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The AM-GM-HM inequality and the Kantorovich inequality for sector matrices

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Abstract. In the present paper, some new inequalities are proved for sector matrices. Among many other results, we show that if $A, B \in S_\alpha$ satisfying $0 < m \leq \Re A, \Re B \leq M$. Then

$$\cos^4 \alpha \psi_{\frac{1}{2}}(h) \Re \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \leq \Re(A \sharp B) \leq \sec^2 \alpha \psi_{\frac{1}{2}}^{-1}(h) \Re \left(\frac{A + B}{2} \right),$$

where $\psi_{\frac{1}{2}}(h) = 1 + \frac{\sqrt{2}(h-1)^2}{4(h+1)^{\frac{3}{2}}}$ and $S_\alpha (0 \leq \alpha < \frac{\pi}{2})$ is considered as the set of all sector matrices. In end, some inequalities for singular values or norms are presented.

1. introduction

Let M and m be scalars and I be the identity operator. Let $\mathbb{B}(\mathcal{H})$ denote C^*-algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. We say $A \in \mathbb{B}(\mathcal{H})$ is self-adjoint, if it satisfies $A = A^*$. An operator A is said to be positive and is denoted by $A \geq 0$ if $\langle Ax, x \rangle \geq 0$ for all $x \in \mathcal{H}$, and A is said to be strictly positive and is denoted by $A > 0$, if $\langle Ax, x \rangle > 0$ for all $x \in \mathcal{H}$. For two self-adjoint operators A and B, $A \geq B$ means $A - B \geq 0$. We say linear map $\Phi : \mathbb{B}(\mathcal{H}) \to \mathbb{B}(\mathcal{H})$ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$. For $A, B \in \mathbb{B}(\mathcal{H})$ such that $A, B > 0$ and $0 \leq \nu \leq 1$, we use the notations $A \sharp_\nu B$, $A \nabla_\nu B$ and $A !_\nu B$ to define the geometric mean, the arithmetic mean and the harmonic mean, respectively, and are defined in the following form:

$$A \sharp_\nu B = A^{\frac{1}{2}} (A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^\nu A^{\frac{1}{2}}, \qquad A \nabla_\nu B = (1 - \nu) A + \nu B$$

and

$$A !_\nu B = ((1 - \nu) A^{-1} + \nu B^{-1})^{-1}.$$

The noncommutative AM-GM-HM inequalities for two strictly positive operators A and B and $0 \leq \nu \leq 1$ have been proved by Bhatia [1] in following form:

$$A !_\nu B \leq A \sharp_\nu B \leq A \nabla_\nu B, \tag{1}$$

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where the second inequality is famous as the operator Young inequality. For special case, when $\nu = \frac{1}{2}$, we have the following inequality:

$$\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \leq A\sharp B \leq \frac{A+B}{2}. \tag{2}$$

Let $\mathbb{M}_n$ denote the set of all $n\times n$ complex matrices. For every $A \in \mathbb{M}_n$, we can write a cartesian decomposition $A = \Re A + i\Im A$, where $\Re A = \frac{A+A^*}{2}$ and $\Im A = \frac{A-A^*}{2i}$ are the real and imaginary parts of A, respectively(see [2, p. 6] and [7, p. 7]). A matrix $A \in \mathbb{M}_n$ is called accretive, if $\Re A$ is positive definite. Also, a matrix $A \in \mathbb{M}_n$ is called accretive-disipative, if both $\Re A$ and $\Im A$ are positive definite. Here, we recall that the numerical range of $A \in \mathbb{M}_n$ is defined by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}.$$

For $0 \leq \alpha < \frac{\pi}{2}$, we define a sector as follows:

$$\mathcal{S}_\alpha = \{z \in \mathbb{C} : \Re z > 0, |\Im z| \leq (\Re z)\tan\alpha\}.$$

We say a matrix $A \in \mathbb{M}_n$ is a sector matrix and write $A \in S_\alpha$, if whose numerical range is contained in sector $\mathcal{S}_\alpha$, i.e. $W(A) \subset \mathcal{S}_\alpha$. Since $W(A) \subset \mathcal{S}_\alpha$ implies that $W(X^*AX) \subset \mathcal{S}_\alpha$ for any nonzero $n \times m$ matrix X, thus $W(A^{-1}) \subset \mathcal{S}_\alpha$, that is, inverse of every sector matrix is a sector matrix. Clearly, a sector matrix is accretive with extra information about the angle α. For more information on sector matrices, the interested reader can refer to [4, 9–11, 13, 18].

