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A study of improved error bounds for Simpson type inequality via fractional integral operator

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Abstract. Fractional integral operators have been studied extensively in the last few decades, and many different types of fractional integral operators have been introduced by various mathematicians. In 1967 Michele Caputo introduced Caputo fractional derivatives, which defined one of these fractional operators, the Caputo Fabrizio fractional integral operator. The main aim of this article is to established the new integral equalities related to Caputo-Fabrizio fractional integral operator. By incorporating this identity and convexity theory to obtained a novel class of Simpson type inequality. In this paper, we present a novel generalization of Simpson type inequality via s-convex and quasi-convex functions. Then, utilizing this identity the bounds of classical Simpson type inequality is improved. Finally, we discussed some applications to Simpson′s quadrature rule.

1. Introduction

Mathematics provides a framework for understanding and analyzing the patterns and structures that underlie the natural world and human-made systems. It allows us to develop and test theories, make predictions and design and optimize solutions to real-world problems. Similarly, concepts in geometry and algebra are related through various properties and theorems, which allow mathematicians to make deductions and prove mathematical statements. In recent years fractional analysis has been around for awhile, but recently it has received a lot of interest in the fields of pure and applied mathematics. The adventure that beginning with the issue of whether a solution exists for fractional order differential equations is established with various kinds of derivative and integral operators. By defining the derivative and integral operators in fractional order the scholars who aimed to suggest more efficient approaches to the analysis of physical phenomena. The basic concepts of fractional calculus have also been used by several authors to obtain new bounds. Some of the most commonly studied fractional integral operators include the Riemann-Liouville, Caputo Fabrizio, Atangana-Baleanu, and the Grunwald-Letnikov fractional integral.

Now a days inequalities like the trapezoid, mind-point and Simpson have recently attracted the interest of scholars. Several scholars acknowledge the expansion and generalization of these integral inequalities.

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For instance some authors presented distinct error estimates for the trapezoidal formula and investigate trapezoidal and fractional trapezoidal-type rules, determine explicit bounds and studied bounded variation functions in [1]-[4], [6]-[7] and [9]. Sarikaya established some new inequalities of the Simpson and trapezoid kinds for functions whose second derivative in absolute value is convex [5]. Kirmaci established midpoint-type inequality for differentiable convex functions [8]. One of the most important and widely requested inequalities is that of Simpson which can be stated as follows:

Suppose $\breve{g} : [\varpi_1, \varpi_2] \to \mathbb{R}$ is a four-times continuously differentiable mapping on (ϖ_1, ϖ_2) and

$$\left\|\breve{g}^{(4)}\right\|_\infty = \sup_{\hat{s}\in(\varpi_1,\varpi_2)} \left|\breve{g}^{(4)}(\hat{s})\right| < \infty,$$

then the following inequality holds:

$$\begin{aligned} &\left|\int_{\varpi_1}^{\varpi_2} \breve{g}(\hat{s})\,d\hat{s} - \frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1) + 4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right) + \breve{g}(\varpi_2)\right]\right| \\ \leq\ & \frac{(\varpi_2-\varpi_1)^5}{2880}\left\|\breve{g}^{(4)}\right\|_\infty . \end{aligned} \tag{1}$$

Some Simpson's type inequalities for functions whose n-th derivative, $n \in \{0,1,2,3\}$ is of bounded variation were established by Pecari'c and Varosanec in [10]. The following inequalities as follows:

Theorem 1.1. *Let $n \in \{0,1,2,3\}$ and let $\breve{g}$ be a real function on $[\varpi_1, \varpi_2]$ such that $\breve{g}^{(n)}$ is function of bounded variation, then the following inequality holds:*

$$\begin{aligned} &\left|\int_{\varpi_1}^{\varpi_2} \breve{g}(\hat{s})\,d\hat{s} - \frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1) + 4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right) + \breve{g}(\varpi_2)\right]\right| \\ \leq\ & C_n(\varpi_2-\varpi_1)^{n+1} \vee_{\varpi_1}^{\varpi_2}\left(\breve{g}^{(n)}\right), \end{aligned} \tag{2}$$

where

$$C_0 = \frac{1}{3}, C_1 = \frac{1}{24}, C_2 = \frac{1}{324}, C_3 = \frac{1}{1152},$$

and $\vee_{\varpi_1}^{\varpi_2}\left(\breve{g}^{(n)}\right)$ is the total variation of $\breve{g}^{(n)}$ on the interval $[\varpi_1, \varpi_2]$.

It is crucial to remember that Dragomir [11] proved the inequality (2) for $n = 0$ in literature. Ghizzetti and Ossicini [12] further proved that the inequality (2) with $n = 3$ holds, then $\breve{g}'''$ is an absolutely continuous mapping with total variation $\vee_{\varpi_1}^{\varpi_2}(\breve{g})$. Recently, some researchers worked on fractional integrals [13]-[18] and [22].

Hudzik et al. [19] considered the class of s-convex functions in the second sense.

