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Generalizations of partial isometries, EP and normal elements in rings

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Abstract. Let a, d be two elements in rings and $a^{\|d}$ be the inverse of a along d. Some characterizations of the equality $a^{\|d} = d$ are given in rings. We also investigate the equivalent conditions for $aa^{\|d} = a^{\|d}a$ to hold, as well as $ad = da$. We prove that $a^{\|d} = d$ if and only if ad is idempotent, when $a \in R^{\|d}$; $aa^{\|d} = a^{\|d}a$ if and only if there exists $t \in R^{-1}$ such that $a^{\|d} = at = ta$, when $a \in R^{\|\bullet d}$; $ad = da$ if and only if $a^{\|d}(a + d) = (a + d)a^{\|d}$, when $a \in R^{\|\bullet d}$. Thus, some well-known results on partial isometries, EP and normal elements in rings are extended to more general settings.

1. Introduction

Throughout this paper, R denotes an associative ring with unity 1 and $\mathbb{N}$ means the set of all positive integers. An involution $*$: $R \to R$ is an anti-isomorphism: $(a^*)^* = a$, $(a + b)^* = a^* + b^*$ and $(ab)^* = b^*a^*$ for all $a, b \in R$. First, we list several types of generalized inverses as follows.

An element $a \in R$ is said to be Moore-Penrose invertible with respect to involution $*$ [12] if the following equations

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (3)\ (ax)^* = ax,\ \ (4)\ (xa)^* = xa$$

have a common solution. Such solution is unique if it exists, and is denoted by $a^{\dagger}$.

The group inverse of $a \in R$ is the element $x \in R$ which satisfies

$$(1)\ axa = a,\ \ (2)\ xax = x,\ \ (5)\ ax = xa.$$

The element x above is unique if it exists and is denoted by $a^{\#}$.

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In 2011, Mary [6] defined a new generalized inverse called the inverse along an element in a ring or semigroup. The element $a \in R$ is said to be invertible along $d \in R$ [6] if there exists $b \in R$ such that

$$bad = d = dab, \ bR \subseteq dR \text{ and } Rb \subseteq \ Rd,$$

i.e.,

$$bab = b, \ bR = dR \text{ and } Rb = Rd.$$

If such b exists, then it is unique and is said to be the inverse of a along d, which will be denoted by $a^{\|d}$. In particular, $a^{\|1} = a^{-1}$, $a^{\|a} = a^{\#}$ and $a^{\|a^*} = a^{\dagger}$. Moreover, if $aa^{\|d}a = a$, then we say that $a^{\|d}$ is an inner inverse of a along d, and a is inner invertible along d.

The symbols R^{-1}, $R^{\dagger}$, $R^{\#}$, $R^{\|d}$ and $R^{\|\bullet d}$ stand for the sets of all invertible, Moore-Penrose invertible, group invertible, invertible along d and inner invertible elements along d in the ring R, respectively.

As we all know, an element $a \in R^{\dagger}$ satisfying $a^* = a^{\dagger}$ is called a partial isometry. An element $a \in R$ is said to be EP if $a \in R^{\dagger}$ and $aa^{\dagger} = a^{\dagger}a$. An element $a \in R$ satisfying $aa^* = a^*a$ is called normal. Many researchers studied the partial isometry, EP and normal elements in different settings, such as complex matrices, Banach algebras and rings [1, 3, 8–11, 15]. Motivated by some known results, in this paper we will consider more general case by using the inverse along an element. Several characterizations of the equality $a^{\|d} = d$ are obtained in rings. We also consider the equivalent conditions for $aa^{\|d} = a^{\|d}a$ to hold, as well as $ad = da$. In particular, in a ring with involutions, if $d = a^*$, then the equalities $a^{\|d} = d$, $aa^{\|d} = a^{\|d}a$ and $ad = da$ become $a^{\dagger} = a^*$, $aa^{\dagger} = a^{\dagger}a$ and $aa^* = a^*a$, respectively. So, some results on partial isometries, EP and normal elements are extended to more general settings.

Next, we will give some lemmas. First, we present the existence criteria for the inverse along an element and the group inverse in rings as follows.

Lemma 1.1. [7, Theorem 2.1] *Let $a, d \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\|d}$.

(ii) $dR \subseteq daR$ *and* $da \in R^{\#}$.

