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Generalized weighted core inverse in Banach *-algebras

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Abstract. In this paper, we introduce the notion of the generalized weighted core inverse in a Banach *-algebra. We characterize this new generalized inverse by using the generalized weighted core decomposition and present the representations by the generalized Drazin inverse. These extend the core-EP inverse and weighted (pseudo) core inverse for complex matrices and linear bounded operators to more general setting.

1. Introduction

Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n \times n$ complex matrices with conjugate transpose $*$. The core-EP inverse of a complex matrix was introduced by Prasad and Mohana (see [13]). A matrix $A \in \mathbb{C}^{n\times n}$ has core-EP inverse X if and only if

$$AX^2 = X, (AX)^* = AX, XA^{k+1} = A^k,$$

where $k = ind(A)$ is the Drazin index of A. Such X is unique, and we denote it by $A^{\textcircled{\dagger}}$. Core-EP inverse is a natural extension of the core inverse which was first studied by Baksalary and Trenkler for a complex matrix in 2010 (see [1]). Behera et al. extended the concept of the core-EP inverse and introduced the notion of weighted core-EP inverse for a complex matrix (see [2]). The weighted core-EP inverse of A is the unique solution to the system:

$$AX^2 = X, (EAX)^* = EAX, XA^{k+1} = A^k,$$

where $k = ind(A)$, and we denote X by $A^{\textcircled{\dagger},E}$, where $E \in \mathbb{C}^{n\times n}$.

Recently, the generalized inverses mentioned above were extended to ring and algebras. A Banach algebra $\mathcal{A}$ is called a Banach *-algebra if there exists an involution $* : x \to x^*$ satisfying $(x+y)^* = x^*+y^*, (\lambda x)^* = \overline{\lambda}x^*, (xy)^* = y^*x^*, (x^*)^* = x$. Rakić et al. generalized the core inverse of a complex matrix to the case of an element in a ring (see [21]). An element a in a Banach *-algebra $\mathcal{A}$ has core inverse if and only if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (ax)^* = ax, xa^2 = a.$$

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If such x exists, it is unique, and denote it by $a^{\textcircled{\#}}$. Gao and Chen extended the concept of the core inverse and introduced the notion of core-EP inverse (i.e., pseudo core inverse) (see [7]). An element $a \in \mathcal{A}$ has core-EP inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (ax)^* = ax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^{\textcircled{D}}$. Many authors have investigated core and core-EP inverses from many different views, e.g., [5, 6, 8, 15, 16, 23, 25, 26].

Mosić et al. introduced and studied weighted core inverse for a ring element (see [18]). Let $a \in \mathcal{A}$ and $e \in \mathcal{A}$ be an invertible Hermitian element (i.e., e is invertible and $e^* = e$). An element $a \in \mathcal{A}$ has e-core inverse if there exist $x \in \mathcal{A}$ such that

$$ax^2 = x, (eax)^* = eax, xa^2 = a.$$

If such x exists, it is unique, and denote it by $a^{e,\textcircled{\#}}$. Let $\mathcal{A}^{e,\textcircled{\#}}$ denote the set of all e-core invertible elements in $\mathcal{A}$. In [29], Zhu and Wang extended the notion of weighted core inverse and introduced weighted pseudo core inverse. An element $a \in \mathcal{A}$ has pseudo e-core inverse if there exist $x \in \mathcal{A}$ and $k \in \mathbb{N}$ such that

$$ax^2 = x, (eax)^* = eax, xa^{k+1} = a^k.$$

If such x exists, it is unique, and denote it by $a^{e,\textcircled{D}}$. We refer the reader for weighted core inverse to [9, 11].

The motivation of this paper is to introduce and study a new kind of generalized inverse as a natural generalization of generalized inverses mentioned above. In Section 2, we introduce generalized weighted core inverse in terms of a new kind of generalized weighted core decomposition. Many new properties of the core-EP inverse and weighted (pesudo) core inverse are thereby obtained.

Definition 1.1. *An element $a \in \mathcal{A}$ has generalized e-core decomposition if there exist $x, y \in \mathcal{A}$ such that*

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}^{e,\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

Here,

$$\mathcal{A}^{qnil} = \{x \in \mathcal{A} \mid \lim_{n\to\infty} \| x^n \|^{\frac{1}{n}} = 0\}.$$

As is well known, $x \in \mathcal{A}^{qnil}$ if and only if $1 + \lambda x \in \mathcal{A}$ is invertible for any $\lambda \in \mathbb{C}$. We prove that $a \in \mathcal{A}$ has generalized e-core decomposition if and only if there exists unique $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Recall that $a \in \mathcal{A}$ has g-Drazin inverse (i.e., generalized Drazin inverse) if there exists $x \in \mathcal{A}$ such that

$$ax^2 = x, ax = xa, a - a^2x \in \mathcal{A}^{qnil}.$$

Such x is unique, if exists, and denote it by a^d. The g-Drazin inverse plays an important role in matrix and operator theory (see [4]). In Section 3, we establish relations between generalized weighted core inverse and g-Drazin inverse for an element in a Banach *-algebra by using involved images. We prove that $a \in \mathcal{A}^{e,\textcircled{d}}$ if and only if $a \in \mathcal{A}^d$ and there exists $x \in \mathcal{A}$ such that

$$xax = x, x\mathcal{A} = a^d\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^d)^*e.$$

In [19], Mosić and Djordjević introduced and studied core-EP inverse for g-Drazin invertible Hilbert space operators. This new generalized inverse was extensively investigated in [14, 20]. Recently, Mosić extended core-EP inverse of bounded linear operators on Hilbert spaces to elements of a C^*-algebra by means of range projections (see [17]). Surprisingly, the preceding result shows that the core-EP inverse of Mosić and Djordjević coincide with the generalized weighted core inverse with weight $e = 1$. New representations of core-EP inverse for Hilbert space operators are thereby added.

Finally, in Section 4, the relations between other generalized weighted inverses for elements in a Banach *-algebra were presented. We prove that $a \in \mathcal{A}^{e,\textcircled{d}}$ if and only if $a \in \mathcal{A}^d$ and a^d has e-core inverse.

Throughout the paper, all Banach *-algebras are complex with an identity. Let $e \in \mathcal{A}$ be an invertible Hermitian element, (i.e., e is invertible and $e^* = e$). $\mathcal{A}^d$ and $\mathcal{A}^{\textcircled{\#}}$ denote the sets of all g-Drazin and core invertible elements in $\mathcal{A}$, respectively. Let $a \in \mathcal{A}^d$. We use a^π to stand for the spectral idempotent $1 - aa^d$.

2. Generalized weighted core decomposition

The aim of this section is to introduce the notion of the generalized weighted core inverse in a Banach *-algebra. We begin with

Theorem 2.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized e-core decomposition.
(2) There exists $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Proof. (1) ⇒ (2) By hypothesis, there exist $z, y \in \mathcal{A}$ such that

$$a = z + y, z^*ey = yz = 0, z \in \mathcal{A}^{e,\oplus}, y \in \mathcal{A}^{qnil}.$$

Set $x = z^{e,\oplus}$. One easily checks that

$$\begin{aligned} ax &= (z+y)z^{e,\oplus} = zz^{e,\oplus} + yz(z^{e,\oplus})^2 = zz^{e,\oplus}, \\ ax^2 &= (z+y)(z^{e,\oplus})^2 = z(z^{e,\oplus})^2 = z^{e,\oplus} = x, \\ z^{e,\oplus}y &= xy = xzxy = xe^{-1}(ezx)y = xe^{-1}(ezx)^*y \\ &= xe^{-1}x^*(z^*ey) = 0. \end{aligned}$$

Now by applying $z^{e,\oplus}y = 0$ and [10, Lemma 1.4], we deduce that

$$axa = (ax)a = zz^{e,\oplus}(z+y) = zz^{e,\oplus}z = z.$$

Then

$$(eax)^* = (ezz^{e,\oplus})^* = ezz^{e,\oplus} = eax,$$
$$a(1-xa) = a - axa = a - z = y \in \mathcal{A}^{qnil}.$$

By using Cline's formula (see [12, Theorem 2.1]), $a - xa^2 = (1-xa)a \in \mathcal{A}^{qnil}$. As $yz = 0$, we see that

$$(a - xa^2)z = [z + y - z^{e,\oplus}(z+y)^2]z = (z - z^{e,\oplus}z^2)z = 0.$$

Thus we have

$$\begin{aligned} ||a^n - xa^{n+1}||^{\frac{1}{n}} &= ||(a-xa^2)a^{n-1}||^{\frac{1}{n}} \\ &= ||(a-xa^2)(z+y)^{n-1}||^{\frac{1}{n}} \\ &= ||(a-xa^2)y^{n-1}||^{\frac{1}{n}} \\ &\le ||a-xa^2||^{\frac{1}{n}}[||y^{n-1}||^{\frac{1}{n-1}}]^{1-\frac{1}{n}}. \end{aligned}$$

Since $y \in \mathcal{A}^{qnil}$, we deduce that

$$\lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0,$$

as required.

