). \qquad (33)
\end{aligned}
$$

Proof. Lemma 2.3 and techniques identical to those outlined in the proof of Theorem 3.5 are used to demonstrate (33). □

Remark 3.7. *In a similar way, we may also derive the weighted versions of Theorem 3.3.*

4. Bounds Associated with the Main Results

This section presents the derivation of two additional identities that will assist us in constructing bounds for our major findings.

Lemma 4.1. *Let Ψ be a differentiable function provided that $\Psi \in C^{n+1}(I)$ and $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_\Omega)$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\Omega)$ be three tuples where $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in I$, for all $\varsigma' = 1, \ldots, \Omega$, $\varrho \in [0, 1]$, $\varpi > 0$. If $\Psi^{n+1} \in L(I)$, then*

$$
\begin{aligned}
&\frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right]\\
&\quad-\frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right.\\
&\quad\left.+(-1)^{n}\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]\\
&=\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\int_0^1\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}(\varrho\tau_{\varsigma'}+(1-\varrho)\varepsilon_{\varsigma'})\right)d\varrho. \qquad (34)
\end{aligned}
$$

Proof. In order to determine our intended outcome, we begin by assuming that

$$
I=\int_0^1\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}(\varrho\tau_{\varsigma'}+(1-\varrho)\varepsilon_{\varsigma'})\right)d\varrho
$$

$$
\begin{aligned}
&= \int_0^1 \varrho^{n-\frac{\varpi}{k}} \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right) d\varrho \\
&\quad - \int_0^1 (1-\varrho)^{n-\frac{\varpi}{k}} \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right) d\varrho \\
&= I_1 - I_2. \qquad (35)
\end{aligned}
$$

Assuming that

$$
\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \varepsilon_{\varsigma'} < \sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \tau_{\varsigma'}
$$

and employing the integration by parts formula, we determine I_1 and I_2 in the following manner:

$$
\begin{aligned}
I_1 &= \int_0^1 \varrho^{n-\frac{\varpi}{k}} \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right) d\varrho \\
&= \left. \frac{\varrho^{n-\frac{\varpi}{k}} \Psi^{(n)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right)}{\sum_{\varsigma'=1}^{\Omega-1} (\varepsilon_{\varsigma'} - \tau_{\varsigma'})} \right|_0^1 \\
&\quad - \frac{n-\frac{\varpi}{k}}{\sum_{\varsigma'=1}^{\Omega-1} (\varepsilon_{\varsigma'} - \tau_{\varsigma'})} \int_0^1 \varrho^{n-\frac{\varpi}{k}-1} \Psi^{(n)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right) d\varrho \\
&= \frac{\Psi^{(n)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \tau_{\varsigma'} \right)}{\sum_{\varsigma'=1}^{\Omega-1} (\varepsilon_{\varsigma'} - \tau_{\varsigma'})} \\
&\quad - \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum_{\varsigma'=1}^{\Omega-1} (\varepsilon_{\varsigma'} - \tau_{\varsigma'}) \right)^{n-\frac{\varpi}{k}+1}} (-1)^n \left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \tau_{\varsigma'} \right)^-} \Psi \right) \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \varepsilon_{\varsigma'} \right). \qquad (36)
\end{aligned}
$$

Likewise,

$$
\begin{aligned}
I_2 &= \int_0^1 (1-\varrho)^{n-\frac{\varpi}{k}} \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right) d\varrho \\
&= \left. \frac{(1-\varrho)^{n-\frac{\varpi}{k}} \Psi^{(n)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right)}{\sum_{\varsigma'=1}^{\Omega-1} (\varepsilon_{\varsigma'} - \tau_{\varsigma'})} \right|_0^1 \\
&\quad + \frac{n-\frac{\varpi}{k}}{\sum_{\varsigma'=1}^{\Omega-1} (\varepsilon_{\varsigma'} - \tau_{\varsigma'})} \int_0^1 (1-\varrho)^{n-\frac{\varpi}{k}-1} \Psi^{(n)} \left(\sum_{\varsigma'=1}^{\Omega} \xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1} \left(\varrho\tau_{\varsigma'} + (1-\varrho)\varepsilon_{\varsigma'} \right) \right) d\varrho
\end{aligned}
$$

$$= -\frac{\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)}{\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})} + \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}+1}}\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^+}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right). \tag{37}$$