(iii) $Rd \subseteq Rad$ *and* $ad \in R^{\#}$.

In this case, $a^{\|d} = d(ad)^{\#} = (da)^{\#}d$.

Lemma 1.2. [16, Lemma 3] and [13, Corollary 1] *Let $a, d \in R$. Then the following statements are equivalent:*

(i) $a \in R^{\|\bullet d}$. (ii) $d \in R^{\|\bullet a}$. (iii) $a \in R^{\|d}$ *and* $d \in R^{\|a}$.

In this case, $aa^{\|d} = d^{\|a}d$ *and* $a^{\|d}a = dd^{\|a}$.

Lemma 1.3. [5, Theorem 1] *Let $a \in R$. Then $a \in R^{\#}$ if and only if $a \in a^2R \cap Ra^2$. In this case, if $a = a^2x = ya^2$, then* $a^{\#} = ax^2 = y^2a = yax$.

The next lemmas gives some equivalent conditions for $aa^{\|d} = a^{\|d}a$, which will play an important role in the sequel.

Lemma 1.4. [2, Theorem 7.1 and 7.3] *Let $a, d \in R$ be such that $a \in R^{\|d}$. Then the following statements are equivalent:*

(i) $aa^{\|d} = a^{\|d}a$.

(ii) $d \in R^{\#}$ *and* $add^{\#} = dd^{\#}a$.

(iii) $da \in Rd$ *and* $ad \in dR$.

Lemma 1.5. [16, Theorem 5 and Lemma 10] *Let $a, d \in R$ be such that $d \in R^{\|a}$. Then, the following statements are equivalent:*

(i) $a \in R^{\|d}$ *and* $aa^{\|d} = a^{\|d}a$.

(ii) $a, d \in R^{\#}$ *and* $aa^{\#} = dd^{\#}$.

(iii) $a \in R^{\|d} \cap R^{\#}$ *and* $a^{\|d} = a^{\#}$.

(iv) $a \in R^{\|d}$, $aR \subseteq dR$ *and* $Ra \subseteq Rd$.

(v) $aR = dR$ *and* $Ra = Rd$.

Lemma 1.6. [14, Proposition 4.2] *Let $a, d \in R$ be such that $a \in R^{\|d}$. Then, $ad = da$ if and only if $aa^{\|d} = a^{\|d}a$ and $da^{\|d} = a^{\|d}d$.*

The following lemma is known as the bicommuting properties of the group inverse.

Lemma 1.7. [4, Theorem 1] *Let $a, d \in R$. If $a \in R^{\#}$ and $da = ad$, then $da^{\#} = a^{\#}d$.*

2. Characterizations for $a^{\|d} = d$ and $aa^{\|d} = a^{\|d}a$

In this section, first we give several characterizations for the equality $a^{\|d} = d$. Then, some new necessary and sufficient conditions for $aa^{\|d} = a^{\|d}a$ are obtained. Next, we present equivalent conditions, which ensure that both $a^{\|d} = d$ and $aa^{\|d} = a^{\|d}a$ hold.

Inspired by [9, Theorem 2.1], we characterize the equality $a^{\|d} = d$ by the idempotent as follows, which generalized the partial isometry.

Theorem 2.1. *Let $a, d \in R$ be such that $a \in R^{\|d}$. Then the following statements are equivalent:*

(i) $a^{\|d} = d$.

(ii) $ad = aa^{\|d}$.

(iii) $da = a^{\|d}a$.

(iv) *da is idempotent.*

(v) *ad is idempotent.*

Proof. (i) ⇒ (ii) is obvious.

(ii) ⇒ (iii). If $ad = aa^{\|d}$, then we have

$$da = a^{\|d}(ad)a = a^{\|d}aa^{\|d}a = a^{\|d}a.$$

(iii) ⇒ (iv). Note that $a^{\|d}a$ is idempotent. Thus, da is idempotent.

(iv) ⇒ (v). By the hypotheses, we get

$$ad = adaa^{\|d} = adadaa^{\|d} = adad = (ad)^2.$$

(v) ⇒ (i). Since ad is idempotent, we deduce that $ad = (ad)^2$ and $(ad)^{\#} = ad$. Then, we obtain

$$d = a^{\|d}ad = a^{\|d}adad = dad,$$

which implies that $a^{\|d} = d(ad)^{\#} = dad = d$ by Lemma 1.1, as required. □

According to Theorem 2.1 and Lemma 1.2, we directly have the following corollary.