Liu et. al [9] and Lin [11] extended the inequalities (2) to sector matrices as follows:

$$\cos^4\alpha\,\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} \leq \Re(A\sharp B) \leq \sec^2\alpha\,\Re\left(\frac{A+B}{2}\right). \tag{3}$$

Main aim of this paper is to give the new double inequalities for two sector matrices A, B satisfying the condition $0 < m \leq \Re A, \Re B \leq M$. These inequalities which refine the inequalities (3) will give in Section 2. In Section 3, we will present a few application corresponding to the obtained results in Section 2. Finally, we will extend the relative entropy and the Kantorovich inequality for sector matrices.

2. AM-GM-HM inequality for sector matrices

Furuichi and Moradi in [5, Eq.(2.6)], based on the well-known Hermite-Hadamard inequality, proved the following inequality for $0 \leq \alpha \leq 1$ and $0 < x \leq 1$

$$\psi_\alpha(x)x^\alpha \leq (1-\alpha) + \alpha x,$$

where $\psi_\alpha(x) = 1 + \frac{2^\alpha\alpha(1-\alpha)(x-1)^2}{(x+1)^{\alpha+1}}$. By putting $\alpha = \frac{1}{2}$, we get

$$\psi_{\frac{1}{2}}(x)x^{\frac{1}{2}} \leq \frac{x+1}{2}, \tag{4}$$

where $\psi_{\frac{1}{2}}(x) = 1 + \frac{\sqrt{2}(x-1)^2}{4(x+1)^{\frac{3}{2}}}$.

Now, using functional calculus, we obtain an analogue of [5, Theorem A] as follows:

Lemma 2.1. *Let $A, B \in \mathbb{B}(\mathcal{H})$ be two strictly positive operators under this condition that $0 < sA \leq B \leq tA$ for positive real numbers $0 < s \leq t$. Then*

$$\min\left\{\psi_{\frac{1}{2}}(s), \psi_{\frac{1}{2}}(t)\right\} A\sharp B \leq \frac{A+B}{2}. \tag{5}$$

Proof. Utilizing the inequality (4), for every strictly positive operator X, we have the following inequality:

$$\min_{s\le x\le t}\psi_{\frac{1}{2}}(x)X^{\frac{1}{2}}\le\frac{X+1}{2}. \tag{6}$$

It is clear that

$$s\le A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\le t.$$

By applying the monotonic property of operator functions for the inequality (6) and the operator $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$, we have the following inequality:

$$\min_{s\le x\le t}\psi_{\frac{1}{2}}(x)\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^{\frac{1}{2}}\le\frac{A^{-\frac{1}{2}}BA^{-\frac{1}{2}}+I}{2}, \tag{7}$$

By multiplying the both sides of the inequality (7) by $A^{\frac{1}{2}}$, we get to the desired result. □

Remark 2.2. *Let $A,B\in\mathbb{B}(\mathcal{H})$ be two strictly positive operators under this condition that $0<m\le A,B\le M$. An simple computation shows that $0<\frac{m}{M}A\le B\le\frac{M}{m}A$. Letting $s=\frac{m}{M}$ and $t=\frac{M}{m}$ in Lemma 2.1, we have*

$$\psi_{\frac{1}{2}}(h)A\sharp B\le\frac{A+B}{2}, \tag{8}$$

where $h=\frac{m}{M}$.

Remark 2.3. *It is clear that $\psi_{\frac{1}{2}}(h)\ge 1$. Therefore,* (8) *is a refinement of* (2).

Now, we are ready to present our main result applying (8). To do it, we need the following Lemmas:

Lemma 2.4. **([10, 11])** *Let $A\in M_n$ with $A\in S_\alpha$. Then we have $\mathfrak{R}(A^{-1})\le\mathfrak{R}^{-1}(A)\le\sec^2(\alpha)\mathfrak{R}(A^{-1})$. The first inequality holds for an accretive matrix $A\in\mathbb{M}_n$.*

