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A helicoidal hypersurfaces family in five-dimensional euclidean space

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Abstract. A family of helicoidal hypersurfaces, denoted as $\mathfrak{x}(u,v,s,t)$, is introduced within the context of the five-dimensional Euclidean space $\mathbb{E}^5$. Matrices for the first and second fundamental forms, the Gauss map, and the shape operator matrix of $\mathfrak{x}$ are derived. Furthermore, by employing the Cayley–Hamilton theorem to define the curvatures of these hypersurfaces, the curvatures are computed specifically for the helicoidal hypersurfaces family $\mathfrak{x}$. Several relationships between the mean and Gauss–Kronecker curvatures of $\mathfrak{x}$ are established. Additionally, the equation $\Delta\mathfrak{x} = \mathcal{A}\mathfrak{x}$ is demonstrated, where $\mathcal{A}$ is a 5×5 matrix in $\mathbb{E}^5$.

1. Introduction

The relationship between ruled (helicoidal) and rotational surfaces elucidated by Bour's theorem, as established in the seminal work of Bour [7]. Do Carmo and Dajczer [9] undertaken a rigorous investigation of helical surfaces within the framework of Bour's theorem in Euclidean 3-space $\mathbb{E}^3$.

Moore [45, 46] conducted an extensive study on general rotational surfaces, encompassing their fundamental properties and characteristics. Ganchev and Milousheva [10] delved into the realm of Minkowski 4-space, exploring the analogous counterparts of these surfaces within that context. Hasanis and Vlachos [24] dedicated their research efforts to the examination of hypersurfaces equipped with harmonic mean curvature vector fields. The concept of affine umbilical surfaces was introduced by Magid et al. [43], while Scharlach [47] made significant contributions to the study of affine geometry pertaining to surfaces and hypersurfaces.

Arslan et al. [1] made notable strides in the understanding of generalized rotational surfaces, extending the theoretical framework associated with these surfaces. In a separate study, Arslan et al. [2] explored the properties and behavior of tensor product surfaces featuring pointwise 1-type Gauss maps.

Ikawa [26, 27] worked the Bour's theorem and Gauss map. Beneki et al. [6] studied the helicoidal surfaces; Güler and Turgut Vanlı [18] served the Bour's theorem; Güler [11] worked the helicoidal surfaces with light-like generating curve; Mira and Pastor [44] presented the helicoidal maximal surfaces; Kim and Yoon [30–32] considered the ruled and rotation surfaces. The readers can see [5, 18–20, 25, 28, 29, 48] for details.

Güler et al. [17] introduced the concept of helicoidal hypersurfaces within the confines of $\mathbb{E}^4$. Güler et al. [16] made substantial contributions by examining the Gauss map and the third Laplace–Beltrami operator associated with rotational hypersurfaces in $\mathbb{E}^4$. Güler [13] achieved some results by identifying rotational

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hypersurfaces that satisfy the condition $\Delta^{\mathbb{I}}R = AR$, where $A \in Mat(4,4)$, within $\mathbb{E}^4$. Moreover, Güler [12] made advancements in the study of the fundamental form $\mathbb{IV}$ and the curvature formulas pertinent to hyperspheres in $\mathbb{E}^4$.

Arslan et al. [4] conducted an in-depth investigation into rotational hypersurfaces within the framework of Euclidean spaces. Güler et al. [21, 22] made contributions by exploring the properties and characteristics of bi-rotational hypersurfaces in $\mathbb{E}^4$ and $\mathbb{E}^4_2$, respectively. Güler [14] considered the helicoidal hypersurfaces constructed by a time-like axis. He [15] also introduced the helicoidal hypersurface determined by a space-like axis in $\mathbb{E}^5_1$.

Within the space forms, an extensive and dedicated research effort spanning over four decades, spearheaded by Chen et al. [8], devoted to the thorough examination of 1-type submanifolds and the associated 1-type Gauss map.