Now, substituting (36) and (37) in (35) and then multiplying both sides by $\frac{\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})}{2}$, we obtain (34). This complete the proof. □

Remark 4.2. *(i) By setting $\tau_1 = \vartheta_1, k = 1$, $\varepsilon_1 = \vartheta_2$ and $\Omega = 2$, then equality* (34) *yields the equality* (2.1) *established in [18].*

(ii) By inserting $\Omega = 2, \varpi = 0, k = 1, n = 1, \tau_1 = \vartheta_1$ and $\varepsilon_1 = \vartheta_2$, then equality (34) *yields the equality* (2.1) *that is derived in [11].*

Lemma 4.1 is now used to generate the subsequent additional results:

Theorem 4.3. *Let Ψ be a differentiable function provided that $\Psi \in C^{n+1}(\mathcal{I})$ and $|\Psi^{n+1}|$ exhibits convexity on the interval $\mathcal{I}$. Additionally, let $\boldsymbol{\xi} = (\xi_1,\ldots,\xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1,\ldots,\tau_\Omega)$, and $\boldsymbol{\varepsilon} = (\varepsilon_1,\ldots,\varepsilon_\Omega)$ be three tuples where $\xi_{\varsigma'},\tau_{\varsigma'},\varepsilon_{\varsigma'} \in I$, for all $\varsigma' = 1,\ldots,\Omega$, and $\varpi > 0$, $\tau_\Omega > \varepsilon_\Omega$. If $\boldsymbol{\tau} \prec \boldsymbol{\xi}$ and $\boldsymbol{\varepsilon} \prec \boldsymbol{\xi}$, then*

$$\left|\frac{\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)}{2} - \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^+}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right) + (-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^-}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]\right|$$

$$\leq \frac{1}{n-\frac{\varpi}{k}+1}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(1-\frac{1}{2^{n-\frac{\varpi}{k}}}\right)\left[\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right| - \frac{\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right| + \sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|}{2}\right]. \tag{38}$$

Proof. Lemma 4.1 is used first, followed by Theorem 2.1 for $\rho_1 = \varrho$, $\rho_2 = 1-\varrho$ and $n = 2$ as a result of convexity of $|\Psi^{(n+1)}|$ to establish the following

$$\left|\frac{\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)}{2} - \frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\right.$$

$$\times\left[\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+(-1)^{n}\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]\Bigg|$$

$$\leq\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\int_{0}^{1}\left|\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)\right|$$

$$\times\left[\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right)\right]d\varrho$$

$$=\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left[\int_{0}^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)\left\{\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|\right.\right.$$

$$\left.-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right)\right\}d\varrho+\int_{\frac{1}{2}}^{1}\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)$$

$$\left.\times\left[\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right)\right]d\varrho\right]$$

$$=\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(I_1+I_2\right). \tag{39}$$

Now, we determine I_1 and I_2, as follows:

$$I_1=\int_{0}^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)\left[\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\right)\right]d\varrho$$

$$=\left(\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|\right)\left(\int_{0}^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)d\varrho\right)-\left[\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|\right.$$

$$\left.\times\int_{0}^{\frac{1}{2}}\varrho\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)d\varrho+\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\int_{0}^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)(1-\varrho)d\varrho\right]$$

$$=\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|\left(\frac{1-2^{\frac{\varpi}{k}-n}}{n-\frac{\varpi}{k}+1}\right)-\left[\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|\left(\int_{0}^{\frac{1}{2}}\varrho(1-\varrho)^{n-\frac{\varpi}{k}}d\varrho-\int_{0}^{\frac{1}{2}}\varrho^{n-\frac{\varpi}{k}+1}d\varrho\right)\right.$$