(2) ⇒ (1) By hypotheses, we have $z \in \mathcal{A}$ such that

$$z = az^2, (eaz)^* = eaz, \lim_{n\to\infty} ||a^n - za^{n+1}||^{\frac{1}{n}} = 0.$$

For any $n \in \mathbb{N}$, we have

$$\begin{aligned} az &= a(az^2) = a^2z^2 = a^2(az^2)z = a^3z^3 \\ &= \cdots = a^nz^n = \cdots = a^{n+1}z^{n+1}. \end{aligned}$$

Hence

$$\begin{aligned} ||z - zaz|| &= ||(az)z - zaz|| \\ &= ||(a^nz^n)z - z(a^{n+1}z^{n+1})|| \\ &= ||(a^n - za^{n+1})z^{n+1}||. \end{aligned}$$

Then

$$\|z - zaz\|^{\frac{1}{n}} \leq \|(a^n - za^{n+1})\|^{\frac{1}{n}}\|z\|^{1+\frac{1}{n}}.$$

We infer that

$$\lim_{n\to\infty} \|z - zaz\|^{\frac{1}{n}} = 0,$$

hence, $z = zaz$.

Set $x = aza$ and $y = a - aza$. Then $a = x + y$. We check that

$$\begin{array}{rcl} (a - za^2)z & = & (a - za^2)az^2 \\ & = & (a - za^2)a^2z^3 \\ & \vdots & \\ & = & (a - za^2)a^{n-1}z^n \\ & = & (a^n - za^{n+1})z^n. \end{array}$$

Therefore

$$\|(a - za^2)z\|^{\frac{1}{n}} \leq \|a^n - za^{n+1}|^{\frac{1}{n}}\|\|z^n\|^{\frac{1}{n}}.$$

Since

$$\lim_{n\to\infty} \|a^n - za^{n+1}\|^{\frac{1}{n}} = 0,$$

we prove that

$$\lim_{n\to\infty} \|(a - za^2)z\|^{\frac{1}{n}} = 0.$$

This implies that $(a - za^2)z = 0$.

We claim that x has e-core inverse. Evidently, we verify that

$$\begin{array}{rcl} zx^2 & = & za(za^2z)a = za^2za = aza = x, \\ xz^2 & = & azaz^2 = az^2 = z, \\ (exz)^* & = & (eazaz)^* = (eaz)^* = eaz = e(aza)z = exz. \end{array}$$

Therefore $x \in \mathcal{A}^{e,\textcircled{\#}}$ and $z = x^{e,\textcircled{\#}}$.

We verify that

$$\begin{array}{rcl} \|(a - za^2)^{n+1}\|^{\frac{1}{n+1}} & = & \|(a - za^2)^n(a - za^2)\|^{\frac{1}{n+1}} \\ & = & \|(a - za^2)^{n-1}(a - za^2)a\|^{\frac{1}{n+1}} \\ & = & \|(a - za^2)^{n-1}a^2\|^{\frac{1}{n+1}} \\ & \vdots & \\ & = & \|(a - za^2)a^n\|^{\frac{1}{n+1}} \\ & \leq & \left[\|a^n - za^{n+1}\|^{\frac{1}{n}}\right]^{\frac{n}{n+1}}\|a^n\|^{\frac{1}{n+1}}. \end{array}$$

Accordingly,

$$\lim_{n\to\infty} \|(a - za^2)^{n+1}\|^{\frac{1}{n+1}} = 0.$$

This implies that $a - za^2 \in \mathcal{A}^{qnil}$. By using Cline's formula, $y = a - aza \in \mathcal{A}^{qnil}$.

Moreover, we see that

$$\begin{array}{rcl} x^*ey & = & (aza)^*e(1 - az)a = a^*(az)^*e^*(1 - az)a \\ & = & a^*(eaz)^*(1 - az)a = 0, \\ & = & a^*(eaz)(1 - az)a = 0, \\ yx & = & (a - aza)aza = a(a - za^2)za = 0. \end{array}$$

Then we have a generalized e-core decomposition $a = x + y$, thus yielding the result. □

Corollary 2.2. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}$ has generalized e-core decomposition.
(2) There exists unique $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Proof. (2) ⇒ (1) This is clear by Theorem 2.1.

(1) ⇒ (2) In view of Theorem 2.1, there exists $x \in \mathcal{A}$ such that

$$x = ax^2, (eax)^* = eax, \lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Claim 1. $\lim_{n\to\infty} ||a^n - a^n x^n a^n||^{\frac{1}{n}} = 0$. We verify that

$$\begin{array}{rcl} ||a^n - a^n x^n a^n||^{\frac{1}{n}} & = & ||a^n - xa^{n+1} + (ax^2)a^{n+1} - a^n x^n a^n||^{\frac{1}{n}} \\ & = & ||a^n - xa^{n+1} + (a^n x^n)xa^{n+1} - a^n x^n a^n||^{\frac{1}{n}} \\ & \leq & ||a^n - xa^{n+1} - (a^n x^n)(a^n - xa^{n+1})||^{\frac{1}{n}} \\ & \leq & ||1 - a^n x^n||^{\frac{1}{n}} ||a^n - xa^{n+1}||^{\frac{1}{n}} \\ & \leq & (1 + ||a||||x||)||a^n - xa^{n+1}||^{\frac{1}{n}}. \end{array}$$

Since

$$\lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = [\lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n+1}}]^{1+\frac{1}{n}} = 0,$$

we get

$$\lim_{n\to\infty} ||a^n - a^n x^n a^n||^{\frac{1}{n}} = 0.$$

Claim 2. Assume that there exists $z \in \mathcal{A}$ such that

$$z = az^2, (eaz)^* = eaz, \lim_{n\to\infty} ||a^n - za^{n+1}||^{\frac{1}{n}} = 0.$$

Then $axa = aza$.