Corollary 2.2. *Let $a, d \in R$ be such that $a \in R^{\|\bullet d}$. Then, $a^{\|d} = d$ if and only if $d^{\|a} = a$.*

The following theorem show that the idempotency of d and $a^{\|d}$ is related to the equality of $a^{\|d} = d$.

Theorem 2.3. *Let $a, d \in R$ be such that $a \in R^{\|d}$ and $d = d^2$. Then, $a^{\|d} = (a^{\|d})^2$ if and only if $a^{\|d} = d$.*

Proof. Suppose that $a^{\|d} = (a^{\|d})^2$. Then,

$$a^{\|d} = d(ad)^{\#} = d^2(ad)^{\#} = d\left(d(ad)^{\#}\right) = da^{\|d}.$$

Thus, we claim that

$$d = daa^{\|d} = \left(daa^{\|d}\right)a^{\|d} = da^{\|d} = a^{\|d}.$$

Conversely, it is clear. □

In what follows, we present new equivalent conditions for $aa^{\|d} = a^{\|d}a$ in terms of the ring units, which extend [3, Theorem 16].

Theorem 2.4. *Let $a, d \in R$ be such that $a \in R^{\|\bullet d}$. Then, the following statements are equivalent:*

(i) $aa^{\|d} = a^{\|d}a$.

(ii) *There exists $t \in R^{-1}$ such that $a^{\|d} = at = ta$.*

(iii) *$a, d \in R^{\#}$ and $d = u^{-1}av$, where $u = a^2 + 1 - aa^{\#}$ and $v = ad + 1 - dd^{\#}$.*

Proof. (i) ⇒ (ii). Suppose that (i) holds. Then, by Lemma 1.5, we get $a \in R^{\#}$ and $a^{\|d} = a^{\#}$. Let $t = (a^{\#})^2 + 1 - aa^{\#}$. Note that

$$t(a^2 + 1 - aa^{\#}) = (a^2 + 1 - aa^{\#})t = 1.$$

So, $t \in R^{-1}$. In addition, it is clear that $a^{\|d} = a^{\#} = at = ta$.

(ii) ⇒ (i). By the hypotheses, we get $d = a^{\|d}ad = atad \in aR$ and $a = a^{\|d}t^{-1} = d(ad)^{\#}t^{-1} \in dR$, which yield $aR = dR$. Similarly, we can deduce that $Ra = Rd$. Using Lemma 1.5, we infer that $aa^{\|d} = a^{\|d}a$.

(i) ⇒ (iii). In view of Lemma 1.5, we see that $a, d \in R^{\#}$ and $aa^{\#} = dd^{\#}$. Note that $u \in R^{-1}$ and $u^{-1} = (a^{\#})^2 + 1 - aa^{\#}$. Then,

$$u^{-1}av = \left((a^{\#})^2 + 1 - aa^{\#}\right)a(ad + 1 - dd^{\#}) = a^{\#}ad + a^{\#}(1 - dd^{\#}) = d.$$

(iii) ⇒ (i). Since $a \in R^{\|d}$ and $d \in R^{\#}$, by [7, Theorem 3.2], we get $v \in R^{-1}$ and $a^{\|d} = dv^{-1}$. Thus, we have

$$a^{\|d} = dv^{-1} = u^{-1}avv^{-1} = u^{-1}a = \left((a^{\#})^2 + 1 - aa^{\#}\right)a = a^{\#},$$

which yields that $aa^{\|d} = a^{\|d}a$. □

Motivated by [10, Theorem 2.1], the second characterization of $aa^{\|d} = a^{\|d}a$ is given by means of the group inverse as follows.

Theorem 2.5. *Let $a, d \in R$ be such that $a \in R^{\|\bullet d} \cap R^{\#}$. Then the following statements are equivalent:*

(i) $aa^{\|d} = a^{\|d}a$.

(ii) $aa^{\#}a^{\|d} = a^{\|d}aa^{\#}$.

(iii) $aa^{\#}d = daa^{\#}$.

(iv) $aa^{\#}d + da^{\#}a = 2d$.

(v) $a^2a^{\|d} = a^{\|d}a^2$.