Lemma 2.5. **([14])** *If $A,B\in\mathbb{M}_n$ be accretive and $0<\lambda<1$. Then*

$$\mathfrak{R}A\sharp_\lambda\mathfrak{R}B\le\mathfrak{R}(A\sharp_\lambda B).$$

Theorem 2.6. *Let $A,B\in S_\alpha$ be such that $0<m\le\mathfrak{R}A,\mathfrak{R}B\le M$. Then*

$$\cos^4\alpha\psi_{\frac{1}{2}}(h)\mathfrak{R}\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\le\mathfrak{R}(A\sharp B)\le\sec^2\alpha\psi_{\frac{1}{2}}^{-1}(h)\mathfrak{R}\left(\frac{A+B}{2}\right). \tag{9}$$

Proof. By [16, Lemma 5], for $\lambda=\frac{1}{2}$, we have

$$\mathfrak{R}(A\sharp B)\le\sec^2\alpha\mathfrak{R}(A)\sharp\mathfrak{R}(B).$$

On the other hand, applying (8) for $\mathfrak{R}(A)$ and $\mathfrak{R}(B)$, we get

$$\mathfrak{R}(A)\sharp\mathfrak{R}(B)\le\psi_{\frac{1}{2}}^{-1}(h)\mathfrak{R}\left(\frac{A+B}{2}\right),$$

where $h=\frac{m}{M}$. From two inequalities above, we obtain the second inequality of (9) as claimed.
If we first apply Lemma 2.5 for special case $\lambda=\frac{1}{2}$ and for A^{-1} and B^{-1} replacement A and B and then use the second inequality of (9), we obtain

$$\mathfrak{R}(A^{-1})\sharp\mathfrak{R}(B^{-1})\le\mathfrak{R}(A^{-1}\sharp B^{-1})\le\sec^2\alpha\psi_{\frac{1}{2}}^{-1}(h)\mathfrak{R}\left(\frac{A^{-1}+B^{-1}}{2}\right),$$

therefore

$$\mathfrak{R}(A^{-1})\sharp\mathfrak{R}(B^{-1}) \leq \sec^2\alpha\psi_{\frac{1}{2}}^{-1}(h)\mathfrak{R}\left(\frac{A^{-1}+B^{-1}}{2}\right).$$

If we take reverse on the both sides of the relation above, it follows that

$$(\mathfrak{R}(A^{-1})\sharp\mathfrak{R}(B^{-1}))^{-1} \geq \cos^2\alpha\psi_{\frac{1}{2}}(h)\mathfrak{R}^{-1}\left(\frac{A^{-1}+B^{-1}}{2}\right),$$

which is equivalent to

$$\sec^2\alpha\psi_{\frac{1}{2}}^{-1}(h)(\mathfrak{R}(A^{-1})\sharp\mathfrak{R}(B^{-1}))^{-1} \geq \mathfrak{R}^{-1}\left(\frac{A^{-1}+B^{-1}}{2}\right). \tag{10}$$

Compute

$$\begin{aligned}
\mathfrak{R}\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1} &\leq \mathfrak{R}^{-1}\left(\frac{A^{-1}+B^{-1}}{2}\right)\text{(by Lemma 2.4)} \\
&\leq \sec^2\alpha\psi_{\frac{1}{2}}^{-1}(h)(\mathfrak{R}(A^{-1})\sharp\mathfrak{R}(B^{-1}))^{-1}\text{(by (10))} \\
&= \sec^2\alpha\psi_{\frac{1}{2}}^{-1}(h)(\mathfrak{R}^{-1}(A^{-1})\sharp\mathfrak{R}^{-1}(B^{-1})) \\
&\leq \sec^4\alpha\psi_{\frac{1}{2}}^{-1}(h)(\mathfrak{R}A\sharp\mathfrak{R}B)\text{ (by Lemma 2.4)} \\
&\leq \sec^4\alpha\psi_{\frac{1}{2}}^{-1}(h)\mathfrak{R}(A\sharp B)\text{ (by Lemma 2.5)},
\end{aligned} \tag{11}$$

where the third inequality follows by the property of geometric mean. This proves the first inequality of (9). □

Remark 2.7. *From $\psi_{\frac{1}{2}}^{-1}(h) \leq 1(\psi_{\frac{1}{2}}(h) \geq 1)$, it is clear that the upper and lower bounds in* (9) *are tighter than ones in* (3).

3. Applications

Here, we present some applications of the inequality (9) such as unitarily invariant norm. A norm $\|\cdot\|_u$ is called an unitarily invariant norm if $\|X\|_u = \|UXV\|_u$ for any unitary matrices U, V and any $X \in \mathbb{M}_n$. We use the symbols $\lambda_j(X)$ and $s_j(X)$ as the j-th largest eigen value and singular value of X, respectively. The following lemmas are known.