Set $a_1 = axa, a_2 = a - a_1$ and $b_1 = aza, b_2 = a - b_1$. As in the proof of Theorem 2.1, we prove that

$$\begin{array}{c} a_1^* e a_2 = a_2 a_1 = 0, a_2 \in \mathcal{A}^{qnil}, \\ b_1^* e b_2 = b_2 b_1 = 0, b_2 \in \mathcal{A}^{qnil}. \end{array}$$

For every $n \in \mathbb{N}$, $a^n = \sum_{i=0}^{n} b_1^i b_2^{n-i}$, and so $(a^n)^* e b_2 = (b_2^n)^* e b_2$. Since $b_2 b_1 = 0$, we have $a^n b_1 (b_1^n)^\# = (b_1)^n b_1 (b_1^n)^\# = b_1$. Since $ax^2 = x$, we have $a^n x^n = ax$. Then

$$\begin{array}{rcl} ||b_1 - a_1||^2 & = & ||b_1 - axa||^2 \\ & = & ||b_1 - axb_1 - axb_2||^2 \\ & = & ||b_1 - axb_1 - e^{-1}(eax)b_2||^2 \\ & = & ||b_1 - axb_1 - e^{-1}(eax)^* b_2||^2 \\ & = & ||b_1 - a^n x^n b_1 - e^{-1}(ea^n x^n)^* b_2||^2 \\ & = & ||b_1 - a^n x^n b_1 - e^{-1}(x^n)^*(a^n)^* e b_2||^2 \\ & = & ||b_1 - a^n x^n a^n b_1 (b_1)^\# - e^{-1}(x^n)^*(a^n)^* e b_2||^2 \\ & = & ||(a^n - a^n x^n a^n) b_1 (b_1)^\# - e^{-1}(x^n)^*(b_2^*)^n e b_2||^2 \\ & \leq & ||a^n - a^n x^n a^n||^2 ||b_1(b_1)^\#||^2 + ||e^{-1}(x^n)^*||^2 ||eb_2||^2 ||(b_2^*)^n||^2 \\ & + & 2||a^n - a^n x^n a^n|| ||(b_2^*)^n|| ||b_1(b_1)^\#|| ||(x^n)^*|| ||b_2||. \end{array}$$

Therefore

$$\begin{array}{rl} & ||b_1 - a_1||^{\frac{2}{n}} \\ \leq & ||a^n - a^n x^n a^n||^{\frac{2}{n}} ||b_1(b_1)^\#||^{\frac{2}{n}} + ||(x^n)^*||^{\frac{2}{n}} ||b_2||^{\frac{2}{n}} ||(b_2^*)^n||^{\frac{2}{n}} \\ + & 2||a^n - a^n x^n a^n||^{\frac{1}{n}} ||(b_2^*)^n||^{\frac{1}{n}} ||b_1(b_1)^\#||^{\frac{2}{n}} ||e^{-1}(x^n)^*||^{\frac{2}{n}} ||eb_2||^{\frac{2}{n}} \\ \leq & ||1 - ax||^{\frac{2}{n}} ||||a||^2 ||b_1(b_1)^\#||^{\frac{2}{n}} + ||x^*||^2 ||b_2||^{\frac{2}{n}} ||(b_2^*)^n||^{\frac{2}{n}} \\ + & 2||1 - ax||^{\frac{1}{n}} ||a||^2 ||(b_2^*)^n||^{\frac{2}{n}} ||b_1(b_1)^\#||^{\frac{2}{n}} ||e^{-1}||^{\frac{2}{n}} ||x^*||^2 ||eb_2||^{\frac{2}{n}}. \end{array}$$

Since $b_2 \in \mathcal{A}^{qnil}$, then $1-\overline{\lambda}b_2 \in \mathcal{A}^{-1}$; hence, $1-\lambda b_2^* \in \mathcal{A}^{-1}$. This implies that $b_2^* \in \mathcal{A}^{qnil}$; hence,

$$\lim_{n\to\infty} \|(b_2^*)^n\|^{\frac{1}{n}} = 0.$$

Accordingly,

$$\lim_{n\to\infty} \|b_1 - a_1\|^{\frac{2}{n}} = 0,$$

and so $a_1 = b_1$.

As in the proof of Theorem 2.1, we verify that $x = (axa)^{e,\#} = (a_1)^{e,\#} = (b_1)^{e,\#} = (aza)^{e,\#} = z$. Accordingly, $x = z$, the result follows. □

We denote x in Corollary 2.2 by $a^{e,\circledS}$, and call it the generalized e-core inverse of a. Let $\mathbb{C}^{n\times n}$ be the Banach algebra of all $n\times n$ complex matrices, with conjugate transpose as the involution. For a complex $A \in \mathbb{C}^{n\times n}$, it follows by [7, Theorem 3.4] that the pseudo core inverse and generalized core inverse coincide with each other, i.e., $A^{\circledD} = A^{e,\circledS}$ for $e = 1$.

Theorem 2.3. *Let $a = x + y$ be the generalized e-core decomposition of $a \in \mathcal{A}$. Then $a^{e,\circledS} = x^{e,\#}$.*

Proof. Let $a = x + y$ be the generalized e-core decomposition of $a \in \mathcal{A}$. Similarly to the proof of Theorem 2.1, $x^{e,\#}$ is the generalized e-core inverse of a. So the theorem is true. □

Corollary 2.4. *Let $a \in \mathcal{A}^{e,\circledS}$. Then the following hold.*

(1) $a^{e,\circledS} = a^{e,\circledS}aa^{e,\circledS}$.
(2) $aa^{e,\circledS} = a^m(a^{e,\circledS})^m$ for any $m \in \mathbb{N}$.

Proof. These are obvious by the proof of Theorem 2.1 and Theorem 2.3. □

We are ready to prove:

Theorem 2.5. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e,\circledS}$ if and only if*

(1) $a \in \mathcal{A}^d$;
(2) There exists $x \in \mathcal{A}$ such that

$$ax^2 = x, (eax)^* = eax, \lim_{n\to\infty} \|a^n - axa^n\|^{\frac{1}{n}} = 0.$$

In this case,

$$\begin{aligned} a^d &= \sum_{n=0}^{\infty} (a^{e,\circledS})^{n+1}(a - aa^{e,\circledS}a)^n[1 - (a - aa^{e,\circledS}a)(a - aa^{e,\circledS}a)^d], \\ a^{e,\circledS} &= aa^dz, \text{ where } az^2 = z, (eaz)^* = eaz; a^{e,\circledS}\mathcal{A} = a^d\mathcal{A}. \end{aligned}$$

Proof. ⇒ As in the proof of Corollary 2.2, there exists $b \in \mathcal{A}$ such that

$$ab^2 = b, (eab)^* = eab, \lim_{n\to\infty} \|a^n - a^nb^na^n\|^{\frac{1}{n}} = 0.$$

In view of Theorem 2.1, there exists $z \in \mathcal{A}$ such that

$$z = az^2, (eaz)^* = eaz, \lim_{n\to\infty} \|a^n - za^{n+1}\|^{\frac{1}{n}} = 0.$$

Set $x = aza$ and $y = a - aza$. By the proof of Theorem 2.1, we have $a = x + y, x \in \mathcal{A}^{e,\#}, y \in \mathcal{A}^{qnil}, x^*ey = yx = 0$.

Clearly, $y \in \mathcal{A}^d$, $y^d = 0$ and $yx = 0$. In view of [18, Theorem 2.1], $x \in \mathcal{A}^{e,\#}$. By using [18, Theorem 2.1], $x^\# = (x^{e,\#})^2 x$. In light of [3, Corollary 3.4], $a = y + x$ has g-Drazin inverse. Since $y^d = 0$ and $x^{e,\#} = z = a^{e,@}$, then $x^d = z^2 x = (a^{e,@})^2 aa^{e,@}a = (a^{e,@})^2 a$. By using [3, Corollary 3.4], we have

$$\begin{aligned} a^d &= \sum_{n=0}^{\infty}(x^d)^{n+1}y^n y^\pi \\ &= \sum_{n=0}^{\infty}((a^{e,@})^2 a)^{n+1}(a - aa^{e,@}a)^n[1 - (a - aa^{e,@}a)(a - aa^{e,@}a)^d]. \end{aligned}$$

Since $a(a^{e,@})^2 = a^{e,@}$ and $(eaa^{e,@})^* = eaa^{e,@}$. Then (2) holds.