Definition 1.2. *A function $\breve{g} : I \subseteq \mathbb{R} \to \mathbb{R}_0 = [0, \infty)$ is said to be s-convex functions if:*

$$\breve{g}(\Upsilon\varpi_1 + (1-\Upsilon)\varpi_2) \leq \Upsilon^s \breve{g}(\varpi_1) + (1-\Upsilon)^s \breve{g}(\varpi_2),$$

$s \in (0,1]$, where $\varpi_1, \varpi_2 \in I$ and $\Upsilon \in [0,1]$.
The notion of quasi-convex is the generalization of convex function.

Definition 1.3. *[20] A function $\breve{g} : I \to \mathbb{R}$ is said to be quasi-convex functions:*

$$\breve{g}(\Upsilon\varpi_1 + (1-\Upsilon)\varpi_2) \leq \sup\{\breve{g}(\varpi_1), \breve{g}(\varpi_2)\},$$

if all $\varpi_1, \varpi_2 \in I$ and $\Upsilon \in [0,1]$.

Definition 1.4. *[21] Let $H^1(\varpi_1,\varpi_2)$ be the Sobolev space of order one defined as*

$$H^1(\varpi_1,\varpi_2)=\left\{g\in L^2(\varpi_1,\varpi_2):g'\in L^2(\varpi_1,\varpi_2)\right\},$$

where

$$L^2(\varpi_1,\varpi_2)=\left\{g(z):\left(\int_{\varpi_1}^{\varpi_2}g^2(z)\,dz\right)^{\frac{1}{2}}<\infty\right\}.$$

Let $\breve{g}\in H^1(\varpi_1,\varpi_2)$, $\varpi_1<\varpi_2$ and $\alpha\in[0,1]$, the nth notion of left derivative in the sense of Caputo-Fabrizio is defined as:

$$\left({}^{C\breve{F}D}_{\varpi_1}D^{\alpha}\breve{g}\right)(\hat{s})=\frac{\beta(\alpha)}{1-\alpha}\int_{\varpi_1}^{\hat{s}}\breve{g}'(\Upsilon)\,e^{\frac{-\alpha(\hat{s}-\Upsilon)^{\alpha}}{1-\alpha}}\,d\Upsilon,$$

$\hat{s}>\alpha$ and the associated integral operator is

$$\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(\hat{s})=\frac{1-\alpha}{\beta(\alpha)}\breve{g}(\hat{s})+\frac{\alpha}{\beta(\alpha)}\int_{\varpi_1}^{\hat{s}}\breve{g}(\Upsilon)\,d\Upsilon,$$

where $\beta(\alpha)>0$ is the normalization function satisfying $\beta(0)=\beta(1)=1$. For $\alpha=0,\alpha=1$, the left derivative is defined as follows, respectively

$$\begin{aligned}\left({}^{C\breve{F}D}_{\varpi_1}D^{0}\breve{g}\right)(\hat{s})&=\breve{g}'(\hat{s})\\ \left({}^{C\breve{F}D}_{\varpi_1}D^{1}\breve{g}\right)(\hat{s})&=\breve{g}(\hat{s})-\breve{g}(\varpi_1).\end{aligned}$$

For the right derivative operator

$$\left({}^{C\breve{F}D}_{\varpi_2}D^{\alpha}\breve{g}\right)(\hat{s})=\frac{\beta(\alpha)}{1-\alpha}\int_{\hat{s}}^{\varpi_2}\breve{g}'(\Upsilon)\,e^{\frac{-\alpha(\Upsilon-\hat{s})^{\alpha}}{1-\alpha}}\,d\Upsilon,$$

$\hat{s}<\varpi_2$ and the associated integral operator is

$$\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(\hat{s})=\frac{1-\alpha}{\beta(\alpha)}\breve{g}(\hat{s})+\frac{\alpha}{\beta(\alpha)}\int_{\hat{s}}^{\varpi_2}\breve{g}(\Upsilon)\,d\Upsilon.$$

Motivated by ongoing studies, we establish novel fractional version of the Simpson type inequality utilizing the Caputo-Fabrizio integral operator. In this paper, to obtain more advanced results, then utilizing this identity the bound of classical Simpson type inequality is improved. Furthermore, obtained novel fractional integral bounds can be much better than some recent acquire bounds. Finally, we discussed some applications to Simpson's quadrature rule.

2. Main results

In this section, we deal with identity, which is necessary to attain our main result.