Lemma 3.1. **([4, 17])** *Let $A \in S_\alpha$. Then*

$$\lambda_j(\mathfrak{R}A) \leq s_j(A) \leq \sec^2\alpha\lambda_j(\mathfrak{R}A), \qquad j = 1, \cdots, n.$$

Lemma 3.2. **([18])** *Let $A \in S_\alpha$. Then*

$$\|\mathfrak{R}(A)\|_u \leq \|A\|_u \leq \sec\alpha\|\mathfrak{R}(A)\|_u.$$

It is trivial that if $A \geq 0$(i. e. $A \in S_0$), then $\omega(A) = \|A\|$. So, we have $\omega(\mathfrak{R}A) = \|\mathfrak{R}A\|$. For $A \in S_\alpha$, Bedrani et al. [3] showed that

$$\omega(\mathfrak{R}A) \leq \omega(A) \leq \sec\alpha\omega(\mathfrak{R}A). \tag{12}$$

Theorem 3.3. *Let $A, B \in S_\alpha$ be such that $0 < m \leq \Re A, \Re B \leq M$. Then, the following inequalities hold:*

$$\cos^6 \alpha \psi_{\frac{1}{2}}(h) s_j \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \leq s_j(A \sharp B) \leq \sec^2 \alpha \psi^{-1}_{\frac{1}{2}}(h) s_j \left(\frac{A+B}{2} \right). \tag{13}$$

Proof. By simple computations

$$\begin{aligned} s_j \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} &\leq \sec^2 \alpha s_j \left(\Re \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \right) \text{(by Lemma 3.1)} \\ &\leq \sec^6 \alpha \psi^{-1}_{\frac{1}{2}}(h) s_j(\Re(A \sharp B)) \text{(by (9))} \\ &\leq \sec^6 \alpha \psi^{-1}_{\frac{1}{2}}(h) s_j(A \sharp B) \text{(by Lemma 3.1)}. \end{aligned}$$

This proves the left-hand side of the inequality (13). Similarly, to prove the right-hand side of the inequality (13), we have

$$\begin{aligned} s_j(A \sharp B) &\leq \sec^2 \alpha s_j(\Re(A \sharp B)) \text{(by Lemma 3.1)} \\ &\leq \sec^4 \alpha \psi^{-1}_{\frac{1}{2}}(h) s_j \left(\Re \left(\frac{A+B}{2} \right) \right) \text{(by (9))} \\ &\leq \sec^4 \alpha \psi^{-1}_{\frac{1}{2}}(h) s_j \left(\frac{A+B}{2} \right) \text{(by Lemma 3.1)}. \end{aligned}$$

This complete the proof of the inequality (13). □

Remark 3.4. *Again since $\psi_{\frac{1}{2}}(h) \geq 1$ or equivalently $\psi^{-1}_{\frac{1}{2}}(h) \leq 1$, thus the first inequality and the second inequality in* (13)*, respectively, refine [9, Eq. (3.5)] and [11, Eq. (13)] .*

For the special case, when A is accretive-dissipative (i.e. both $\Re A$ and $\Im A$ are positive), we have $e^{-i\frac{\pi}{4}} A \in S_{\frac{\pi}{4}}$. As an immediate result we have the following corollary:

Corollary 3.5. *Let $A, B \in \mathbb{M}_n$ be accretive-dissipative satisfying $0 < m \leq \Re A, \Re B \leq M$. Then*

$$8\psi_{\frac{1}{2}}(h) s_j \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \leq s_j(A \sharp B) \leq 2\psi^{-1}_{\frac{1}{2}}(h) s_j \left(\frac{A+B}{2} \right).$$

The next Theorem is a norm version of (9) for unitarily invariant norms.