$\Leftarrow$ By hypothesis, there exists some $z \in \mathcal{A}$ such that

$$az^2 = z, (eaz)^* = eaz, \lim_{n\to\infty} ||a^n - a^n z^n a^n||^{\frac{1}{n}} = 0.$$

Let $x = aa^d z$. We claim that $a^{e,@} = x$. One directly verifies that

$$\begin{aligned} ||az - ax|| &= ||(a - a^2 a^d)z|| = ||(1 - aa^d)az|| \\ &= ||(1 - aa^d)^n a^n z^n|| = ||(a - a^2 a^d)^n z^n|| \le ||(a - a^2 a^d)^n|| ||z^n||, \end{aligned}$$

Then

$$\lim_{n\to\infty} ||az - ax||^{\frac{1}{n}} = 0;$$

hence, $ax = az$, and so $(eax)^* = (eaz)^* = eaz = eax$.

$$\begin{aligned} x - ax^2 &= aa^d z - (az)x = aa^d z - azaa^d z \\ &= (aa^d z - az^2) + az(1 - aa^d)z = (a^2 a^d z^2 - az^2) \\ &+ az(1 - aa^d)z = (aa^d - 1)az^2 + az(1 - aa^d)z \\ &= (az - 1)(1 - aa^d)z. \end{aligned}$$

Since $z = az^2$, by induction, we have $x - ax^2 = (az-1)(1-aa^d)a^n z^{n+1}$; hence, $||x - ax^2|| \le ||az-1|| ||(a - a^2 a^d)^n|| ||z||^{n+1}$. Since

$$\lim_{n\to\infty} ||(a - a^2 a^d)^n||^{\frac{1}{n}} = 0,$$

we deduce that

$$\lim_{n\to\infty} ||x - ax^2||^{\frac{1}{n}} = 0.$$

This implies that $ax^2 = x$.

Observing that

$$\begin{aligned} & ||a^n - xa^{n+1}||^{\frac{1}{n}} \\ = & ||(a^n - a^{n+1}a^d) + (a^d a^n a - a^d a^n z^n a^n a)||^{\frac{1}{n}} \\ \le & ||a^n - a^{n+1}a^d||^{\frac{1}{n}} + ||a^d||^{\frac{1}{n}} ||a^n - a^n z^n a^n||^{\frac{1}{n}} ||a||^{\frac{1}{n}}, \end{aligned}$$

we have

$$\lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

This completes the proof. □

Corollary 2.6. *Let $A \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix. Then the following are equivalent:*

(1) A has pseudo E-core inverse.
(2) There exists $X \in \mathbb{C}^{n\times n}$ such that

$$AX^2 = X, (EAX)^* = EAX, \lim_{n\to\infty} ||A^n - AXA^n||^{\frac{1}{n}} = 0.$$

Proof. For the Banach algebra $\mathbb{C}^{n\times n}$, the generalized weighted core inverse and weighted pseudo core inverse coincide with each other. This completes the proof by Theorem 2.5. □

In [24], Wang et al. introduced and studied right pseudo e-core inverse. An element $a \in \mathcal{A}$ is right pseudo e-core invertible if there exist $x \in \mathcal{A}$ and some positive integer k such that $axa^k = a^k, x = ax^2$ and $(eax)^* = eax$. By using Theorem 2.1 and Theorem 2.5, every g-Drazin and right pseudo e-core invertible element in a Banach *-algebra is the sum of an e-core invertible and nilpotent elements.

3. Characterizations by using involved images

The aim of this section is to characterize generalized weighted core inverse of an element in a Banach *-algebra by its involved images.

Lemma 3.1. *Let $a \in \mathcal{A}^d$. Then*
$$\lim_{n\to\infty} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} = 0.$$

Proof. Let $x = a - a^d a^2$. Then $x \in \mathcal{A}^{qnil}$. For any $\lambda \in \mathbb{C}$, we have $1 - \overline{\lambda}x \in \mathcal{A}^{-1}$, and so $1 - \lambda x^* \in \mathcal{A}^{-1}$. This implies that $x^* \in \mathcal{A}^{qnil}$. We easily check that

$$\begin{aligned} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} &= \|(1-a^d a)^*(a^n)^*\| = \|[(1-a^d a)^n]^*(a^n)^*\| \\ &= \|[(a-a^d a^2)^n]^*\| = \|(x^*)^n\|. \end{aligned}$$

Since $x^* \in \mathcal{A}^{qnil}$, we have
$$\lim_{n\to\infty} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} = 0.$$
□

Lemma 3.2. *Let $a \in \mathcal{A}^{e,ⓓ}$. Then*
$$\lim_{n\to\infty} \|(a^n - aa^{e,ⓓ}a^n)^*\|^{\frac{1}{n}} = 0.$$

Proof. Construct x, y, z as in the proof of Theorem 2.1. Then

$$\lim_{n\to\infty} \|(a^n - xa^{n+1})^*\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(y^*)^n\|^{\frac{1}{n}} = 0.$$

Similarly to Corollary 2.2, we check that

$$\|(a^n - a^n x^n a^n)^*\|^{\frac{1}{n}} \leq (1 + \|a^*\|\|x^*\|)\|(a^n - xa^{n+1})^*\|^{\frac{1}{n}}.$$

Therefore
$$\lim_{n\to\infty} \|(a^n - a^n (a^{e,ⓓ})^n a^n)^*\|^{\frac{1}{n}} = 0.$$

In view of Corollary 2.4, we have
$$\lim_{n\to\infty} \|(a^n - aa^{e,ⓓ}a^n)^*\|^{\frac{1}{n}} = 0,$$

as asserted. □

We are ready to prove:

Theorem 3.3. *Let $a \in \mathcal{A}$. Then $a \in \mathcal{A}^{e,ⓓ}$ if and only if*

(1) $a \in \mathcal{A}^d$;
(2) There exists $x \in \mathcal{A}$ such that
$$xax = x, x\mathcal{A} = a^d\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^d)^*e.$$

In this case, $a^{e,\circledast} = x$.

Proof. $\Longrightarrow$ Choose $x = a^{e,\circledast}$. By using Corollary 2.4 and Theorem 2.5, we have $xax = x, x\mathcal{A} = a^d\mathcal{A}$.

Since $ax = a^n x^n$,we have

$$\begin{array}{rcl} x & = & xax = xe^{-1}(eax)^* = xe^{-1}(ea^n x^n)^* = xe^{-1}(x^n)^*(a^n)^*e \\ & = & xe^{-1}(x^n)^*(a^n - a^d a^{n+1})^*e + xe^{-1}(x^n)^*(a^{n+1})^*(a^d)^*e \\ & = & xe^{-1}(x^n)^*(a^n - a^d a^{n+1})^*e + xe^{-1}(a^{n+1}x^n)^*(a^d)^*e \\ & = & xe^{-1}(x^n)^*(a^n - a^d a^{n+1})^*e + xe^{-1}(a^2x)^*(a^d)^*e. \end{array}$$

Hence,

$$\begin{array}{rcl} \|x - xe^{-1}(a^2x)^*(a^d)^*e\|^{\frac{1}{n}} & = & \|xe^{-1}(x^n)^*(a^n - a^d a^{n+1})^*e\|^{\frac{1}{n}} \\ & \leq & \|xe^{-1}(x^n)^*\|^{\frac{1}{n}}\|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}}\|e\|^{\frac{1}{n}}. \end{array}$$

By virtue of Lemma 3.1,

$$\lim_{n\to\infty} \|(a^n - a^d a^{n+1})^*\|^{\frac{1}{n}} = 0,$$

we have

$$\lim_{n\to\infty} \|x - xe^{-1}(a^2x)^*(a^d)^*e\|^{\frac{1}{n}} = 0,$$

and so $x = xe^{-1}(a^2x)^*(a^d)^*e$. Then $\mathcal{A}x \subseteq \mathcal{A}(a^d)^*e$.