The aim of this research is to investigate the helicoidal hypersurfaces family $\mathfrak{x} = \mathfrak{x}(u, v, s, t)$ in five-dimensional Euclidean space $\mathbb{E}^5$. The objectives include analyzing its fundamental forms, Gauss map, and shape operator matrix, calculating its curvatures, and establishing relationships between mean and Gauss–Kronecker curvatures. Additionally, the study aims to explore the mathematical connection between $\mathfrak{x}$ and a 5×5 matrix $\mathcal{A}$ through the Laplace–Beltrami operator of $\mathfrak{x}$. The future research endeavors will focus on exploring the practical applications of the core discoveries in this paper. The main aim is to integrate concepts derived from soliton theory, submanifold theory, and other relevant results mentioned in references [17, 18, 21, 22, 30–42]. Through this approach to investigate the most promising avenues that can advance the research objectives.

In Section 2, we present a comprehensive overview of the fundamental concepts and principles of five-dimensional Euclidean geometry. Specifically, we delve into the establishment of the first and second fundamental form matrices, Gauss map, and the shape operator matrix applicable to hypersurfaces residing in $\mathbb{E}^5$.

Moving forward, in Section 3, we precisely define the concept of a helicoidal hypersurfaces family within the domain of $\mathbb{E}^5$. Subsequently, in Section 4, we introduce the curvatures of hypersurfaces by leveraging the Cayley−Hamilton theorem. Furthermore, we present the curvature formulas and perform the computation of the curvatures associated with the family $\mathfrak{x}$. We also establish pertinent relationships concerning the mean and Gauss–Kronecker curvatures of $\mathfrak{x}$.

Finally, in the last section, we unveil the intriguing result $\Delta\mathfrak{x} = \mathcal{A}\mathfrak{x}$, where Δ denotes the Laplace–Beltrami operator, $\mathcal{A}$ represents a 5×5 matrix.

2. Preliminaries

We introduce the first and second fundamental forms, Gauss map $\mathbf{G}$, the shape operator matrix $\mathbb{S}$, curvature formulas $\mathcal{K}_i$, the mean curvature $\mathcal{K}_1$, and the Gauss–Kronecker curvature $\mathcal{K}_4$ of a hypersurface $\mathfrak{x}$ in Euclidean 5-space $\mathbb{E}^5$. We identify a vector $\overrightarrow{\alpha}$ with its transpose in this work.

The following definitions and notations contribute to the understanding and study of Euclidean 5-space and its geometric properties. We assume $\mathfrak{x} = \mathfrak{x}(u, v, s, t)$ be an immersion from $M^4 \subset \mathbb{E}^4$ to $\mathbb{E}^5$.

Definition 2.1. *A Euclidean dot product of* $\overrightarrow{x^1} = (x^1_1, ..., x^1_5)$, $\overrightarrow{x^2} = (x^2_1, ..., x^2_5)$ *of* $\mathbb{E}^5$ *is given by*

$$\overrightarrow{x^1} \cdot \overrightarrow{x^2} = \sum_{i=1}^{5} x^1_i x^2_i.$$

Definition 2.2. *A quadruple vector product of* $\mathbb{E}^5$ *is defined by*

$$\overrightarrow{x^1} \times \overrightarrow{x^2} \times \overrightarrow{x^3} \times \overrightarrow{x^4} = \det \begin{pmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \\ x^1_1 & x^1_2 & x^1_3 & x^1_4 & x^1_5 \\ x^2_1 & x^2_2 & x^2_3 & x^2_4 & x^2_5 \\ x^3_1 & x^3_2 & x^3_3 & x^3_4 & x^3_5 \\ x^4_1 & x^4_2 & x^4_3 & x^4_4 & x^4_5 \end{pmatrix},$$

where e_i, $i = 1, ..., 5$, are the base elements of $\mathbb{E}^5$.