Theorem 3.6. *Let $A, B \in \mathbb{S}_\alpha$ be such that $0 < m \leq \Re A, \Re B \leq M$. Then for any unitarily invariant norm $\|.\|_u$, the following inequalities hold:*

$$\cos^5 \alpha \psi_{\frac{1}{2}}(h) \left\| \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \right\|_u \leq \|A \sharp B\|_u \leq \sec^3 \alpha \psi^{-1}_{\frac{1}{2}}(h) \left\| \frac{A+B}{2} \right\|_u. \tag{14}$$

Proof. By Lemma 3.2, with the left-side of the inequality (9), we have

$$\begin{aligned} \left\| \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \right\|_u &\leq \sec \alpha \left\| \Re \left(\frac{A^{-1} + B^{-1}}{2} \right)^{-1} \right\|_u \\ &\leq \sec^5 \alpha \psi^{-1}_{\frac{1}{2}}(h) \|\Re(A \sharp B)\|_u \\ &\leq \sec^5 \alpha \psi^{-1}_{\frac{1}{2}}(h) \|A \sharp B\|_u. \end{aligned}$$

Analogously, Lemma 3.2 with the right-side of the inequality (9) follow that

$$\begin{aligned}\|A\sharp B\|_u &\le \sec\alpha\|\Re(A\sharp B)\|_u \\ &\le \sec^3\alpha\psi_{\frac{1}{2}}^{-1}(h)\left\|\Re\left(\frac{A+B}{2}\right)\right\|_u \\ &\le \sec^3\alpha\psi_{\frac{1}{2}}^{-1}(h)\left\|\frac{A+B}{2}\right\|_u.\end{aligned}$$

□

Remark 3.7. *It is obvious that* $\psi_{\frac{1}{2}}(h) \ge 1$. *That is, the inequality (14) implies [9, Eq. (3.6)] and [11, Eq. (14)].*

Corollary 3.8. *Let* $A, B \in \mathbb{M}_n$ *be accretive-dissipative such that* $0 < m \le \Re A, \Re B \le M$. *Then*

$$4\sqrt{2}\psi_{\frac{1}{2}}(h)\left\|\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right\|_u \le \|A\sharp B\|_u \le 2\sqrt{2}\psi_{\frac{1}{2}}^{-1}(h)\left\|\frac{A+B}{2}\right\|_u.$$

We finish this section by presenting the other application of the inequality (9). The neat lemma is needed to prove it.

Lemma 3.9. **([7, 10])** *If* $A \in S_\alpha$. *Then*

$$\det(\Re A) \le |\det A| \le \sec^n\alpha\det(\Re A).$$

The first inequality is known as the Ostrowski-Taussky inequality.

Theorem 3.10. *Let* $A, B \in S_\alpha$ *with* $0 < m \le \Re A, \Re B \le M$. *Then,*

$$\begin{aligned}\cos^{4n}\alpha\psi_{\frac{1}{2}}(h)\det\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right) &\le |\det(A\sharp B)| \\ &\le \sec^{3n}\alpha\psi_{\frac{1}{2}}^{-1}(h)\det\left(\Re\left(\frac{A+B}{2}\right)\right).\end{aligned} \tag{15}$$

Proof. By Lemma 3.9,

$$|\det(A\sharp B)| \le \sec^n\alpha\det(\Re(A\sharp B)).$$

On the other hand, by taking determinan from the right-side of the inequality (9) and making use the property of determinan, we derive

$$\det(\Re(A\sharp B)) \le \sec^{2n}\alpha\psi_{\frac{1}{2}}^{-1}(h)\det\left(\Re\left(\frac{A+B}{2}\right)\right).$$

Combining two latter relations, we obtain

$$|\det(A\sharp B)| \le \sec^{3n}\alpha\psi_{\frac{1}{2}}^{-1}(h)\det\left(\Re\left(\frac{A+B}{2}\right)\right).$$

The inequality above yields the right-side of the desired inequality.
Based on Lemma 3.9,

$$|\det(A\sharp B)| \ge \det(\Re(A\sharp B)).$$

With help of the left-side of the inequality (9),

$$\det(\Re(A\sharp B)) \ge \det\left(\Re\left(\frac{A^{-1}+B^{-1}}{2}\right)^{-1}\right)\sec^{4n}\alpha\psi_{\frac{1}{2}}(h).$$

From two inequalities above, we result the first inequality of (15). □

Remark 3.11. *A standard argument like that Remark 3.7 follows that the double inequalities* (15) *refine [15, Theorem 3.3], for* $\lambda = \frac{1}{2}$.

4. Relative operator entropy and operator Kantorovich inequality

For two strictly positive operators A and B, the relative operator entropy is defined by Fujii et. al [8]

$$\mathcal{S}(A|B) := A^{\frac{1}{2}} \log(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}.$$

Raissouli et. al [14] recently extended the definition above and defined the relative operator entropy of two accretive operators A and B via the following formula:

$$\mathcal{S}(A|B) = \int_0^1 \frac{A!_t B - A}{t} dt.$$

In the same time, they obtain following remarkable property about the relative operator entropy of two accretive operators:

$$\Re(\mathcal{S}(A|B)) \geq \mathcal{S}(\Re A|\Re B). \tag{16}$$

The next Lemma give a reverse of (16).