One directly checks that

$$\begin{array}{rcl} (a^d)^*e & = & ((a^d)^{n+1})^*(a^n)^*e = ((a^d)^{n+1})^*(ea^n)^*, \\ (a^d)^*eax & = & ((a^d)^{n+1})^*(a^n)^*eax = ((a^d)^{n+1})^*(a^n)^*(eax)^* \\ & = & ((a^d)^{n+1})^*(eaxa^n)^* = ((a^d)^{n+1})^*(ea^n x^n a^n)^*. \end{array}$$

Then we derive

$$\begin{array}{rcl} \|(a^d)^*\mathrm{e}-(a^d)^*eax\|^{\frac{1}{n}} & = & \|((a^d)^{n+1})^*(ea^n)^* - ((a^d)^{n+1})^*(ea^n x^n a^n)^*\|^{\frac{1}{n}} \\ & = & \|((a^d)^{n+1})^*(a^n - a^n x^n a^n)^*e\|^{\frac{1}{n}} \\ & \leq & \|((a^d)^{n+1})^*\|^{\frac{1}{n}}\|(a^n - a^n x^n a^n)^*\|^{\frac{1}{n}}\|e\|^{\frac{1}{n}}. \end{array}$$

In light of Lemma 3.2, we see that

$$\lim_{n\to\infty} \|(a^n - a^n x^n a^n)^*\|^{\frac{1}{n}} = 0.$$

Then we have

$$\lim_{n\to\infty} \|(a^d)^*e - (a^d)^*eax\|^{\frac{1}{n}} = 0,$$

and so $(a^d)^*e = (a^d)^*eax$. Hence $\mathcal{A}(a^d)^*e \subseteq \mathcal{A}x$. Therefore $\mathcal{A}x = \mathcal{A}(a^d)^*e$, as required.

$\Longleftarrow$ By hypothesis, there exists $x \in \mathcal{A}$ such that

$$xax = x, x\mathcal{A} = a^d\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^d)^*e.$$

We claim that $a^{e,\circledast} = x$.

Step 1.

$$\lim_{n\to\infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

Write $a^d = xy$ for some $y \in \mathcal{A}$. For any $n \in \mathbb{N}$, we have

$$\begin{array}{rcl} a^n & = & (a^n - a^d a^{n+1}) + a^d a^{n+1}, \\ xa^{n+1} & = & xa(a^n - a^d a^{n+1}) + xaa^d a^{n+1} \\ & = & xa(a^n - a^d a^{n+1}) + (xax)ya^{n+1} \\ & = & xa(a^n - a^d a^{n+1}) + (xy)a^{n+1} \\ & = & xa(a^n - a^d a^{n+1}) + a^d a^{n+1}. \end{array}$$

Hence,

$$a^n - xa^{n+1} = (1 - xa)(a^n - a^d a^{n+1}),$$

and so

$$||a^n - xa^{n+1}||^{\frac{1}{n}} \leq ||1 - xa||^{\frac{1}{n}} ||a^n - a^d a^{n+1}||^{\frac{1}{n}},$$

we have

$$\lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Step 2. $(eax)^* = eax$.

Since $\mathcal{A}x = \mathcal{A}(a^d)^*e$, we have $x^*\mathcal{A} = ea^d\mathcal{A}$. Write $ea^d = x^*y$ for some $y \in \mathcal{A}$. As $xax = x$, we get $x^*a^*x^* = x^*$, and so $(ax)^*x^* = x^*$. Then $(ax)^*ea^d = ea^d$. Since $x\mathcal{A} = a^d\mathcal{A}$, we can find $z \in \mathcal{A}$ such that $x = a^d z$. Then $(ax)^*e(ax) = (ax)^*ea(a^d z) = [(ax)^*ea^d](az) = eaa^d z = eax$. Hence $(eax)^* = [(ax)^*e(ax)]^* = (ax)^*e(ax) = eax$.

Step 3. $ax^2 = x$.

By the argument above, $(ax)^*ea^d = ea^d$. Hence $ea^d = (eax)^*a^d = eaxa^d$. As $e \in \mathcal{A}^{-1}$, we obtain $a^d = axa^d$. Since $x\mathcal{A} = a^d\mathcal{A}$, there exists some $s \in \mathcal{A}$ such that $x = a^d s$. Hence $x = a^d s = (axa^d)s = (ax)(a^d s) = ax^2$. This completes the proof. □

Corollary 3.4. *Let $a \in \mathcal{A}$. Then a has pseudo e-core inverse if and only if*

(1) $a \in \mathcal{A}^{e,\circledast}$;
(2) a has Drazin inverse.

Proof. $\Longrightarrow$ In view of [28, Theorem 3.2], a has generalized e-core inverse. By virtue of [28, Lemma 3.3], a has Drazin inverse.

$\Longrightarrow$ Since a has generalized e-core inverse, by Theorem 3.3, there exists $x \in \mathcal{A}$ such that

$$xax = x, x\mathcal{A} = a^d\mathcal{A}, \mathcal{A}x = \mathcal{A}(a^d)^*e.$$

As a has Drazin inverse, we have $a^d = a^D$. Let $n = ind(a)$. Then $a^n = a^{n+1}a^d, aa^d = a^d a$ and $a^d = a(a^d)^2$. Hence, $a^d = a^n[(a^d)^{n+1}]$ and $a^n = a^d a^{n+1}$. Then $a^n\mathcal{A} = a^d\mathcal{A}$. On the other hand, we have

$$(a^d)^* = [(a^d)^{n+1}]^*(a^n)^*, (a^n)^* = (a^{n+1})^*(a^d)^*.$$

Therefore $\mathcal{A}(a^d)^*e = \mathcal{A}a^n e$, and so $\mathcal{A}x = \mathcal{A}(a^n)^*e$. This implies that a has pseudo e-core inverse, as asserted. □

Let $R(X)$ represent the range space of a complex matrix X and X^T be the transpose of X. We improve [2, Theorem 3.5] and provide a new characterizations of E-core inverse for any complex matrix.

Corollary 3.5. *Let $A \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix. Then the following are equivalent:*

(1) $X = A^{\textcircled{\dagger},E}$.
(2) $XAX = X, R(X) = R(A^D), R(X^T) = R\Big((A^D)^*E\Big)^T$.

Proof. This is immediate from Corollary 3.4 and Theorem 3.3. □

Theorem 3.3 infers that the generalized e-core inverse for $e = 1$ and core-EP inverse for a bounded linear operator over a Hilbert space and an element in a ring introduced in [19, 20] coincide with each other. If a and x satisfy the equations $a = axa$ and $(eax)^* = eax$, then x is called $(e, 1, 3)$-inverse of a and is denoted by $a_e^{(1,3)}$. We use $\mathcal{A}_e^{(1,3)}$ to stand for the set of all $(e, 1, 3)$-invertible elements in $\mathcal{A}$. We now derive

Theorem 3.6. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{e,\circledast}$.
(2) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}_e^{(1,3)}$.
(3) $a \in \mathcal{A}^d$ and there exists an idempotent $q \in \mathcal{A}$ such that $a^d\mathcal{A} = q\mathcal{A}$ and $(eq)^* = eq$.

In this case, $a^{e,\circledast} = (a^d)^2(a^d)_e^{(1,3)} = a^d q$.

Proof. (1) ⇒ (2) In view of Theorem 3.3, $a \in \mathcal{A}^d$. Let $x = a^{e,\circledast}$. By using Theorem 3.3 again, we have $xax = x$ and $(eax)^* = eax$. Let $z = a^2x$. Then we check that

$$\begin{array}{rcl} \|a^d z a^d - a^d\|^{\frac{1}{n}} & = & \|a^d(a^2x)a^d - a^d\|^{\frac{1}{n}} \\ & = & \|a^d a^2(xa^{n+1})(a^d)^{n+2} - a^d a^2 a^n (a^d)^{n+2}\|^{\frac{1}{n}} \\ & \leq & \|a^d a^2\|^{\frac{1}{n}} \|a^n - xa^{n+1}\|^{\frac{1}{n}} \|a^d\|^{1+\frac{2}{n}}. \end{array}$$

Therefore

$$\lim_{n\to\infty} \|a^d z a^d - a^d\|^{\frac{1}{n}} = 0.$$

This implies that $a^d z a^d = a^d$. In view of Theorem 2.5, $a^d\mathcal{A} = x\mathcal{A}$. Then $aa^d x = x$, and so

$$\begin{array}{rcl} ea^d z & = & ea^d(a^2x) = e(a^d a^2 x) = eax, \\ (ea^d z)^* & = & (eax)^* = eax = ea^d z. \end{array}$$

Therefore $a^d \in \mathcal{A}_e^{(1,3)}$, as required.