$$\left.+\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\left(\int_{0}^{\frac{1}{2}}(1-\varrho)^{n-\frac{\varpi}{k}+1}d\varrho-\int_{0}^{\frac{1}{2}}(1-\varrho)\varrho^{n-\frac{\varpi}{k}}d\varrho\right)\right]$$

$$=\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|\left(\frac{1-2^{\frac{\varpi}{k}-n}}{n-\frac{\varpi}{k}+1}\right)-\left[\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|\right.$$

$$\left.\times\left(\frac{1}{(n-\frac{\varpi}{k}+1)(n-\frac{\varpi}{k}+2)}-\frac{2^{\frac{\varpi}{k}-n-1}}{n-\frac{\varpi}{k}+1}\right)+\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\left(\frac{1}{n-\frac{\varpi}{k}+2}-\frac{2^{\frac{\varpi}{k}-n-1}}{n-\frac{\varpi}{k}+1}\right)\right] \tag{40}$$

and

$$I_2=\int_{\frac{1}{2}}^{1}\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)\left[\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\right)\right]d\varrho$$

$$= \left(\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|\right)\left(\int_{\frac{1}{2}}^{1}\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)d\varrho\right)-\left[\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|\right.$$

$$\left.\times\int_{\frac{1}{2}}^{1}\varrho\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)d\varrho+\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\int_{\frac{1}{2}}^{1}\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)(1-\varrho)d\varrho\right]$$

$$= \sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|\left(\frac{1-2^{\frac{\varpi}{k}-n}}{n-\frac{\varpi}{k}+1}\right)-\left[\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|\left(\int_{\frac{1}{2}}^{1}\varrho^{n-\frac{\varpi}{k}+1}d\varrho-\int_{\frac{1}{2}}^{1}\varrho(1-\varrho)^{n-\frac{\varpi}{k}}d\varrho\right)\right.$$

$$\left.+\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\left(\int_{\frac{1}{2}}^{1}(1-\varrho)\varrho^{n-\frac{\varpi}{k}}d\varrho-\int_{\frac{1}{2}}^{1}(1-\varrho)^{n-\frac{\varpi}{k}+1}d\varrho\right)\right]$$

$$= \sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|\left(\frac{1-2^{\frac{\varpi}{k}-n}}{n-\frac{\varpi}{k}+1}\right)-\left[\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|\left(\frac{1}{n-\frac{\varpi}{k}+2}-\frac{2^{\frac{\varpi}{k}-n-1}}{n-\frac{\varpi}{k}+1}\right)\right.$$

$$\left.+\sum_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|\left(\frac{1}{(n-\frac{\varpi}{k}+1)(n-\frac{\varpi}{k}+2)}-\frac{2^{\frac{\varpi}{k}-n-1}}{n-\frac{\varpi}{k}+1}\right)\right]. \tag{41}$$

By substituting (40) and (41) in (39), we obtain (38). This ends the proof. □

Remark 4.4. *When we substitute $\Omega = 2, k = 1, \tau_1 = \vartheta_1, \varepsilon_1 = \vartheta_2$, then inequality* (38) *gives*

$$\left|\frac{\Psi^{(n)}(\vartheta_1)+\Psi^{(n)}(\vartheta_2)}{2}-\frac{\Gamma(n-\varpi+1)}{2(\vartheta_2-\vartheta_1)^{n-\varpi}}\left[{}^{c}D^{\varpi}_{\vartheta_1^+}\Psi(\vartheta_2)+(-1)^{n\,c}D^{\varpi}_{\vartheta_2^-}\Psi(\vartheta_1)\right]\right|$$

$$\leq\frac{\vartheta_2-\vartheta_1}{n-\varpi+1}\left(1-\frac{1}{2^{n-\varpi}}\right)\left[\frac{|\Psi^{(n+1)}(\vartheta_1)|+|\Psi^{(n+1)}(\vartheta_2)|}{2}\right],$$

which is mentioned in [18].