Lemma 2.1. *Let $\breve{g}:[\varpi_1,\varpi_2]\to\mathbb{R}$ be four times differentiable mapping on (ϖ_1,ϖ_2) and $\breve{g}^{(4)}\in L[\varpi_1,\varpi_2]$, then the following equality holds:*

$$\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]$$
$$-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)$$

$$= \frac{(\varpi_2-\varpi_1)^4}{2304}\int_0^1 \Upsilon^3(4-3\Upsilon)\left[\breve{g}^{(4)}(u)+\breve{g}^{(4)}(v)\right]d\Upsilon,$$

and

$$u=\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right),\ v=\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right).$$

Proof. Let

$$\begin{aligned} I &= \int_0^1 \Upsilon^3(4-3\Upsilon)\,\breve{g}^{(4)}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \\ &\quad +\int_0^1 \Upsilon^3(4-3\Upsilon)\,\breve{g}^{(4)}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \\ &= I_1+I_2. \end{aligned}$$

Integration by parts, we have

$$\begin{aligned} I_1 &= \int_0^1 \Upsilon^3(4-3\Upsilon)\,\breve{g}^{(4)}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{24}{\varpi_2-\varpi_1}\int_0^1\left(\Upsilon^2-\Upsilon^3\right)\breve{g}'''\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{24}{\varpi_2-\varpi_1}\left[\frac{-2}{\varpi_2-\varpi_1}\left(\Upsilon^2-\Upsilon^3\right)\breve{g}''\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\Big|_0^1\right. \\ &\quad \left.+\frac{2}{\varpi_2-\varpi_1}\int_0^1\left(2\Upsilon-3\Upsilon^2\right)\breve{g}''\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\right] \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{48}{(\varpi_2-\varpi_1)^2}\int_0^1\left(2\Upsilon-3\Upsilon^2\right)\breve{g}''\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{48}{(\varpi_2-\varpi_1)^2}\left[\frac{-2}{\varpi_2-\varpi_1}\left(2\Upsilon-3\Upsilon^2\right)\breve{g}'\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\Big|_0^1\right. \\ &\quad \left.+\frac{2}{\varpi_2-\varpi_1}\int_0^1(2-6\Upsilon)\,\breve{g}'\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\right] \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right) \\ &\quad +\frac{96}{(\varpi_2-\varpi_1)^3}\int_0^1(2-6\Upsilon)\,\breve{g}'\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{96}{(\varpi_2-\varpi_1)^3} \\ &\quad \left[\frac{-2}{\varpi_2-\varpi_1}(2-6\Upsilon)\,\breve{g}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\Big|_0^1\right. \\ &\quad \left.-\frac{12}{\varpi_2-\varpi_1}\int_0^1\breve{g}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\right] \\ &= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{768}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right) \\ &\quad +\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_2)-\frac{1152}{(\varpi_2-\varpi_1)^4}\int_0^1\breve{g}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon \end{aligned}$$

$$
\begin{aligned}
&= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{768}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)\\
&\quad+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_2)-\frac{2304}{(\varpi_2-\varpi_1)^5}\int_{\frac{\varpi_1+\varpi_2}{2}}^{\varpi_2}\breve{g}(u)\,du. \qquad (3)
\end{aligned}
$$

Similarly, we get

$$
\begin{aligned}
I_2 &= \int_0^1 \Upsilon^3(4-3\Upsilon)\,\breve{g}^{(4)}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\\
&= \frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{24}{\varpi_2-\varpi_1}\int_0^1\left(\Upsilon^2-\Upsilon^3\right)\breve{g}'''\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\\
&= \frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{48}{(\varpi_2-\varpi_1)^2}\int_0^1\left(2\Upsilon-3\Upsilon^2\right)\breve{g}''\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\\
&= \frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)\\
&\quad-\frac{96}{(\varpi_2-\varpi_1)^3}\int_0^1(2-6\Upsilon)\,\breve{g}'\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\\
&= \frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{96}{(\varpi_2-\varpi_1)^3}\\
&\quad\times\left[\frac{2}{\varpi_2-\varpi_1}\,(2-6\Upsilon)\,\breve{g}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)\Bigg|_0^1\right.\\
&\quad\left.-\frac{12}{\varpi_2-\varpi_1}\int_0^1\breve{g}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\right]\\
&= \frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{768}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)\\
&\quad+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_1)-\frac{1152}{(\varpi_2-\varpi_1)^4}\int_0^1\breve{g}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)d\Upsilon\\
&= \frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{768}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)\\
&\quad+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_1)-\frac{2304}{(\varpi_2-\varpi_1)^5}\int_{\varpi_1}^{\frac{\varpi_1+\varpi_2}{2}}\breve{g}(u)\,du. \qquad (4)
\end{aligned}
$$

By adding the equality (3) and (4), we get

$$
\begin{aligned}
&I_1+I_2\\
&= \frac{-2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{768}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)\\
&\quad+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_2)+\frac{2}{\varpi_2-\varpi_1}\breve{g}'''\left(\frac{\varpi_1+\varpi_2}{2}\right)-\frac{96}{(\varpi_2-\varpi_1)^3}\breve{g}'\left(\frac{\varpi_1+\varpi_2}{2}\right)\\
&\quad+\frac{768}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_1)-\frac{2304}{(\varpi_2-\varpi_1)^5}\left(\int_{\varpi_1}^{\frac{\varpi_1+\varpi_2}{2}}\breve{g}(u)\,du+\int_{\frac{\varpi_1+\varpi_2}{2}}^{\varpi_2}\breve{g}(u)\,du\right)\\
&= \frac{1536}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_1)+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_2)\\
&\quad-\frac{2304}{(\varpi_2-\varpi_1)^5}\int_{\varpi_1}^{\varpi_2}\breve{g}(u)\,du. \qquad (5)
\end{aligned}
$$