(2) ⇒ (1) Let $x = (a^d)^2(a^d)_e^{(1,3)}$. Then we verify that

$$xax = (a^d)^2(a^d)_e^{(1,3)} a (a^d)^2 (a^d)_e^{(1,3)} = (a^d)^2(a^d)_e^{(1,3)} a^d (a^d)_e^{(1,3)} = (a^d)^2(a^d)_e^{(1,3)} = x.$$

Clearly, $x\mathcal{A} \subseteq a^d\mathcal{A}$. Also we see that $a^d = (a^d)^2 a = [(a^d)^2(a^d)_e^{(1,3)}]a^d a = xa^d a$; hence, $a^d\mathcal{A} \subseteq x\mathcal{A}$. This implies that $x\mathcal{A} = a^d\mathcal{A}$.

We easily verify that

$$\begin{array}{rcl} x & = & (a^d)^2(a^d)_e^{(1,3)} = (a^d e^{-1})[ea^d(a^d)_e^{(1,3)}] \\ & = & (a^d e^{-1})[ea^d(a^d)_e^{(1,3)}]^* = (a^d e^{-1})[(a^d)_e^{(1,3)}]^*(a^d)^* e; \end{array}$$

hence, $\mathcal{A}x \subseteq \mathcal{A}(a^d)^* e$. Also we see that

$$\begin{array}{rcl} (a^d)^* e & = & [a^d(a^d)_e^{(1,3)} a^d]^* e = [(ea^d(a^d)_e^{(1,3)})a^d]^* \\ & = & (a^d)^*(ea^d(a^d)_e^{(1,3)}) = [(a^d)^* ea](a^d)^2(a^d)_e^{(1,3)} \\ & = & [(a^d)^* ea]x, \end{array}$$

and then $\mathcal{A}(a^d)^* e \subseteq \mathcal{A}x$. Accordingly, $\mathcal{A}x = \mathcal{A}(a^d)^* e$. Therefore $a \in \mathcal{A}^{e,\circledast}$ by Theorem 3.3.

(2) ⇒ (3) Since $a^d \in \mathcal{A}_e^{(1,3)}$, we have $a^d = a^d(a^d)_e^{(1,3)} a^d$ and $(ea^d(a^d)_e^{(1,3)})^* = ea^d(a^d)_e^{(1,3)}$. Let $q = a^d(a^d)_e^{(1,3)}$. Then $a^d\mathcal{A} = q\mathcal{A}, q^2 = q$ and $(eq)^* = eq$, as desired.

(3) ⇒ (2) Set $x = a^d q$. Then $ax = aa^d q = q$, and so $(eax)^* = (eq)^* = eq = eax$. Moreover, we have

$$ax^2 = (aa^d)qa^d q = qa^d q = q(aa^d)a^d q = (aa^d)a^d q = a^d q = x.$$

We verify that

$$\begin{array}{rcl} \|a^n - xa^{n+1}\| & = & \|(a^n - (a^d q)a^d a^{n+2}) - [x(a^{n+1} - a^d a^{n+2})]\| \\ & \leq & \|a^n - a^d a^{n+1}\| + \|x\|\|a^{n+1} - a^d a^{n+2}\| \\ & \leq & \Big(1 + \|x\|\|a\|\Big)\|a^n - a^d a^{n+1}\| \\ & = & \Big(1 + \|x\|\|a\|\Big)\|a^n(1 - a^d a)^n\| \\ & = & \Big(1 + \|x\|\|a\|\Big)\|(a - a^d a^2)^n\|. \end{array}$$

Since $a - a^d a^2 \in \mathcal{A}^{qnil}$, we have

$$\lim_{n\to\infty} \|(a - a^d a^2)^n\|^{\frac{1}{n}} = 0.$$

Therefore

$$\lim_{n\to\infty} \|a^n - xa^{n+1}\|^{\frac{1}{n}} = 0.$$

In view of Theorem 2.1, $a \in \mathcal{A}^{e,\circledast}$. In this case, $a^{e,\circledast} = a^d q = (a^d)^2(a^d)_e^{(1,3)}$. □

Corollary 3.7. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) a has pseudo e-core inverse.
(2) $a \in \mathcal{A}^D$ and $a^D \in \mathcal{A}_e^{(1,3)}$.
(3) $a \in \mathcal{A}^D$ and there exists an idempotent $q \in \mathcal{A}$ such that $a^D\mathcal{A} = q\mathcal{A}$ and $(eq)^* = eq$.

In this case, $a^{e,\textcircled{D}} = (a^D)^2(a^D)_e^{(1,3)} = a^D q$.

Proof. As $a \in \mathcal{A}^D$, we have $a^d = a^D$. Therefore we complete the proof by Theorem 3.6. □

4. Relations with other weighted generalized inverses

The aim of this section is to establish the relations between generalized weighted core inverse and other weighted generalized inverses. We come now to the demonstration for which this section has been developed.

Theorem 4.1. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{e,\textcircled{d}}$.
(2) $a \in \mathcal{A}^d$ and $a^d \in \mathcal{A}^{e,\textcircled{\#}}$.

In this case,

$$a^{e,\textcircled{d}} = (a^d)^2(a^d)^{e,\textcircled{\#}}.$$

Proof. (1) ⇒ (2) In view of Theorem 2.1, $a \in \mathcal{A}^d$ and we have a generalized e-core decomposition, i.e., there exist $x, y \in \mathcal{A}$ such that

$$a = x + y, x^*ey = yx = 0, x \in \mathcal{A}^{e,\textcircled{\#}}, y \in \mathcal{A}^{qnil}.$$

We verify that

$$\begin{aligned} ||a^2(a^d)^{e,\textcircled{d}} - a^d[a^2(a^d)^{e,\textcircled{d}}]^2|| &= ||a^2(a^d)^{e,\textcircled{d}} - a(a^d)^{e,\textcircled{d}}a^2(a^d)^{e,\textcircled{d}}|| \\ &= ||a[a^n - (a^d)^{e,\textcircled{d}}a^{n+1}][(a^d)^{e,\textcircled{d}}]^n|| \\ &\leq ||a||||a^n - (a^d)^{e,\textcircled{d}}a^{n+1}||||(a^d)^{e,\textcircled{d}}||^n. \end{aligned}$$

Since

$$\lim_{n\to\infty} ||a^n - (a^d)^{e,\textcircled{d}}a^{n+1}||^{\frac{1}{n}} = 0,$$

we deduce that

$$\lim_{n\to\infty} ||a^d[a^2(a^d)^{e,\textcircled{d}}]^2 - a^2(a^d)^{e,\textcircled{d}}||^{\frac{1}{n}} = 0.$$

Hence $a^d[a^2(a^d)^{e,\textcircled{d}}]^2 = a^2(a^d)^{e,\textcircled{d}}$. Furthermore, we have

$$\begin{aligned} a^d[a^2(a^d)^{e,\textcircled{d}}] &= a(a^d)^{e,\textcircled{d}}, \\ (ea^d(a^2(a^d)^{e,\textcircled{d}}))^* &= (ea(a^d)^{e,\textcircled{d}})^* = ea^d(a^2(a^d)^{e,\textcircled{d}}), \\ a^2(a^d)^{e,\textcircled{d}}(a^d)^2 &= a^2a^d = a^d. \end{aligned}$$

Accordingly, $a^d \in \mathcal{A}^{e,\textcircled{\#}}$ and $(a^d)^{e,\textcircled{\#}} = a^2(a^d)^{e,\textcircled{d}}$, as required.