Theorem 4.5. *Let Ψ be a differentiable function provided that $\Psi \in C^{n+1}(\mathcal{I})$ and $|\Psi^{(n+1)}|^q$ $(q > 1)$ exhibits convexity on the interval $\mathcal{I}$. Additionally, let $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_\Omega)$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\Omega)$ be three tuples where $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in I$, for all $\varsigma' = 1, \ldots, \Omega$, and $\varpi > 0$, $\tau_\Omega > \varepsilon_\Omega$. If $\boldsymbol{\tau} \prec \boldsymbol{\xi}$, $\boldsymbol{\varepsilon} \prec \boldsymbol{\xi}$, then*

$$\left|\frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right]-\frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\varepsilon_{\varsigma'}-\tau_{\varsigma'}\right)\right)^{n-\frac{\varpi}{k}}}\right.$$

$$\left.\times\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^+}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^-}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]\right|$$

$$\leq\frac{1}{n-\frac{\varpi}{k}+1}\sum_{\varsigma'=1}^{\Omega-1}\left|\varepsilon_{\varsigma'}-\tau_{\varsigma'}\right|\left(1-\frac{1}{2^{n-\frac{\varpi}{k}}}\right)\left[\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|^q-\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\tau_{\varsigma'})\right|^q+\sum\limits_{\varsigma'=1}^{\Omega-1}\left|\Psi^{(n+1)}(\varepsilon_{\varsigma'})\right|^q}{2}\right]^{\frac{1}{q}}. \tag{42}$$

Proof. Lemma 4.1 is used first, followed by power mean inequality, to establish the following:

$$\begin{aligned}
&\Bigg|\frac{1}{2}\left[\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)+\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right]-\frac{k\Gamma_k(n-\frac{\varpi}{k}+k)}{2\left(\sum_{\varsigma'=1}^{\Omega-1}\left(\varepsilon_{\varsigma'}-\tau_{\varsigma'}\right)\right)^{n-\frac{\varpi}{k}}}\\
&\times\left[\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)+(-1)^{n}\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]\Bigg|\\
&\leq\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\int_0^1\left|\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right|d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1\left|\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right|\right.\\
&\left.\times\left|\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}(\varrho\tau_{\varsigma'}+(1-\varrho)\varepsilon_{\varsigma'})\right)\right|^{q}d\varrho\right)^{\frac{1}{q}}\\
&=\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\int_0^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)d\varrho+\int_{\frac{1}{2}}^1\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)d\varrho\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right|\left|\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}(\varrho\tau_{\varsigma'}+(1-\varrho)\varepsilon_{\varsigma'})\right)\right|^{q}d\varrho\right)^{\frac{1}{q}}.
\end{aligned}\tag{43}$$

Through the utilization of Theorem 2.1 for $\rho_1=\varrho$, $\rho_2=1-\varrho$ and $n=2$ in (43) as a result of convexity of $|\Psi^{n+1}|^q$, we have

$$\begin{aligned}
&=\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\int_0^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)d\varrho+\int_{\frac{1}{2}}^1\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)d\varrho\right)^{1-\frac{1}{q}}\\
&\times\left(\int_0^1\left|\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right|\right)\left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q\right.\right.\\
&\left.\left.+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right)d\varrho\right)^{\frac{1}{q}}\\
&=\frac{1}{2}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\int_0^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)d\varrho+\int_{\frac{1}{2}}^1\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)d\varrho\right)^{1-\frac{1}{q}}\\
&\times\left[\int_0^{\frac{1}{2}}\left((1-\varrho)^{n-\frac{\varpi}{k}}-\varrho^{n-\frac{\varpi}{k}}\right)\left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q\right.\right.\right.\\
&\left.\left.+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right)\right)d\varrho+\int_{\frac{1}{2}}^1\left(\varrho^{n-\frac{\varpi}{k}}-(1-\varrho)^{n-\frac{\varpi}{k}}\right)\left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q\right.\\
&\left.\left.-\left(\varrho\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q+(1-\varrho)\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right)\right)d\varrho\right]^{\frac{1}{q}}.
\end{aligned}\tag{44}$$

Thus, the completion of (42) is reached after computing the integrals mentioned in (44). □

To get more outcomes, we introduce the following lemma.