Multiplying $\frac{\alpha(\varpi_2-\varpi_1)^5}{2304\beta(\alpha)}$ by (5) and subtracting $\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)$ on both sides, we get

$$\begin{aligned}
&(I_1+I_2)\frac{\alpha(\varpi_2-\varpi_1)^5}{2304\beta(\alpha)}-\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\\
=\;&\frac{1536}{(\varpi_2-\varpi_1)^4}\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)\frac{\alpha(\varpi_2-\varpi_1)^5}{2304\beta(\alpha)}+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_1)\frac{\alpha(\varpi_2-\varpi_1)^5}{2304\beta(\alpha)}\\
&+\frac{384}{(\varpi_2-\varpi_1)^4}\breve{g}(\varpi_2)\frac{\alpha(\varpi_2-\varpi_1)^5}{2304\beta(\alpha)}-\frac{\alpha}{\beta(\alpha)}\int_{\varpi_1}^{\varpi_2}\breve{g}(u)\,du-\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\\
=\;&\frac{2\alpha(\varpi_2-\varpi_1)}{3\beta(\alpha)}+\frac{1\alpha(\varpi_2-\varpi_1)}{6\beta(\alpha)}\breve{g}(\varpi_1)+\frac{1\alpha(\varpi_2-\varpi_1)}{6\beta(\alpha)}\breve{g}(\varpi_2)\\
&-\left(\frac{\alpha}{\beta(\alpha)}\int_{\varpi_1}^{k}\breve{g}(u)\,du-\frac{(1-\alpha)}{\beta(\alpha)}\breve{g}(k)+\frac{\alpha}{\beta(\alpha)}\int_{k}^{\varpi_2}\breve{g}(u)\,du-\frac{(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right)\\
=\;&\frac{2\alpha(\varpi_2-\varpi_1)}{3\beta(\alpha)}+\frac{\alpha(\varpi_2-\varpi_1)}{6\beta(\alpha)}\breve{g}(\varpi_1)+\frac{\alpha(\varpi_2-\varpi_1)}{6\beta(\alpha)}\breve{g}(\varpi_2)\\
&-\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right).
\end{aligned}$$

Thus, we have

$$\begin{aligned}
&\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\\
&-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\\
=\;&\frac{(\varpi_2-\varpi_1)^4}{2304}\int_0^1\Upsilon^3(4-3\Upsilon)\left[\breve{g}^{(4)}(u)+\breve{g}^{(4)}(v)\right]d\Upsilon.
\end{aligned}$$

This completes the proof. □

Theorem 2.2. *Under the assumption of Lemma 2.1. If $\left|\breve{g}^{(4)}\right|$ is s-convex on $[\varpi_1,\varpi_2]$, for some fixed $s\in(0,1]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\le\;&\frac{(\varpi_2-\varpi_1)^4}{2304}\left[\frac{3\times 2^{5-s}\left(5-2^{2+s}+s+2^{2+s}s\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\right]\left[\left|\breve{g}^{(4)}(\varpi_1)\right|+\left|\breve{g}^{(4)}(\varpi_2)\right|\right].
\end{aligned}$$

Proof. By using the Lemma 2.1 and since $\breve{g}$ is s-convexity, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\le\;&\frac{(\varpi_2-\varpi_1)^4}{2304}\left[\int_0^1\Upsilon^3(4-3\Upsilon)\left|\breve{g}^{(4)}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\right|d\Upsilon\right.\\
&\left.+\int_0^1\Upsilon^3(4-3\Upsilon)\left|\breve{g}^{(4)}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)\right|d\Upsilon\right]
\end{aligned}$$

$$
\begin{aligned}
&\leq \frac{(\varpi_2-\varpi_1)^4}{2304}\left[\int_0^1 \Upsilon^3(4-3\Upsilon)\left(\left(\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_1)\right|+\left(1-\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_2)\right|\right)d\Upsilon\right.\\
&\quad\left.+\int_0^1 \Upsilon^3(4-3\Upsilon)\left(\left(\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_2)\right|+\left(1-\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_1)\right|\right)d\Upsilon\right]\\
&\leq \frac{(\varpi_2-\varpi_1)^4}{2304}\left[\frac{2^{-s}(s+8)}{(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_1)\right|\right.\\
&\quad\frac{2^{-s}\left(432+3\times 2^{7+s}(s-1)+2s-59s^2\times 14s^3-s^4\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_2)\right|+\frac{2^{-s}(s+8)}{(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_2)\right|\\
&\quad\left.+\frac{2^{-s}\left(432+3\times 2^{7+s}(s-1)+2s-59s^2\times 14s^3-s^4\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_1)\right|\right]\\
&\leq \frac{(\varpi_2-\varpi_1)^4}{2304}\left[\frac{3\times 2^{5-s}\left(5-2^{2+s}+s+2^{2+s}s\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\right]\left[\left|\breve{g}^{(4)}(\varpi_1)\right|+\left|\breve{g}^{(4)}(\varpi_2)\right|\right].
\end{aligned}
$$