(2) ⇒ (1) Set $x = (a^d)^2(a^d)^{e,\textcircled{\#}}$. Then we check that

$$[a^d(a^d)^{e,\textcircled{\#}}a][(a^d)^2(a^d)^{e,\textcircled{\#}}] = [a^d(a^d)^{e,\textcircled{\#}}]^2 = a^d(a^d)^{e,\textcircled{\#}},$$

hence, $[ea^d(a^d)^{e,\textcircled{\#}}ax]^* = [ea^d(a^d)^{e,\textcircled{\#}}]^* = ea^d(a^d)^{e,\textcircled{\#}} = ea^d(a^d)^{e,\textcircled{\#}}ax$. Moreover, we see that

$$\begin{aligned} a^d(a^d)^{e,\textcircled{\#}}ax^2 &= a^d(a^d)^{e,\textcircled{\#}}(a^d)^2(a^d)^{e,\textcircled{\#}} = (a^d)^2(a^d)^{e,\textcircled{\#}} = x, \\ x[a^d(a^d)^{e,\textcircled{\#}}a]^2 &= (a^d)^2(a^d)^{e,\textcircled{\#}}[a^d(a^d)^{e,\textcircled{\#}}a]^2 = (a^d)^2a(a^d)^{e,\textcircled{\#}}a = a^d(a^d)^{e,\textcircled{\#}}a. \end{aligned}$$

Then $a^d(a^d)^{e,\textcircled{\#}}a \in \mathcal{A}^{e,\textcircled{\#}}$ and $[a^d(a^d)^{e,\textcircled{\#}}a]^{e,\textcircled{\#}} = (a^d)^2(a^d)^{e,\textcircled{\#}}$.

Write $a = a_1 + a_2$, where $a_1 = a^d(a^d)^{e,\textcircled{\#}}a$ and $a_2 = a - a^d(a^d)^{e,\textcircled{\#}}a$. It is easy to verify that

$$\begin{aligned} a_2a_1 &= [a - a^d(a^d)^{e,\textcircled{\#}}a]a^d(a^d)^{e,\textcircled{\#}}a \\ &= aa^d(a^d)^{e,\textcircled{\#}}a - a^d a(a^d)^{e,\textcircled{\#}}a = 0, \\ a_1^*ea_2 &= a^*(a^d(a^d)^{e,\textcircled{\#}})^*e[a - a^d(a^d)^{e,\textcircled{\#}}a] \\ &= a^*(ea^d(a^d)^{e,\textcircled{\#}})^*[a - a^d(a^d)^{e,\textcircled{\#}}a] \\ &= a^*ea^d(a^d)^{e,\textcircled{\#}}[a - a^d(a^d)^{e,\textcircled{\#}}a] \\ &= a^*ea^d(a^d)^{e,\textcircled{\#}}[1 - a^d(a^d)^{e,\textcircled{\#}}]a = 0. \end{aligned}$$

Moreover, we check that $(a-a^2a^d)[1-a^d(a^d)^{e,\textcircled{\#}}] = a-a^2a^d \in \mathcal{A}^{qnil}$. By using Cline's formula (see [12, Theorem 2.1], we have

$$\begin{aligned} [1 - a^d(a^d)^{e,\textcircled{\#}}]a &= [1 - a^d(a^d)^{e,\textcircled{\#}}]a - [1 - a^d(a^d)^{e,\textcircled{\#}}]a^da^2 \\ &= [1 - a^d(a^d)^{e,\textcircled{\#}}](a - a^2a^d) \in \mathcal{A}^{qnil}. \end{aligned}$$

Thus, $a_2 = a - a^d(a^d)^{e,\textcircled{\#}}a = a - a^d(a^d)^{e,\textcircled{\#}} \in \mathcal{A}^{qnil}$. Therefore $a = a_1 + a_2$ is the generalized e-core decomposition of a. Then $a^{e,\textcircled{d}} = a_1^{e,\textcircled{\#}} = (a^d)^2(a^d)^{e,\textcircled{\#}}$. □

As an immediate consequence, we provide a new formula of the weighted core-EP inverse which also extend [2, Theorem 3.23] and [23, Corollary 3.4].

Corollary 4.2. *Let $A \in \mathbb{C}^{n\times n}$ and $E \in \mathbb{C}^{n\times n}$ be an invertible Hermitian matrix. Then*

$$A^{\textcircled{\dagger},E} = (A^D)^2(A^D)^{E,\textcircled{\#}}.$$

Proof. This is obvious by Theorem 2.4. □

Example 4.3.

Let $A = \begin{pmatrix} 0 & 1 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathbb{C}^{4\times 4}$. We take the involution on $\mathbb{C}^{4\times 4}$ as the conjugate transpose. Then $A^D = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Moreover, we have $(A^D)^{E,\textcircled{\#}} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. By using Corollary 4.2, we compute that

$$A^{\textcircled{\dagger},E} = (A^D)^2(A^D)^{E,\textcircled{\#}} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Evidently, we check that

$$A(A^{\textcircled{\dagger},E})^2 = A^{\textcircled{\dagger},E}, (EAA^{\textcircled{\dagger},E})^* = EAA^{\textcircled{\dagger},E}, A^{\textcircled{\dagger},E}A^4 = A^3.$$

If a and x satisfy the equations $a = axa$ and $(exa)^* = exa$, x is called $(e,1,4)$-inverse of a and is denoted by $a_e^{(1,4)}$. We use $\mathcal{A}_e^{(1,4)}$ to stand for sets of all $(e,1,4)$ invertible elements in $\mathcal{A}$.

Lemma 4.4. *Let $f \in \mathcal{A}$ be an idempotent. Then the following are equivalent:*

(1) $f \in \mathcal{A}_e^{(1,3)}$.
(2) $f^\pi \in \mathcal{A}_e^{(1,4)}$.

Proof. It is obtained by [27, Lemma 4.1].

(2) ⇒ (1) This is proved in the similar way. □

Theorem 4.5. *Let $a \in \mathcal{A}$. Then the following are equivalent:*

(1) $a \in \mathcal{A}^{e,\textcircled{d}}$.
(2) $a \in \mathcal{A}^d$ and $aa^d \in \mathcal{A}_e^{(1,3)}$.
(3) $a \in \mathcal{A}^d$ and $a^\pi \in \mathcal{A}_e^{(1,4)}$.

In this case, $a^{e,\textcircled{d}} = a^d(aa^d)_e^{(1,3)} = a^d[1-(a^\pi)_e^{(1,4)}a^\pi]$.

Proof. (1) ⇒ (2) In view of Theorem 2.5, $a \in \mathcal{A}^d$ and $aa^da^{e,\textcircled{d}} = a^{e,\textcircled{d}}$. For any $m \in \mathbb{N}$, we check that

$$\begin{array}{rcl} ||a^{e,\textcircled{d}}aa^d - a^d|| & = & ||a^{e,\textcircled{d}}a^{m+1}(a^d)^{m+1} - a^m(a^d)^{m+1}|| \\ & \leq & ||a^m - a^{e,\textcircled{d}}a^{m+1}||||(a^d)^{m+1}||. \end{array}$$

Since

$$\lim_{m\to\infty} ||a^m - a^{e,\textcircled{d}}a^{m+1}||^{\frac{1}{m}} = 0,$$

we have

$$\lim_{m\to\infty} ||a^{e,\textcircled{d}}aa^d - a^d||^{\frac{1}{m}} = 0.$$

Hence $a^{e,\textcircled{d}}aa^d = a^d$, and then

$$\begin{array}{rcl} (aa^d)(aa^{e,\textcircled{d}}) & = & aa^{e,\textcircled{d}}, \\ (e(aa^d)(aa^{e,\textcircled{d}}))^* & = & e(aa^d)(aa^{e,\textcircled{d}}), \\ (aa^d)(aa^{e,\textcircled{d}})(aa^d) & = & aa^d. \end{array}$$

Accordingly, $aa^d \in \mathcal{A}_e^{(1,3)}$, as required.