Theorem 4.1. *Let $A, B \in S_\alpha$. Then*

$$\Re(\mathcal{S}(A|B)) \leq \sec^2 \alpha \mathcal{S}(\Re A|\Re B). \tag{17}$$

Proof. By the first inequality of Lemma 2.4,

$$\Re(tA^{-1} + (1-t)B^{-1})^{-1} \leq \Re^{-1}(tA^{-1} + (1-t)B^{-1}) = (t\Re A^{-1} + (1-t)\Re B^{-1}))^{-1}. \tag{18}$$

Now, the second inequality of Lemma 2.4 ensure us that

$$t\Re^{-1}(A) + (1-t)\Re^{-1}(B) \leq \sec^2 \alpha (t\Re A^{-1} + (1-t)\Re B^{-1}).$$

Taking reverse from the latter inequality, we get

$$(t\Re A^{-1} + (1-t)\Re B^{-1})^{-1} \leq \sec^2 \alpha (t\Re^{-1} A + (1-t)\Re^{-1} B)^{-1}. \tag{19}$$

Applying (18), together with (19), we have

$$\Re(tA^{-1} + (1-t)B^{-1})^{-1} \leq \sec^2 \alpha (t\Re^{-1} A + (1-t)\Re^{-1} B)^{-1}. \tag{20}$$

Making use definition of the relative operator entropy of two accretive operators and using (20), it follows that

$$\begin{aligned}\Re(\mathcal{S}(A|B)) &= \int_0^1 \frac{\Re(tA^{-1} + (1-t)B^{-1})^{-1} - \Re A}{t} dt \\ &\leq \int_0^1 \sec^2 \alpha \frac{(t\Re^{-1} A + (1-t)\Re^{-1} B)^{-1} - \Re A}{t} dt \\ &= \sec^2 \alpha \mathcal{S}(\Re A|\Re B).\end{aligned}$$

This complete the proof. □

It is well known that for two positive operators A and B, the informational monotonicity property of relative operator entropy satisfies $\Phi(S(A|B)) \leq S(\Phi(A)|\Phi(B))$ for all unital positive linear maps Φ. Aplying (17), a sectorial operator version of the previous inequality stands below. Throughout of this section, Φ is a unital positive linear map.

Theorem 4.2. *Let $A, B \in S_\alpha$. Then*

$$\Re(\Phi(S(A|B))) \leq \sec^2\alpha \Re S(\Phi(A)|\Phi(B)).$$

Proof. We have the following chain of inequalities

$$\begin{aligned} \Re(\Phi(S(A|B))) &= \Phi(\Re(S(A|B))) \\ &\leq \sec^2\alpha \Phi(S(\Re A|\Re B))(\text{by}(17)) \\ &\leq \sec^2\alpha S(\Phi(\Re A)|\Phi(\Re B)) \\ &= \sec^2\alpha S(\Re\Phi(A)|\Re\Phi(B)) \\ &\leq \sec^2\alpha \Re S(\Phi(A)|\Phi(B)). \end{aligned}$$

□

Corollary 4.3. *Let $A, B \in S_\alpha$. Then*

$$\omega(\Phi(S(A|B))) \leq \sec^3\alpha\, \omega(S(\Phi(A)|\Phi(B))).$$

Proof. We compute

$$\begin{aligned} \omega(\Phi(S(A|B))) &\leq \sec\alpha\, \omega(\Re\Phi(S(A|B)))(\text{by}(12)) \\ &= \sec\alpha \|\Re\Phi(S(A|B))\| \\ &\leq \sec^3\alpha \|\Re S(\Phi(A)|\Phi(B))\|(\text{by Theorem 4.2}) \\ &= \sec^3\alpha\, \omega(\Re S(\Phi(A)|\Phi(B))) \\ &\leq \sec^3\alpha\, \omega(S(\Phi(A)|\Phi(B))).(\text{by}(12)) \end{aligned}$$