(vi) $a^2a^{\|d} + a^{\|d}a^2 = 2a$.

(vii) $a^{\#} = a(a^{\|d})^2 = (a^{\|d})^2a$.

(viii) $a^{\|d}d = a^{\#}d$ *and* $da^{\|d} = da^{\#}$.

(ix) $aa^{\|d}da = da^2a^{\|d}$ *and* $a^{\|d}a^2d = ada^{\|d}a$.

(x) $a^{\|d} = a^{\#}$.

Proof. (i) ⇒ (ii) − (x). Note that $a \in R^{\|\bullet d} \cap R^{\#}$ and $aa^{\|d} = a^{\|d}a$. In view of Lemma 1.5, we obtain $a^{\#} = a^{\|d}$ and $aa^{\#} = dd^{\#}$. Then, it is easy to check that items (ii) − (x) hold.

(ii) ⇒ (i). Suppose that $aa^{\#}a^{\|d} = a^{\|d}aa^{\#}$. Then, we get

$$aa^{\|d} = a(aa^{\#}a^{\|d}) = aa^{\|d}aa^{\#} = aa^{\#} = a^{\#}a = (a^{\#}aa^{\|d})a = a^{\|d}aa^{\#}a = a^{\|d}a.$$

(iii) ⇒ (i). Since $a \in R^{\|\bullet d}$ and $a \in R^{\#}$, by Lemma 1.4, it follows that $dd^{\|a} = d^{\|a}d$. So, in view of Lemma 1.2, we get $aa^{\|d} = a^{\|d}a$.

(iv) ⇒ (i). By the item (iv), we get

$$\begin{aligned} d &= 2d - d = a^{\|d}a(2d) - d = a^{\|d}a(aa^{\#}d + da^{\#}a) - d \\ &= a^{\|d}ad + a^{\|d}ada^{\#}a - d = daa^{\#}. \end{aligned}$$

Similarly, we have $d = aa^{\#}d$. Thus, $aa^{\#}d = daa^{\#}$, which means item (iii) holds.

(v) ⇒ (i). Suppose that $a^2a^{\|d} = a^{\|d}a^2$. Then, we get

$$\begin{aligned} aa^{\|d} &= a^{\#}a^2a^{\|d} = a^{\#}a^{\|d}a^2 = (a^{\#})^2aa^{\|d}a^2 = (a^{\#})^2a^2 = a^{\#}a \\ &= aa^{\#} = aa(a^{\#})^2 = a^2a^{\|d}a(a^{\#})^2 = a^{\|d}a^2a^{\#} \\ &= a^{\|d}a. \end{aligned}$$

(vi) ⇒ (i). From $a^2a^{\|d} + a^{\|d}a^2 = 2a$, it follows that

$$\begin{aligned} aa^{\#} &= 2aa^{\#} - aa^{\#} = (a^2a^{\|d} + a^{\|d}a^2)a^{\#} - aa^{\#} \\ &= a^2a^{\|d}a(a^{\#})^2 + a^{\|d}a - aa^{\#} \\ &= aa^{\#} + a^{\|d}a - aa^{\#} \\ &= a^{\|d}a. \end{aligned}$$

Dually, $a^{\#}a = aa^{\|d}$. So, $aa^{\|d} = a^{\|d}a$.

(vii) ⇒ (i). By the hypotheses, it follows that

$$aa^{\#} = aa(a^{\|d})^2 = aa(a^{\|d})^2aa^{\|d} = aa^{\#}aa^{\|d} = aa^{\|d}.$$

Similarly, we have $a^{\#}a = a^{\|d}a$. Therefore, $aa^{\|d} = a^{\|d}a$.

(viii) ⇒ (i). Note that $da^{\|d} = da^{\#}$, $aa^{\|d}a = a$ and $aa^{\|d} = d^{\|a}d$. Then, we have

$$a^{\#} = a(a^{\#})^2 = aa^{\|d}a(a^{\#})^2 = aa^{\|d}a^{\#} = d^{\|a}da^{\#} = d^{\|a}da^{\|d} = aa^{\|d}a^{\|d} = a(a^{\|d})^2.$$

Also, we have $a^{\#} = (a^{\|d})^2a$. So, item (vii) holds.