Lemma 4.6. *Let Ψ be a differentiable function provided that $\Psi \in C^{n+1}(\mathcal{I})$ and $\boldsymbol{\xi} = (\xi_1, \dots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \dots, \tau_\Omega)$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_\Omega)$ be three tuples where $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in I$, for all $\varsigma' = 1, \dots, \Omega$, $\varrho \in [0,1]$, $\varpi > 0$. If $\Psi^{n+1} \in L(I)$, then*

$$\begin{aligned}
&\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right.\\
&\left.+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]-\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\\
&=\frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\left[\int_0^1\varrho^{n-\frac{\varpi}{k}}\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)d\varrho\right.\\
&\left.-\int_0^1\varrho^{n-\frac{\varpi}{k}}\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)d\varrho\right]. \qquad (45)
\end{aligned}$$

Proof. We demonstrate (45) by using techniques identical to those outlined in the proof of Lemma 4.1. □

Remark 4.7. *When we insert $\Omega = 2, k = 1$, $\tau_1 = \vartheta_1$ and $\varepsilon_1 = \vartheta_2$ in* (45)*, then we acquire the equality* (3.1) *that is established in [26].*

The subsequent results are now derived using Lemma 4.6.

Theorem 4.8. *Let Ψ be a differentiable function provided that $\Psi \in C^{n+1}(\mathcal{I})$ and $|\Psi^{n+1}|$ exhibits convexity on the interval $\mathcal{I}$. Additionally, let $\boldsymbol{\xi} = (\xi_1, \dots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \dots, \tau_\Omega)$, and $\boldsymbol{\varepsilon} = (\varepsilon_1, \dots, \varepsilon_\Omega)$ be three tuples where $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in \mathcal{I}$, for all $\varsigma' = 1, \dots, \Omega$, and $\varpi > 0$ $\tau_\Omega > \varepsilon_\Omega$. If $\boldsymbol{\tau} \prec \boldsymbol{\xi}$, $\boldsymbol{\varepsilon} \prec \boldsymbol{\xi}$, then*

$$\begin{aligned}
&\left|\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right.\right.\\
&\left.\left.+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]-\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\right|\\
&\leq\frac{1}{2(n-\frac{\varpi}{k}+1)}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left[\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\frac{1}{2}\left[\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|+\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right]\right]. \qquad (46)
\end{aligned}$$

Proof. Lemma 4.6 is used first, followed by Theorem 2.1 for $\rho_1 = \frac{2-\varrho}{2}$, $n = 2$, and $\rho_2 = \frac{\varrho}{2}$ to establish the following

$$\left|\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right.\right.$$

$$+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\Bigg]-\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\Bigg|$$

$$\leq \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left[\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left(\sum_{\varsigma'=1}^{\Omega}\left|\Psi^{(n+1)}(\xi_{\varsigma'})\right|-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right)\right)d\varrho\right.$$
$$\left.+\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|\right)\right)d\varrho\right]$$
$$=\frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left[\frac{\sum\limits_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|}{n-\frac{\varpi}{k}+1}-\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|}{n-\frac{\varpi}{k}+1}+\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|}{2(n-\frac{\varpi}{k}+2)}-\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|}{2(n-\frac{\varpi}{k}+2)}\right.$$

$$\left.+\frac{\sum\limits_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|}{n-\frac{\varpi}{k}+1}-\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|}{n-\frac{\varpi}{k}+1}+\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|}{2(n-\frac{\varpi}{k}+2)}-\frac{\sum\limits_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|}{2(n-\frac{\varpi}{k}+2)}\right]$$
$$=\frac{1}{2(n-\frac{\varpi}{k}+1)}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left[\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\frac{1}{2}\left[\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|+\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right]\right].$$

Consequently, the desired proof is accomplished. □

Corollary 4.9. *We gain the following inequality that is derived in [11], by fixing $\varpi = 0$, $n = 1$, $\Omega = 2$, $k = 1$, $\tau_1 = \vartheta_1$, and $\varepsilon_1 = \vartheta_2$ in* (46).

$$\left|\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}\Psi(u)du-f\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right|\leq\frac{\left(|\vartheta_2-\vartheta_1|\right)\left(|\Psi'(\vartheta_1)|+|\Psi'(\vartheta_2)|\right)}{8}.$$