This completes the proof. □

Corollary 2.3. *If we put $s = 1$ in Theorem 2.2, then we have*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\leq\ & \frac{(\varpi_2-\varpi_1)^4}{5760}\left[\left|\breve{g}^{(4)}(\varpi_1)\right|+\left|\breve{g}^{(4)}(\varpi_2)\right|\right].
\end{aligned}
$$

Remark 2.4. *We note that the error bounds in (1) and (2) have improved.*

Theorem 2.5. *Under the assumption of Lemma 2.1. If $\left|\breve{g}^{(4)}\right|$ is s-convex on $[\varpi_1, \varpi_2]$, for some fixed $s \in (0, 1]$ and $q \geq 1$, then the following inequality holds:*

$$
\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\leq\ & \frac{(\varpi_2-\varpi_1)^4}{2304}\left(\frac{2}{5}\right)^{1-\frac{1}{q}}\left[\left(\frac{2^{-s}(s+8)}{(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_1)\right|^q\right.\right.\\
&\left.+\frac{2^{-s}\left(432+3\times 2^{7+s}(s-1)+2s-59s^2\times 14s^3-s^4\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_2)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{2^{-s}(s+8)}{(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_2)\right|^q\right.\\
&\left.\left.+\frac{2^{-s}\left(432+3\times 2^{7+s}(s-1)+2s-59s^2\times 14s^3-s^4\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_1)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}
$$

Proof. By using the Lemma 2.1, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\leq\;& \frac{(\varpi_2-\varpi_1)^4}{2304}\left[\int_0^1 \Upsilon^3(4-3\Upsilon)\breve{g}^{(4)}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\right.\\
&\left.+\int_0^1 \Upsilon^3(4-3\Upsilon)\breve{g}^{(4)}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)\right]d\Upsilon\\
\leq\;& \frac{(\varpi_2-\varpi_1)^4}{2304}\left[\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\right)^{1-\frac{1}{q}}\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\left|\breve{g}^{(4)}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\right|^q\right)^{\frac{1}{q}}d\Upsilon\right.\\
&\left.+\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\right)^{1-\frac{1}{q}}\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\left|\breve{g}^{(4)}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)\right|^q d\Upsilon\right)^{\frac{1}{q}}\right]\\
\leq\;& \frac{(\varpi_2-\varpi_1)^4}{2304}\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\right)^{1-\frac{1}{q}}\times\\
&\left[\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\left(\left(\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_1)\right|^q+\left(1-\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_2)\right|^q\right)d\Upsilon\right)^{\frac{1}{q}}\right.\\
&\left.+\left(\int_0^1 \Upsilon^3(4-3\Upsilon)\left(\left(\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_2)\right|^q+\left(1-\frac{\Upsilon}{2}\right)^s\left|\breve{g}^{(4)}(\varpi_1)\right|^q\right)d\Upsilon\right)^{\frac{1}{q}}\right]\\
\leq\;& \frac{(\varpi_2-\varpi_1)^4}{2304}\left(\frac{2}{5}\right)^{1-\frac{1}{q}}\left[\left(\frac{2^{-s}(s+8)}{(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_1)\right|^q\right.\right.\\
&\left.+\frac{2^{-s}\left(432+3\times 2^{7+s}(s-1)+2s-59s^2\times 14s^3-s^4\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_2)\right|^q\right)^{\frac{1}{q}}\\
&+\left(\frac{2^{-s}(s+8)}{(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_2)\right|^q\right.\\
&\left.\left.+\frac{2^{-s}\left(432+3\times 2^{7+s}(s-1)+2s-59s^2\times 14s^3-s^4\right)}{(s+1)(s+2)(s+3)(s+4)(s+5)}\left|\breve{g}^{(4)}(\varpi_1)\right|^q\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This completes the proof. □

Theorem 2.6. *Under the assumption of Lemma 2.1. If $\left|\breve{g}^{(4)}\right|$ is quasi-convex on $[\varpi_1,\varpi_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\leq\;& \frac{(\varpi_2-\varpi_1)^4}{2880}\left[\sup\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|,\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}\right].
\end{aligned}$$