□

Corollary 4.4. *Let $A, B \in \mathbb{M}_n$ be accretive-dissipative. Then*

$$\omega(\Phi(S(A|B))) \leq 2\sqrt{2}\omega(S(\Phi(A)|\Phi(B))).$$

Corollary 4.5. *Let $A, B \in S_\alpha$. Then*

$$\|\Phi(S(A|B))\|_u \leq \sec^3\alpha \|S(\Phi(A)|\Phi(B))\|_u.$$

Proof. We estimate

$$\begin{aligned} \|\Phi(S(A|B))\|_u &\leq \sec\alpha \|\Re\Phi(S(A|B))\|_u(\text{by Lemma 3.2}) \\ &\leq \sec^3\alpha \|\Re S(\Phi(A)|\Phi(B))\|_u(\text{by Theorem 4.2}) \\ &\leq \sec^3\alpha \|S(\Phi(A)|\Phi(B))\|_u.(\text{by Lemma 3.2}) \end{aligned}$$

□

Corollary 4.6. *Let $A, B \in \mathbb{M}_n$ be accretive-dissipative. Then*

$$\|\Phi(S(A|B))\|_u \leq 2\sqrt{2}\|S(\Phi(A)|\Phi(B))\|_u.$$

Throughout of this section, $K(h) = \frac{(M+m)^2}{4Mm}$ with $h = \frac{M}{m}$ is Kantorovich constant, where M, m are positive real numbers. For $A \in \mathbb{M}_n$ such that $0 < m \leq A \leq M$, Marshall and Olkin [12] obtained an operator Kantorovich inequality as follows:

$$\Phi(A^{-1}) \leq K(h)\Phi^{-1}(A),$$

where Φ is a positive unital linear map. The next Lemma is an extension of Kantorovich operator inequality.

Theorem 4.7. *Let $A \in S_\alpha$ be such that $0 < m \leq \Re A \leq M$. Then*

$$\Re\Phi(A^{-1}) \leq K(h)\sec^2\alpha \Re\Phi^{-1}(A).$$

Proof. The desired inequality concludes by the computation of the following chain of the inequalities:

$$\begin{aligned}\Re\Phi(A^{-1}) &= \Phi(\Re A^{-1})(\text{by [16, Lemma 1]})\\ &\leq \Phi(\Re^{-1}A)(\text{by the first inequality of Lemma 2.4})\\ &\leq K(h)\Phi^{-1}(\Re A)(\text{by the Kantorovich inequality })\\ &= K(h)\Re^{-1}\Phi(A)\\ &\leq K(h)\sec^2\Re\Phi^{-1}(A)(\text{by the second inequality of Lemma 2.4}).\end{aligned}$$

□

Corollary 4.8. *Let $A \in S_\alpha$ be such that $m \leq \Re A \leq M$. Then*

$$\left\|\Phi(A^{-1})\right\|_u \leq K(h)\sec^3\alpha \left\|\Phi^{-1}(A)\right\|_u.$$

Proof. By applying an simple computation, we obtain

$$\begin{aligned}\left\|\Phi(A^{-1})\right\|_u &\leq \sec\alpha \left\|\Re\Phi(A^{-1})\right\|_u (\text{by 12})\\ &\leq K(h)\sec^3\alpha \left\|\Re\Phi^{-1}(A)\right\|_u (\text{by Lemma 4.7})\\ &\leq K(h)\sec^3\alpha \left\|\Phi^{-1}(A)\right\|_u .(\text{by 12})\end{aligned}$$

□

Corollary 4.9. *Let $A \in \mathbb{M}_n$ be accretive-dissipative with $0 < m \leq \Re A \leq M$. Then*

$$\|\Phi(A^{-1})\| \leq 2\sqrt{2}K(h)\|\Phi^{-1}(A)\|.$$

Corollary 4.10. *Let $A \in S_\alpha$ be such that $m \leq \Re A \leq M$. Then*

$$\omega(\Phi(A^{-1})) \leq K(h)\sec^3\alpha\omega(\Phi^{-1}(A)).$$

Proof. We have

$$\begin{aligned}\omega(\Phi(A^{-1})) &\leq \sec\alpha\omega(\Re\Phi(A^{-1}))(\text{by 12})\\ &= \sec\alpha \left\|\Re\Phi(A^{-1})\right\|\\ &\leq K(h)\sec^3(\theta)\left\|\Re\Phi^{-1}(A)\right\| (\text{by Lemma 4.7})\\ &= K(h)\sec^3(\theta)\omega(\Re\Phi^{-1}(A))\\ &\leq K(h)\sec^3(\theta)\omega(\Phi^{-1}(A)).(\text{by 12})\end{aligned}$$

□

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