Theorem 4.10. *Let Ψ a differentiable function provided that $\Psi\in C^{n+1}(\mathcal{I})$ and $|\Psi^{(n+1)}|^q$ exhibits convexity on the interval $\mathcal{I}$ such that $q>1$, $\frac{1}{p}+\frac{1}{q}=1$. Additionally, let $\boldsymbol{\xi}=(\xi_1,\ldots,\xi_\Omega)$, $\boldsymbol{\tau}=(\tau_1,\ldots,\tau_\Omega)$, and $\boldsymbol{\varepsilon}=(\varepsilon_1,\ldots,\varepsilon_\Omega)$ be three tuples where $\xi_{\varsigma'},\tau_{\varsigma'},\varepsilon_{\varsigma'}\in\mathcal{I}$, for all $\varsigma'=1,\ldots,\Omega$, and $\varpi>0$, $\tau_\Omega>\varepsilon_\Omega$. If $\boldsymbol{\tau}\prec\boldsymbol{\xi}$, $\boldsymbol{\varepsilon}\prec\boldsymbol{\xi}$, then*

$$\Bigg|\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right.$$
$$+(-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\Bigg]-\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\Bigg|$$
$$\leq\frac{1}{16}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\frac{4}{np-\frac{\varpi}{k}p+1}\right)^{\frac{1}{p}}\left[4^{\frac{1}{q}}\cdot 2\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\left(3^{\frac{1}{q}}+1\right)\left(\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|\right.\right.$$
$$\left.\left.+\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right)\right]. \tag{47}$$

Proof. Lemma 4.6 is used first, followed by Hölder inequality to obtain the below inequality:

$$\left| \frac{2^{n-\frac{\varpi}{k}-1} k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}} \left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}} \Psi \right) \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'} \right) \right.\right.$$
$$\left.\left. +(-1)^n \left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}} \Psi \right) \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'} \right) \right] - \Psi^{(n)} \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right) \right) \right|$$

$$\leq \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}| \left[\left(\int_0^1 \varrho^{(n-\frac{\varpi}{k})p} d\varrho \right)^{\frac{1}{p}} \left(\int_0^1 \left| \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'} + \frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'} \right) \right) \right|^q d\varrho \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_0^1 \varrho^{(n-\frac{\varpi}{k})p} d\varrho \right)^{\frac{1}{p}} \left(\int_0^1 \left| \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'} + \frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'} \right) \right) \right|^q d\varrho \right)^{\frac{1}{q}} \right]$$

$$= \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}| \left(\int_0^1 \varrho^{(n-\frac{\varpi}{k})p} d\varrho \right)^{\frac{1}{p}} \left[\left(\int_0^1 \left| \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'} + \frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'} \right) \right) \right|^q d\varrho \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_0^1 \left| \Psi^{(n+1)} \left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'} - \left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'} + \frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'} \right) \right) \right|^q d\varrho \right)^{\frac{1}{q}} \right]. \tag{48}$$

Now, setting $n = 2$, $\rho_1 = \frac{2-\varrho}{2}$, $\rho_2 = \frac{\varrho}{2}$ to utilize Theorem 2.1 in (48), we obtain

$$= \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}| \left(\frac{1}{np-\frac{\varpi}{k}p+1} \right)^{\frac{1}{p}}$$
$$\times \left[\left(\int_0^1 \left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q - \left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q + \frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q \right) \right) d\varrho \right)^{\frac{1}{q}} \right.$$
$$\left. + \left(\int_0^1 \left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q - \left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q + \frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q \right) \right) d\varrho \right)^{\frac{1}{q}} \right]$$
$$= \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}| \left(\frac{1}{np-\frac{\varpi}{k}p+1} \right)^{\frac{1}{p}}$$
$$\times \left[\left[\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q - \frac{1}{4}\left(3\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q + \sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q \right) \right]^{\frac{1}{q}} \right.$$
$$\left. + \left[\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q - \frac{1}{4}\left(3\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q + \sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q \right) \right]^{\frac{1}{q}} \right].$$

Minkwoski's inequality is utilized to get

$$= \frac{1}{16}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\frac{4}{np-\frac{\varpi}{k}p+1}\right)^{\frac{1}{p}}\left[4^{\frac{1}{q}}\cdot 2\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|-\left(3^{\frac{1}{q}}+1\right)\left(\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})| + \sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|\right)\right].$$

This concludes the proof. □

Remark 4.11. *We get inequality* (36) *mentioned in [48] by fixing* $\Omega = 2$ *and* $k = 1$ *in* (47).