Proof. By using the Lemma 2.1 and since $\breve{g}$ is quasi-convex, we have

$$\begin{aligned}
&\left|\frac{1}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right.\\
&\left.-\frac{\beta(\alpha)}{\alpha(\varpi_2-\varpi_1)}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right|\\
\leq\ &\frac{(\varpi_2-\varpi_1)^4}{2304}\left[\left(\int_0^1\Upsilon^3(4-3\Upsilon)\left|\breve{g}^{(4)}\left(\varpi_1\frac{\Upsilon}{2}+\varpi_2\left(1-\frac{\Upsilon}{2}\right)\right)\right|\right)d\Upsilon\right.\\
&\left.+\left(\int_0^1\Upsilon^3(4-3\Upsilon)\left|\breve{g}^{(4)}\left(\varpi_2\frac{\Upsilon}{2}+\varpi_1\left(1-\frac{\Upsilon}{2}\right)\right)\right|d\Upsilon\right)\right]\\
\leq\ &\frac{(\varpi_2-\varpi_1)^4}{2304}\left[\int_0^1\Upsilon^3(4-3\Upsilon)\sup\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|,\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}d\Upsilon\right.\\
&\left.+\left(\int_0^1\Upsilon^3(4-3\Upsilon)\sup\left\{\left|\breve{g}^{(4)}(\varpi_2)\right|,\left|\breve{g}^{(4)}(\varpi_1)\right|\right\}d\Upsilon\right)\right]\\
\leq\ &\frac{(\varpi_2-\varpi_1)^4}{2880}\left[\sup\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|,\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}\right].
\end{aligned}$$

This completes the proof. □

Lemma 2.7. *Let $\breve{g}:[\varpi_1,\varpi_2]\to\mathbb{R}$ be four times differentiable mapping on (ϖ_1,ϖ_2) and $\breve{g}^{(4)}\in L[\varpi_1,\varpi_2]$, then the following equality holds:*

$$\begin{aligned}
&-\frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\\
&+\frac{\beta(\alpha)}{\alpha}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\\
=\ &(\varpi_2-\varpi_1)^5\int_0^1 p(\Upsilon)\,\breve{g}^{(4)}(\Upsilon\varpi_1+(1-\Upsilon)\varpi_2)\,d\Upsilon
\end{aligned}$$

and

$$p(\Upsilon)=\begin{cases}\frac{1}{24}\Upsilon^3\left(\Upsilon-\frac{2}{3}\right) & \Upsilon\in\left[0,\frac{1}{2}\right]\\[2mm] \frac{1}{24}(\Upsilon-1)^3\left(\Upsilon-\frac{1}{3}\right), & \Upsilon\in\left(\frac{1}{2},1\right]\end{cases},$$

where $\beta(\alpha)>0$ is a normalization function.

Proof. The required identity can be easily obtained by using changing of variable in Lemma 2.1. □

Theorem 2.8. *Under the assumption of Lemma 2.7. If $\left|\breve{g}^{(4)}\right|$ is quasi-convex on $[\varpi_1,\varpi_2]$, then the following inequality holds:*

$$\begin{aligned}
&\left|\frac{\beta(\alpha)}{\alpha}\left(\left({}^{C\breve{F}}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{C\breve{F}}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right.\\
&\left.-\frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right|\\
\leq\ &\frac{(\varpi_2-\varpi_1)^5}{5760}\left[\sup\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|,\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\right\}\right. \qquad (6)\\
&\left.+\sup\left\{\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|,\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}\right].
\end{aligned}$$

Proof. By using the Lemma 2.7 and since $\breve{g}$ is quasi-convex, we have

$$\begin{aligned}
&\left|\frac{\beta(\alpha)}{\alpha}\left(\left({}^{CF}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{CF}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)\right.\\
&\left.+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)-\frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right|\\
= \ & (\varpi_2-\varpi_1)^5\left|\int_0^1 p(\Upsilon)\breve{g}^{(4)}(\Upsilon\varpi_1+(1-\Upsilon)\varpi_2)\,d\Upsilon\right|\\
\leq \ & (\varpi_2-\varpi_1)^5\left[\int_0^{\frac{1}{2}}\left|\frac{1}{24}\Upsilon^3\left(\Upsilon-\frac{2}{3}\right)\right|\left|\breve{g}^{(4)}(\Upsilon\varpi_1+(1-\Upsilon)\varpi_2)\right|d\Upsilon\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\frac{1}{24}(\Upsilon-1)^3\left(\Upsilon-\frac{1}{3}\right)\right|\left|\breve{g}^{(4)}(\Upsilon\varpi_1+(1-\Upsilon)\varpi_2)\right|d\Upsilon\right]\\
\leq \ & (\varpi_2-\varpi_1)^5\left[\int_0^{\frac{1}{2}}\left|\frac{1}{24}\Upsilon^3\left(\Upsilon-\frac{2}{3}\right)\right|\sup\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|,\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\right\}d\Upsilon\right.\\
&\left.+\int_{\frac{1}{2}}^1\left|\frac{1}{24}(\Upsilon-1)^3\left(\Upsilon-\frac{1}{3}\right)\right|\sup\left\{\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|,\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}d\Upsilon\right]\\
\leq \ & \frac{(\varpi_2-\varpi_1)^5}{5760}\left[\sup\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|,\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\right\}\right.\\
&\left.+\sup\left\{\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|,\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}\right].
\end{aligned}$$