(ix) ⇒ (i). Since $a \in R^{\|\bullet d}$, then $d \in R^{\|a}$, which yields $d^{\|a}da = a$. Note that $aa^{\|d}da = da^2a^{\|d}$. So, we conclude that

$$\begin{aligned} a &= aaa^{\#} = d^{\|a}daaa^{\#} = d^{\|a}(daaa^{\|d})aa^{\#} = d^{\|a}aa^{\|d}daaa^{\#} \\ &= d^{\|a}(aa^{\|d}da) = (d^{\|a}da)aa^{\|d} = a^2a^{\|d}. \end{aligned}$$

Similarly, using $a^{\|d}a^2d = ada^{\|d}a$ we can get $a = a^{\|d}a^2$. Thus, $a^{\|d}a^2 = a^2a^{\|d}$, which gives that item (v) is satisfied, as required.

(x) ⇒ (i). It is clear. □

Now, we present the following several conditions to ensure that both $a^{\|d} = d$ and $aa^{\|d} = a^{\|d}a$ hold, when $a \in R^{\|\bullet d} \cap R^{\#}$.

Theorem 2.6. *Let $a, d \in R$ be such that $a \in R^{\|\bullet d} \cap R^{\#}$ and $n \in \mathbb{N}$. Then the following statements are equivalent:*

(i) $a^{\|d} = d$ *and* $aa^{\|d} = a^{\|d}a$.

(ii) $a^{\|d} = d$ *and* $a^n d = da^n$.

(iii) $d = a^{\#}$.

(iv) $a^n d = a^{\|d}a^n$.

(v) $da^n = a^n a^{\|d}$.

Proof. (i) ⇒ (ii). By item (i), we get $ad = da$, which gives that $a^n d = da^n$, for $n \in \mathbb{N}$.

(ii) ⇒ (iii). Note that $a^{\|d} = d$ is equivalent to $dad = d$. In addition, since $a \in R^{\|\bullet d}$, by Corollary 2.2, we get $d^{\|a} = a$, which gives $ada = a$. From $a^n d = da^n$ and $a \in R^{\#}$, it follows that

$$ad = (a^{\#})^{n-1}a^n d = (a^{\#})^{n-1}da^n = (a^{\#})^n(ada)a^{n-1} = (a^{\#})^n aa^{n-1} = a^{\#}a.$$

Similarly, $da = aa^{\#}$. So, $ad = da$. Therefore, by the definition of the group inverse, we infer that $a^{\#} = d$.

(iii) ⇒ (iv). If $a \in R^{\#}$ and $d = a^{\#}$, then we have $dad = a^{\#}aa^{\#} = a^{\#} = d$, which implies $a^{\|d} = d$. Note that $ad = aa^{\#} = a^{\#}a = da$. Then, $a^n d = da^n = a^{\|d}a^n$, for $n \in \mathbb{N}$.

(iv) ⇒ (v). Applying the hypotheses, we have

$$ad = (a^{\#})^{n-1}(a^n d) = (a^{\#})^{n-1}a^{\|d}a^n = (a^{\#})^n(aa^{\|d}a)a^{n-1} = (a^{\#})^n aa^{n-1} = aa^{\#},$$

which means that ad is idempotent. By Theorem 2.1, we claim that $a^{\|d} = d$. Thus,

$$da^n = a^{\|d}a^n = a^n d = a^n a^{\|d}.$$

(v) ⇒ (i). Since $da^n = a^n a^{\|d}$, we have

$$a = a^{n-1}a(a^{\#})^{n-1} = a^{n-1}aa^{\|d}a(a^{\#})^{n-1} = a^n a^{\|d}a(a^{\#})^{n-1} = da^n a(a^{\#})^{n-1} = da^2,$$

which gives that $da = da^2a^{\#} = aa^{\#}$ is idempotent. So, $a^{\|d} = d$, i.e. $d^{\|a} = a$. Thus, $ada = a$. Then, we deduce that

$$\begin{aligned} a &= (a^{\#})^{n-1}aa^{n-1} = (a^{\#})^{n-1}adaa^{n-1} = (a^{\#})^{n-1}a(da^n) \\ &= (a^{\#})^{n-1}a(a^n a^{\|d}) = a^2a^{\|d} = a^2 d. \end{aligned}$$

Thus, by Lemma 1.3 we get $a^{\#} = dad = d = a^{\|d}$, which implies $aa^{\|d} = a^{\|d}a$, as required. □

3. Characterizations for $ad = da$ and $da^{\|d} = a^{\|d}d$

In this section, the equality $ad = da$ is characterized by certain conditions involving the inverse along an element and the group inverse. The results obtained can be seen as the generalizations of the normal elements.