Theorem 4.12. *Let* Ψ *be a differentiable function provided that* $\Psi \in C^{n+1}(\mathcal{I})$ *and* $|\Psi^{(n+1)}|^q$ $(q > 1)$ *exhibits convexity on the interval* $\mathcal{I}$. *Additionally, let* $\boldsymbol{\xi} = (\xi_1, \ldots, \xi_\Omega)$, $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_\Omega)$, *and* $\boldsymbol{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_\Omega)$ *be three tuples where* $\xi_{\varsigma'}, \tau_{\varsigma'}, \varepsilon_{\varsigma'} \in \mathcal{I}$, *for all* $\varsigma' = 1, \ldots, \Omega$, *and* $\varpi > 0$, $\tau_\Omega > \varepsilon_\Omega$. *If* $\boldsymbol{\tau} \prec \boldsymbol{\xi}$, $\boldsymbol{\varepsilon} \prec \boldsymbol{\xi}$, *then*

$$\Bigg|\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right) + (-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]-\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\Bigg|$$

$$\leq \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\frac{1}{n-\frac{\varpi}{k}+1}\right)^{1-\frac{1}{q}}\Bigg[\Bigg(\frac{1}{n-\frac{\varpi}{k}+1}\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q - \left(\frac{n-\frac{\varpi}{k}+3}{2(n-\frac{\varpi}{k}+1)(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q + \frac{1}{2(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right)\Bigg)^{\frac{1}{q}} + \Bigg(\frac{1}{n-\frac{\varpi}{k}+1}\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q - \left(\frac{n-\frac{\varpi}{k}+3}{2(n-\frac{\varpi}{k}+1)(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q + \frac{1}{2(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q\right)\Bigg)^{\frac{1}{q}}\Bigg]. \tag{49}$$

Proof. Lemma 4.6 is utilized first, followed by power mean inequality, to obtain

$$\Bigg|\frac{2^{n-\frac{\varpi}{k}-1}k\Gamma_k(n-\frac{\varpi}{k}+k)}{\left(\sum\limits_{\varsigma'=1}^{\Omega-1}(\varepsilon_{\varsigma'}-\tau_{\varsigma'})\right)^{n-\frac{\varpi}{k}}}\left[\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{+}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right) + (-1)^n\left({}^{c}D^{\varpi,k}_{\left(\sum\limits_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum\limits_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)^{-}}\Psi\right)\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right]-\Psi^{(n)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\sum_{\varsigma'=1}^{\Omega-1}\left(\frac{\tau_{\varsigma'}+\varepsilon_{\varsigma'}}{2}\right)\right)\Bigg|$$

$$\leq \frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left[\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left|\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}d\varrho\right)^{1-\frac{1}{q}}\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left|\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right]$$

$$=\frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\frac{1}{n-\frac{\varpi}{k}+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left|\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}\right)\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left|\Psi^{(n+1)}\left(\sum_{\varsigma'=1}^{\Omega}\xi_{\varsigma'}-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\varepsilon_{\varsigma'}+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}\tau_{\varsigma'}\right)\right)\right|^q d\varrho\right)^{\frac{1}{q}}\right]. \tag{50}$$