This completes the proof. □

Corollary 2.9. *If $\breve{g}$ is decreasing, then we have*

$$\begin{aligned}
&\left|\frac{\beta(\alpha)}{\alpha}\left(\left({}^{CF}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{CF}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right.\\
&\left.-\frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right|\\
\leq \ & \frac{(\varpi_2-\varpi_1)^5}{5760}\left[\left\{\left|\breve{g}^{(4)}(\varpi_1)\right|+\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|\right\}\right].
\end{aligned} \tag{7}$$

Corollary 2.10. *If $\breve{g}$ is increasing, then we have*

$$\begin{aligned}
&\left|\frac{\beta(\alpha)}{\alpha}\left(\left({}^{CF}_{\varpi_1}I^{\alpha}\breve{g}\right)(k)+\left({}^{CF}I^{\alpha}_{\varpi_2}\breve{g}\right)(k)\right)+\frac{2(1-\alpha)}{\beta(\alpha)}\breve{g}(k)\right.\\
&\left.-\frac{(\varpi_2-\varpi_1)}{6}\left[\breve{g}(\varpi_1)+4\breve{g}\left(\frac{\varpi_1+\varpi_2}{2}\right)+\breve{g}(\varpi_2)\right]\right|\\
\leq \ & \frac{(\varpi_2-\varpi_1)^5}{5760}\left[\left\{\left|\breve{g}^{(4)}\left(\frac{\varpi_1+\varpi_2}{2}\right)\right|+\left|\breve{g}^{(4)}(\varpi_2)\right|\right\}\right].
\end{aligned} \tag{8}$$

Corollary 2.11. *Let $\breve{g}$ as in Theorem 2.8. If $\left|\breve{g}^{(4)}\right|$ is exits, continuous and $\left\|\breve{g}^{(4)}\right\|_\infty = \sup_{\hat{s}\in(\varpi_1,\varpi_2)}\left|\breve{g}^{(4)}(\hat{s})\right| < \infty$, then the inequality (6) reduced to (1).*

Remark 2.12. *We note that the error bounds in (1) and (2) are improved.*

3. Application to Simpson's Formula

Let d is the partition of the interval $[\varpi_1, \varpi_2], d : \varpi_1 = \hat{s}_0 < \hat{s}_1 < \hat{s}_2 < ..... < \hat{s}_{n-1} < \hat{s}_n = \varpi_2, h_i = \frac{(\hat{s}_{i+1}-\hat{s}_i)}{2}$ and let the Simpson formula

$$S(\breve{g}, d) = \sum_{i=0}^{n-1} \frac{\breve{g}(\hat{s}_i) + 4\breve{g}(\hat{s}_i + h_i) + \breve{g}(\hat{s}_{i+1})}{6}(\hat{s}_{i+1} - \hat{s}_i). \tag{9}$$

If the mapping $\breve{g} : [\varpi_1, \varpi_2] \to \mathbb{R}$ is a differentiable such that $\breve{g}^{(4)}(\hat{s})$ exists on (ϖ_1, ϖ_2) and $M = \sup_{\hat{s}\in(\varpi_1,\varpi_2)} \left|\breve{g}^{(4)}(\hat{s})\right| < \infty$, then

$$I = \int_{\varpi_1}^{\varpi_2} \breve{g}(\hat{s})\, d\hat{s} = S(\breve{g}, d) + E_S(\breve{g}, d), \tag{10}$$

where the approximation error $E_S(\breve{g}, d)$ of the interval I by Simpson Formula $S(\breve{g}, d)$ satisfies:

$$\left|E_{ii}(\breve{g}, d)\right| \leq \frac{M}{2880} \sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^5. \tag{11}$$

Proposition 3.1. *Under the assumption of Lemma 2.1. If* $\left|\breve{g}^{(4)}\right|$ *is s-convex on* $[\varpi_1, \varpi_2]$, *for some fixed* $s \in (0, 1]$, *then the following inequality holds:*

$$\left|E_S(\breve{g}, d)\right| \leq \frac{1}{5760} \sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^4 \times \left[\left|\breve{g}^{(4)}(\hat{s}_i)\right| + \left|\breve{g}^{(4)}(\hat{s}_{i+1})\right|\right].$$

Proof. Applying the corollary 2.3 on the subinterval $[\hat{s}_i, \hat{s}_i + 1]$, $(i = 0, 1, 2, 3....n - 1)$ of the division d and $\alpha = 1, \beta(0) = \beta(1) = 1$, we have

$$\begin{aligned} &\left|\int_{\hat{s}i}^{\hat{s}_{i+1}} \breve{g}(\hat{s})\, d\hat{s} + \frac{(\hat{s}_{i+1} - \hat{s}_i)}{6} \left[\breve{g}(\hat{s}_i) + 4\breve{g}\left(\frac{(\hat{s}_{i+1} - \hat{s}_i)}{2}\right) + \breve{g}(\hat{s}_{i+1})\right]\right| \\ \leq\; & \frac{1}{5760} \sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^4 \left[\left|\breve{g}^{(4)}(\hat{s}_i)\right| + \left|\breve{g}^{(4)}(\hat{s}_{i+1})\right|\right]. \end{aligned}$$