We begin with the following theorem, which extends [11, Theorem 2.1].

Theorem 3.1. *Let $a, d \in R$ be such that $a \in R^{\|\bullet d}$. Then, the following statements are equivalent:*

(i) $ad = da$.

(ii) $d^{\|a}a^{\|d} = a^{\|d}d^{\|a}$

(iii) $a^{\|d}(a + d) = (a + d)a^{\|d}$.

Proof. (i) ⇒ (ii). Suppose that $ad = da$. From Lemma 1.6 and Lemma 1.2, it follows that $aa^{\|d} = a^{\|d}a$ and $ad^{\|a} = d^{\|a}a$, which imply

$$d^{\|a}a^{\|d} = d^{\|a}dd^{\|a}a^{\|d} = aa^{\|d}d^{\|a}a^{\|d} = a^{\|d}d^{\|a}a^{\|d}a = a^{\|d}d^{\|a}dd^{\|a} = a^{\|d}d^{\|a}.$$

(ii) ⇒ (iii). Since $d^{\|a}a^{\|d} = a^{\|d}d^{\|a}$, we get

$$da^{\|d} = da^{\|d}aa^{\|d} = d(a^{\|d}d^{\|a})d = dd^{\|a}a^{\|d}d = a^{\|d}aa^{\|d}d = a^{\|d}d.$$

Note that $add^{\|a} = a = d^{\|a}da$. So, we obtain

$$aa^{\|d} = add^{\|a}a^{\|d} = ada^{\|d}d^{\|a} = aa^{\|d}dd^{\|a} = aa^{\|d}a^{\|d}a$$

and

$$a^{\|d}a = a^{\|d}d^{\|a}da = d^{\|a}a^{\|d}da = d^{\|a}da^{\|d}a = aa^{\|d}a^{\|d}a,$$

which means $aa^{\|d} = a^{\|d}a$. Hence, item (iii) holds.

(iii) ⇒ (i). Applying $a^{\|d}(a+d) = (a+d)a^{\|d}$, we get

$$\begin{aligned} aa^{\|d}da &= (a^2 + aa^{\|d}da) - a^2 = (aa^{\|d}aa + aa^{\|d}da) - a^2 \\ &= aa^{\|d}(a+d)a - a^2 = a(a+d)a^{\|d}a - a^2 \\ &= ada^{\|d}a. \end{aligned}$$

So, we get the following equation:

$$da^{\|d} = da^{\|d}aa^{\|d} = a^{\|d}(ada^{\|d}a)a^{\|d} = a^{\|d}(aa^{\|d}da)a^{\|d} = a^{\|d}d,$$

which together with $a^{\|d}(a+d) = (a+d)a^{\|d}$, yield $aa^{\|d} = a^{\|d}a$. Then, by Lemma 1.6, we claim that $ad = da$. □

Next, we continue to investigate the equivalent conditions for $ad = da$ to hold.

Theorem 3.2. *Let $a, d \in R$ be such that $a \in R^{\|\bullet d} \cap R^{\#}$. Then, the following statements are equivalent:*

(i) $ad = da$.

(ii) $a^2d = ada$ *and* $da^2 = ada$;

(iii) $da^{\|d} = a^{\#}d$ *and* $a^{\|d}d = da^{\#}$.

(iv) $d = ada^{\|d}$ *and* $d = a^{\|d}da$.

(v) $d = ada^{\#}$ *and* $d = a^{\#}da$.

(vi) $d^2a^{\#} = da^{\#}d$ *and* $a^{\#}d^2 = da^{\#}d$.

(vii) $da^{\#}a^{\|d} = a^{\|d}da^{\#}$ *and* $a^{\|d}a^{\#}d = a^{\#}da^{\|d}$.

Proof. (i) ⇒ (ii) − (vii). From the condition $ad = da$, we get $a^{\#} = a^{\|d}$ by Lemma 1.5 and $da^{\#} = a^{\#}d$ by Lemma 1.7, which directly imply that items (ii) − (vii) hold.