Now, applying Theorem 2.1 for the values $n=2$, $\rho_1=\frac{2-\varrho}{2}$ and $\rho_2=\frac{\varrho}{2}$ in (50), as a result of convex nature of $|\Psi^{(n+1)}|^q$, we have

$$=\frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\frac{1}{n-\frac{\varpi}{k}+1}\right)^{1-\frac{1}{q}}$$

$$\times\left[\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right)\right)d\varrho\right)^{\frac{1}{q}}\right.$$

$$\left.+\left(\int_0^1 \varrho^{n-\frac{\varpi}{k}}\left(\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q-\left(\frac{2-\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q+\frac{\varrho}{2}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q\right)\right)d\varrho\right)^{\frac{1}{q}}\right]$$

$$=\frac{1}{4}\sum_{\varsigma'=1}^{\Omega-1}|\varepsilon_{\varsigma'}-\tau_{\varsigma'}|\left(\frac{1}{n-\frac{\varpi}{k}+1}\right)^{1-\frac{1}{q}}\left[\left(\frac{1}{n-\frac{\varpi}{k}+1}\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q\right.\right.$$

$$\left.-\left(\frac{n-\frac{\varpi}{k}+3}{2(n-\frac{\varpi}{k}+1)(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q+\frac{1}{2(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right)\right)^{\frac{1}{q}}$$

$$+\left(\frac{1}{n-\frac{\varpi}{k}+1}\sum_{\varsigma'=1}^{\Omega}|\Psi^{(n+1)}(\xi_{\varsigma'})|^q-\left(\frac{n-\frac{\varpi}{k}+3}{2(n-\frac{\varpi}{k}+1)(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\varepsilon_{\varsigma'})|^q\right.\right.$$

$$\left.\left.\left.+\frac{1}{2(n-\frac{\varpi}{k}+2)}\sum_{\varsigma'=1}^{\Omega-1}|\Psi^{(n+1)}(\tau_{\varsigma'})|^q\right)\right)^{\frac{1}{q}}\right].$$

□

Thus, the proof comes to an end.

Corollary 4.13. *The below given inequality is obtained by fixing $n = 1$, $k = 1$, $\varpi = 0$ and $\Omega = 2$ in* (49).

$$\left|\frac{1}{\varepsilon_1-\tau_1}\int_{\vartheta_1+\vartheta_2-\varepsilon_1}^{\vartheta_1+\vartheta_2-\tau_1}\Psi(u)du-\Psi\left(\vartheta_1+\vartheta_2-\frac{\tau_1+\varepsilon_1}{2}\right)\right|\leq\frac{|\varepsilon_1-\tau_1|}{2^{\frac{3q-1}{q}}}\left[\left(\frac{|\Psi'(\vartheta_1)|+|\Psi'(\vartheta_2)|}{2}\right.\right.$$
$$\left.\left.-\frac{2|\Psi'(\tau_1)|^q+|\Psi'(\varepsilon_1)|^q}{2}\right)^{\frac{1}{q}}+\left(\frac{|\Psi'(\vartheta_1)|+|\Psi'(\vartheta_2)|}{2}-\frac{2|\Psi'(\varepsilon_1)|^q+|\Psi'(\tau_1)|^q}{2}\right)^{\frac{1}{q}}\right].$$

Remark 4.14. *Similar procedures can be used to establish the weighted forms of the results found in this section.*

5. Conclusion

At present, special attention is being given to produce such ideas, which are further used to connect various fields of science. Motivated by this fact, we developed a unified form of the Hermite-Hadamard-Jensen-Mercer type inequalities in a discrete and continuous sense. The idea which became the basis for this new work is known as the theory of majorization. First of all, the desired inequalities were derived by using convexity of the function, three majorized tuples and $k-$Caputo fractional derivative operators. These inequalities generalized the existing inequalities in literature which can be verified from the remarks given at the end of each result. The new derived inequalities were also presented in weighted forms by using Lemma 2.2 and Lemma 2.3. Furthermore, two new identities were discovered which enabled us to derive bounds for the discrepancy of terms of the main inequalities. One can observe that the concepts of fractional calculus added more beauty to the results of the Hermite-Hadamard-Jensen-Mercer type inequalities when used along with the majorization concept. This work can be recognized as the application of the majorization theory and can be used with other fractional operators such as Katugampola, Hadamard and conformable fractional operators to obtain further results in conticrete form.

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