(2) ⇒ (1) Let $x = a^d(aa^d)_e^{(1,3)}$. Then we verify that

$$\begin{array}{rcl} ax & = & aa^d(aa^d)_e^{(1,3)}, (eax)^* = eax, \\ ax^2 & = & [aa^d(aa^d)_e^{(1,3)}aa^d]a^d(aa^d)_e^{(1,3)} = a(a^d)^2(aa^d)_e^{(1,3)} = x, \end{array}$$

$$\begin{array}{rl} & ||a^n - xa^{n+1}|| \\ = & ||a^n - a^daa^da^{n+1} + a^daa^d(aa^d)_e^{(1,3)}aa^da^{n+1} - xa^{n+1}|| \\ \leq & ||a^n - a^da^{n+1}|| + ||a^daa^d(aa^d)_e^{(1,3)}aa^da^{n+1} - xa^{n+1}|| \\ \leq & ||a^n - a^da^{n+1}|| + ||a^d(aa^d)_e^{(1,3)}a^da^{n+2} - a^daa^d(aa^d)_e^{(1,3)}a^{n+1}|| \\ \leq & ||a^n - a^da^{n+1}|| + ||x||||a^da^{n+2} - a^{n+1}|| \\ \leq & ||a^n - a^da^{n+1}||[1 + ||x||||a||]. \end{array}$$

Hence,

$$\lim_{n\to\infty} ||a^n - xa^{n+1}||^{\frac{1}{n}} = 0.$$

Therefore $a \in \mathcal{A}^{e,\textcircled{d}}$. In this case, $a^{e,\textcircled{d}} = a^d(aa^d)_e^{(1,3)}$.

(2) ⇔ (3) In view of Lemma 4.4, $aa^d \in \mathcal{A}_e^{(1,3)}$ if and only if $a^\pi = 1 - aa^d \in \mathcal{A}_e^{(1,4)}$, as desired. □

Corollary 4.6. *Let $a, b \in \mathcal{A}$. If $\alpha = ab \in \mathcal{A}^{e,\textcircled{d}}$, then the following are equivalent:*

(1) $\beta = ba \in \mathcal{A}^{e,\textcircled{d}}$.
(2) $b\alpha^{e,\textcircled{d}}a \in \mathcal{A}_e^{(1,3)}$.

Proof. In view of Theorem 3.3, $\beta = ba \in \mathcal{A}^d$. By virtue of Theorem 4.5, $\beta = ba \in \mathcal{A}^{e,\textcircled{a}}$ if and only if $\beta\beta^d \in \mathcal{A}_e^{(1,3)}$. Clearly, we have $\alpha a = a\beta$. Write $y_1 = \alpha^d$ and $y_2 = \beta^d$. Then we have

$$\alpha y_1 = y_1\alpha, y_1\alpha y_1 = y_1, \lim_{k\to\infty} \|\alpha^k - \alpha^{k+1}y_1\|^{\frac{1}{k}} = 0,$$

$$\beta y_2 = y_2\beta, y_2\beta y_2 = y_2, \lim_{k\to\infty} \|\beta^k - \beta^{k+1}y_2\|^{\frac{1}{k}} = 0.$$

Hence, we check that

$$\begin{array}{rcl} y_1a & = & (y_1^{k+1}\alpha^k)a = y_1^{k+1}(\alpha^k a) = y_1^{k+1}(a\beta^k) \\ & = & y_1^{k+1}a[\beta^k - \beta^{k+1}y_2] + y_1^{k+1}(a\beta^{k+1})y_2 \\ & = & y_1^{k+1}a[\beta^k - \beta^{k+1}y_2] + y_1^{k+1}(\alpha^{k+1}a)y_2 \\ & = & y_1^{k+1}a[\beta^k - \beta^{k+1}y_2] + (y_1^{k+1}\alpha^{k+1})ay_2 \\ & = & y_1^{k+1}a[\beta^k - \beta^{k+1}y_2] + (y_1\alpha)ay_2. \end{array}$$

Thus,

$$\|y_1a - y_1\alpha ay_2\|^{\frac{1}{k}} \leq \|y_1\|^{1+\frac{1}{k}}\|a\|^{\frac{1}{k}}\|\beta^k - \beta^{k+1}y_2\|^{\frac{1}{k}}.$$

Therefore

$$\lim_{k\to\infty} \|y_1a - y_1\alpha ay_2\|^{\frac{1}{k}} = 0,$$

and so $y_1a = y_1\alpha ay_2$.

Interchanging α with β and y_1 with y_2, dually, we check that

$$\begin{array}{rcl} ay_2 & = & a(\beta^k y_2^{k+1}) = (a\beta^k)y_2^{k+1} = (\alpha^k a)y_2^{k+1} = \alpha^k(ay_2^{k+1}) \\ & = & (\alpha^k - y_1\alpha^{k+1})(ay_2^{k+1}) + y_1(\alpha^{k+1}a)y_2^{k+1} \\ & = & (\alpha^k - y_1\alpha^{k+1})(ay_2^{k+1}) + y_1(a\beta^{k+1})y_2^{k+1} \\ & = & (\alpha^k - y_1\alpha^{k+1})(ay_2^{k+1}) + y_1a(\beta^{k+1}y_2^{k+1}) \\ & = & (\alpha^k - y_1\alpha^{k+1})(ay_2^{k+1}) + y_1a\beta y_2 \\ & = & (\alpha^k - y_1\alpha^{k+1})(ay_2^{k+1}) + y_1\alpha ay_2. \end{array}$$

This implies that $ay_2 = y_1\alpha ay_2$. Therefore $\alpha^d a = a\beta^d$, and then $\beta\beta^d = b(a\beta^d) = b\alpha^d a$. Hence, $\beta\beta^d \in \mathcal{A}_e^{(1,3)}$ if and only if $b\alpha^d a \in \mathcal{A}_e^{(1,3)}$, as desired. □

Corollary 4.7. *Let $a, b \in \mathcal{A}$. If $\alpha = 1 - ab \in \mathcal{A}^{e,\textcircled{a}}$, then the following are equivalent:*

(1) $\beta = 1 - ba \in \mathcal{A}^{e,\textcircled{a}}$.
(2) $b\alpha^\pi(1 - \alpha\alpha^\pi)^{-1}a \in \mathcal{A}_e^{(1,4)}$.

Proof. By virtue of Theorem 3.3, $1 - ab \in \mathcal{A}^d$. In view of Jacobson's Lemma (see [4, Corollary 6.4.14]), $\beta = 1 - ba \in \mathcal{A}^d$. In light of Theorem 4.5, $\beta = 1 - ba \in \mathcal{A}^{e,\textcircled{a}}$ if and only if $\beta^\pi \in \mathcal{A}^{(1,4)}$. Since $b\alpha = \beta b$, as in the proof of Corollary 4.6, we prove that $b\alpha^d = \beta^d b$. Then $b\alpha\alpha^d = \beta\beta^d b$, and so $b\alpha^\pi = \beta^\pi b$. By induction, we have $b(\alpha\alpha^\pi)^i = b\alpha^i\alpha^\pi = (\beta\beta^\pi)^i b$ for any $i \in \mathbb{N}$. Accordingly,

$$\begin{array}{rl} & b\alpha^\pi(1 - \alpha\alpha^\pi)^{-1}a \\ = & b\alpha^\pi[1 + \alpha\alpha^\pi + (\alpha\alpha^\pi)^2 + \cdots]a \\ = & \beta^\pi b[1 + \alpha\alpha^\pi + (\alpha\alpha^\pi)^2 + \cdots]a \\ = & \beta^\pi[1 + \beta\beta^\pi + (\beta\beta^\pi)^2 + \cdots]ba \\ = & \beta^\pi[1 + \beta\beta^\pi + (\beta\beta^\pi)^2 + \cdots](1 - \beta) \\ = & \beta^\pi(1 - \beta\beta^\pi)[1 + \beta\beta^\pi + (\beta\beta^\pi)^2 + \cdots] \\ = & \beta^\pi(1 - \beta\beta^\pi)(1 - \beta\beta^\pi)^{-1} \\ = & \beta^\pi. \end{array}$$

Therefore $\beta^\pi \in \mathcal{A}_e^{(1,4)}$ if and only if $b\alpha^\pi(1 - \alpha\alpha^\pi)^{-1}a \in \mathcal{A}_e^{(1,4)}$, as desired. □

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