Summing over i from 0 to $n - 1$ and let that $\left|\breve{g}^{(4)}\right|$ is s-convex, by the triangle inequality, we deduce that

$$\left|\int_{\varpi_1}^{\varpi_2} \breve{g}(\hat{s})\, d\hat{s} - S(\breve{g}, d)\right| \leq \frac{1}{5760} \sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^4 \left[\left|\breve{g}^{(4)}(\hat{s}_i)\right| + \left|\breve{g}^{(4)}(\hat{s}_{i+1})\right|\right].$$

This completes the proof. □

Proposition 3.2. *Under the assumption of Lemma 2.1. If* $\left|\breve{g}^{(4)}\right|$ *is quasi-convex on* $[\varpi_1, \varpi_2]$, *then the following inequality holds:*

$$\begin{aligned} \left|E_S(\breve{g}, d)\right| \leq\; & \frac{1}{5760} \sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^5 \left[\sup\left\{\left|\breve{g}^{(4)}(\hat{s}_i)\right|, \left|\breve{g}^{(4)}\left(\frac{\hat{s}_i + \hat{s}_{i+1}}{2}\right)\right|\right\}\right. \\ & \left. + \sup\left\{\left|\breve{g}^{(4)}\left(\frac{\hat{s}_i + \hat{s}_{i+1}}{2}\right)\right|, \left|\breve{g}^{(4)}(\hat{s}_{i+1})\right|\right\}\right]. \end{aligned}$$

Proof. Applying the Theorem 2.8 on the subinterval $[\hat{s}_i, \hat{s}_i + 1]$, $(i = 0, 1, 2, 3....n-1)$ of the division d and $\alpha = 1, \beta(0) = \beta(1) = 1$, we have

$$\begin{aligned}
&\left|\int_{\hat{s}i}^{\hat{s}_{i+1}} \breve{g}(\hat{s})\, d\hat{s} - \frac{(\hat{s}_{i+1} - \hat{s}_i)}{6}\left[\breve{g}(\hat{s}_i) + 4\breve{g}\left(\frac{(\hat{s}_{i+1} - \hat{s}_i)}{2}\right) + \breve{g}(\hat{s}_{i+1})\right]\right| \\
\leq\ & \frac{1}{5760}\sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^5 \left[\sup\left\{\left|\breve{g}^{(4)}(\hat{s}_i)\right|, \left|\breve{g}^{(4)}\left(\frac{\hat{s}_i + \hat{s}_{i+1}}{2}\right)\right|\right\}\right. \\
&\left. + \sup\left\{\left|\breve{g}^{(4)}\left(\frac{\hat{s}_i + \hat{s}_{i+1}}{2}\right)\right|, \left|\breve{g}^{(4)}(\hat{s}_{i+1})\right|\right\}\right].
\end{aligned}$$

Summing over i from 0 to $n-1$ and let that $\left|\breve{g}^{(4)}\right|$ is quasi-convex, by the triangle inequality, we deduce that

$$\begin{aligned}
&\left|\int_{\varpi_1}^{\varpi_2} \breve{g}(\hat{s})\, d\hat{s} - S(\breve{g}, d)\right| \\
\leq\ & \frac{1}{5760}\sum_{i=0}^{n-1} (\hat{s}_{i+1} - \hat{s}_i)^5 \left[\sup\left\{\left|\breve{g}^{(4)}(\hat{s}_i)\right|, \left|\breve{g}^{(4)}\left(\frac{\hat{s}_i + \hat{s}_{i+1}}{2}\right)\right|\right\}\right. \\
&\left. + \sup\left\{\left|\breve{g}^{(4)}\left(\frac{\hat{s}_i + \hat{s}_{i+1}}{2}\right)\right|, \left|\breve{g}^{(4)}(\hat{s}_{i+1})\right|\right\}\right].
\end{aligned}$$

This completes the proof. □

4. Conclusion

Fractional calculus is a fascinating subject with many applications in the modelling of natural problems. Using news techniques, and operators of fractional calculus, several scholar have generalized a variety of inequalities. In this article, we established the Simpson type inequality using the caputo fabrizio fractional operator. Then, utilizing this identity the bound of classical Simpson type inequality is improved. Furthermore, obtained novel fractional integral bounds can be much better than some recent acquire bounds. Finally, we discussed some applications to Simpson's quadrature rule. In the future, scholars may explore inequalities of the Simpson-Mercer type, Jensen-Mercer type, and Hermite-Hadamard-Mercer type with modified Caputo-Fabrizio fractional operators and modified A-B fractional operators.

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