(ii) ⇒ (i). Since $a^2d = ada$, $da^2 = ada$ and $a \in R^{\#}$, then by Lemma 1.7, we get $ada^{\#} = a^{\#}ad$ and $daa^{\#} = a^{\#}da$. Thus, we obtain

$$\begin{aligned} d &= a^{\|d}ad = a^{\|d}a(a^{\#}ad) = a^{\|d}aada^{\#} = a^{\|d}a(ada)a^{\|d}a^{\#} \\ &= a^{\|d}adaaa^{\|d}a^{\#} = da(aa^{\|d}a)(a^{\#})^2 = da^2(a^{\#})^2 \\ &= daa^{\#}. \end{aligned}$$

So, $ad = (ada)a^{\#} = daaa^{\#} = da$.

(iii) ⇒ (i). By the hypotheses, we see that

$$\begin{aligned} a^{\#} &= a(a^{\#})^2 = aa^{\|d}a(a^{\#})^2 = d^{\|a}da(a^{\#})^2 = d^{\|a}da^{\#} = d^{\|a}a^{\|d}d \\ &= d^{\|a}(a^{\|d}d)aa^{\|d} = d^{\|a}da^{\#}aa^{\|d} = aa^{\|d}aa^{\#}a^{\|d} \\ &= aa^{\#}a^{\|d}, \end{aligned}$$

which gives that $daa^{\#} = daaa^{\#}a^{\|d} = daa^{\|d} = d$. On the other hand,

$$\begin{aligned} a^{\#} &= (a^{\#})^2 a = (a^{\#})^2 aa^{\|d} a = (a^{\#})^2 add^{\|a} = a^{\#}dd^{\|a} = da^{\|d}d^{\|a} \\ &= a^{\|d}a(da^{\|d})d^{\|a} = a^{\|d}aa^{\#}dd^{\|a} = a^{\|d}aa^{\#}a^{\|d}a \\ &= a^{\|d}aa^{\#}, \end{aligned}$$

which yields $da^{\#} = da^{\|d}aa^{\#} = a^{\#}(daa^{\#}) = a^{\#}d$. Therefore, we claim that $ad = da$.

(iv) ⇒ (i). Applying $d = ada^{\|d}$ and $d = a^{\|d}da$, we get

$$da^{\|d} = ada^{\|d}a^{\|d} = a^{\#}a(ada^{\|d})a^{\|d} = a^{\#}(ada^{\|d}) = a^{\#}d,$$

and

$$a^{\|d}d = a^{\|d}a^{\|d}da = a^{\|d}(a^{\|d}da)aa^{\#} = (a^{\|d}da)a^{\#} = da^{\#}.$$

Thus, item (iii) holds.

(v) ⇒ (i). Suppose that $d = ada^{\#}$ and $d = a^{\#}da$. Then, $da^{\#} = a^{\|d}(ada^{\#}) = a^{\|d}d$ and $a^{\#}d = (a^{\#}da)a^{\|d} = da^{\|d}$. This means that item (iii) is satisfied.

(vi) ⇒ (i). Let $d^2a^{\#} = da^{\#}d$ and $a^{\#}d^2 = da^{\#}d$. Then, we get

$$\begin{aligned} a^{\#}d &= a(a^{\#})^2 d = aa^{\|d}a(a^{\#})^2 d = d^{\|a}da(a^{\#})^2 d = d^{\|a}(da^{\#}d) \\ &= d^{\|a}dda^{\#} = d^{\|a}(dda^{\#})aa^{\#} = d^{\|a}da^{\#}daa^{\#} \\ &= d^{\|a}da(a^{\#})^2 daa^{\#} = aa^{\|d}a(a^{\#})^2 daa^{\#} \\ &= a^{\#}daa^{\#}. \end{aligned}$$

So, we can conclude that

$$d = a^{\|d}ad = a^{\|d}a^2(a^{\#}d) = a^{\|d}a^2a^{\#}daa^{\#} = a^{\|d}adaa^{\#} = daa^{\#}.$$

Similarly, we get $d = aa^{\#}d$. According to $d = daa^{\#}$, we have

$$aa^{\|d} = a(da)^{\#}d = a(da)^{\#}daa^{\#} = aa^{\|d}aa^{\#} = aa^{\#},$$

which implies $a = a(aa^{\#}) = aaa^{\|d} = a^2a^{\|d}$. Note that $d = aa^{\#}d$ and $aa^{\#} = aa^{\|d}$. Therefore, we get

$$\begin{aligned} d &= aa^{\|d}d = aa^{\|d}aa^{\|d}d = aa^{\|d}d^{\|a}dd = aa^{\|d}d^{\|a}ddaa^{\|d} = aa^{\|d}d^{\|a}(dda^{\#})a^2a^{\|d} \\ &= aa^{\|d}d^{\|a}da^{\#}da^2a^{\|d} = aa^{\|d}aa^{\|d}a^{\#}da^2a^{\|d} = aa^{\|d}a^{\#}d(a^2a^{\|d}) \\ &= a^{\#}da. \end{aligned}$$

Similarly, we get $d = ada^{\#}$. Therefore, item (v) holds.

(vii) ⇒ (i). Suppose that $da^{\#}a^{\|d} = a^{\|d}da^{\#}$ and $a^{\|d}a^{\#}d = a^{\#}da^{\|d}$. Then, we have

$$\begin{aligned} (a^{\#})^2 &= a(a^{\#})^3 = aa^{\|d}a(a^{\#})^3 = d^{\|a}da(a^{\#})^3 = d^{\|a}d(a^{\#})^2aa^{\#} \\ &= d^{\|a}d(a^{\#})^2aa^{\|d}aa^{\#} = d^{\|a}(da^{\#}a^{\|d})aa^{\#} = d^{\|a}a^{\|d}da^{\#}aa^{\#} \\ &= d^{\|a}(a^{\|d}da^{\#}) = d^{\|a}da^{\#}a^{\|d} = d^{\|a}da(a^{\#})^2a^{\|d} = aa^{\|d}a(a^{\#})^2a^{\|d} \\ &= a^{\#}a^{\|d}. \end{aligned}$$

Therefore, we deduce that

$$daa^{\#} = da^2(a^{\#})^2 = da^2a^{\#}a^{\|d} = daa^{\|d} = d,$$

which implies that

$$da^{\#} = d(a^{\#})^2a = d(a^{\#})^2aa^{\|d}a = (da^{\#}a^{\|d})a = a^{\|d}(da^{\#}a) = a^{\|d}d.$$

Similarly, we can deduce that $a^{\#}d = da^{\|d}$. So, item (iii) holds, as required. □

Let us recall that an element $a \in R^{\dagger}$ satisfying $a^*a^{\dagger} = a^{\dagger}a^*$ is called star-dagger. In [9, Theorem 3.1], Mosić and Djordjević investigated several sufficient conditions for Moore-Penrose invertible element a in the ring with involution to be star-dagger. Motivated by this result, we consider the following corresponding sufficient conditions for $da^{\|d} = a^{\|d}d$ to hold, when $a \in R^{\|d}$.

Theorem 3.3. *Let $a, d \in R$ be such that $a \in R^{\|d}$. Consider the following conditions:*

(i) $a^{\|d} = (a^{\|d})^2$.

(ii) $d = (a^{\|d})^2$.

(iii) $a^{\|d} = d^2$.

(iv) $d = da^{\|d}$.

(v) $d = a^{\|d}d$.

If one of the conditions (i) − (v) *holds, then $da^{\|d} = a^{\|d}d$.*

Proof. (i). If $a^{\|d} = (a^{\|d})^2$, then

$$da^{\|d} = daa^{\|d}a^{\|d} = daa^{\|d} = d = a^{\|d}ad = (a^{\|d})^2ad = a^{\|d}(a^{\|d}ad) = a^{\|d}d.$$

(ii). Suppose that $d = (a^{\|d})^2$. Then,

$$da^{\|d} = daa^{\|d}a^{\|d} = dad = (a^{\|d})^2ad = a^{\|d}(a^{\|d}ad) = a^{\|d}d.$$

(iii). By the condition $a^{\|d} = d^2$, we get $da^{\|d} = dd^2 = d^2d = a^{\|d}d$.

(iv). Suppose that $d = da^{\|d}$. Then, we conclude that

$$a^{\|d} = (da)^{\#}d = (da)^{\#}daa^{\|d} = (a^{\|d})^2.$$

So, item (i) holds. Thus, $da^{\|d} = a^{\|d}d$.

(v). It is similar to the proof of item (iv). □

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