Definition 2.3. *For a hypersurface $\mathfrak{x}$ in 5-space, first and second fundamental form matrices, resp., are given by*

$$\mathbb{I} = \begin{pmatrix} E & F & A & D \\ F & G & B & J \\ A & B & C & Q \\ D & J & Q & S \end{pmatrix}, \quad \mathbb{II} = \begin{pmatrix} L & M & P & X \\ M & N & T & Y \\ P & T & V & Z \\ X & Y & Z & I \end{pmatrix},$$

with

$$\begin{aligned} \det \mathbb{I} &= \left(EG - F^2\right)\left(CS - Q^2\right) + \left(J^2 - GS\right)A^2 + \left(D^2 - ES\right)B^2 \\ &\quad +2\left((CF - AB)DJ + (EB - FA)JQ + (GA - FB)DQ\right) \\ &\quad -\left(EJ^2 + GD^2\right)C + 2FABS, \\ \det \mathbb{II} &= \left(LN - M^2\right)\left(IV - Z^2\right) + \left(Y^2 - IN\right)P^2 + \left(X^2 - IL\right)T^2 \\ &\quad +2\left((VM - PT)XY + (LT - MP)YZ + (NP - MT)XZ\right) \\ &\quad -\left(LY^2 + NX^2\right)V + 2MIPT, \end{aligned}$$

where the components of the matrices described by

$$\begin{array}{lllll} E = \mathfrak{x}_u \cdot \mathfrak{x}_u, & F = \mathfrak{x}_u \cdot \mathfrak{x}_v, & A = \mathfrak{x}_u \cdot \mathfrak{x}_s, & D = \mathfrak{x}_u \cdot \mathfrak{x}_t, & G = \mathfrak{x}_v \cdot \mathfrak{x}_v, \\ B = \mathfrak{x}_v \cdot \mathfrak{x}_s, & J = \mathfrak{x}_v \cdot \mathfrak{x}_t, & C = \mathfrak{x}_s \cdot \mathfrak{x}_s, & Q = \mathfrak{x}_s \cdot \mathfrak{x}_t, & S = \mathfrak{x}_t \cdot \mathfrak{x}_t, \\ L = \mathfrak{x}_{uu} \cdot \mathbb{G}, & M = \mathfrak{x}_{uv} \cdot \mathbb{G}, & P = \mathfrak{x}_{us} \cdot \mathbb{G}, & X = \mathfrak{x}_{ut} \cdot \mathbb{G}, & N = \mathfrak{x}_{vv} \cdot \mathbb{G}, \\ T = \mathfrak{x}_{vs} \cdot \mathbb{G}, & Y = \mathfrak{x}_{vt} \cdot \mathbb{G}, & V = \mathfrak{x}_{ss} \cdot \mathbb{G}, & Z = \mathfrak{x}_{st} \cdot \mathbb{G}, & I = \mathfrak{x}_{tt} \cdot \mathbb{G}. \end{array}$$

Here, $\mathfrak{x}_u = \frac{\partial \mathfrak{x}}{\partial u}$, $\mathfrak{x}_{uv} = \frac{\partial^2 \mathfrak{x}}{\partial u \partial v}$, $\mathfrak{x}_{vv} = \frac{\partial^2 \mathfrak{x}}{\partial v^2}$, etc., and

$$\mathbb{G} = \frac{\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_s \times \mathfrak{x}_t}{\|\mathfrak{x}_u \times \mathfrak{x}_v \times \mathfrak{x}_s \times \mathfrak{x}_t\|}$$

denotes the Gauss map of $\mathfrak{x}$.

Definition 2.4. *Computing $\mathbb{I}^{-1} \cdot \mathbb{II}$, the shape operator matrix is given by $\mathbb{S} = \frac{1}{\det \mathbb{I}} \begin{pmatrix} s_{ij} \end{pmatrix}_{4 \times 4}$ with the following components*

$$\begin{aligned} s_{11} &= AJ^2P - CJ^2L - B^2LS + B^2XD + CJMD - BJPD - BMQD \\ &\quad -CGXD + GPQD + ABMS - ABJX - AJMQ + BJLQ \\ &\quad +BJLQ - CFMS + CGLS - AGPS + BFPS + CFJX \\ &\quad +AGQX - BFQX - FJPQ + FMQ^2 - GLQ^2, \end{aligned}$$

$$\begin{aligned} s_{12} &= AJ^2T - CJ^2M - B^2MS + B^2YD + CJND - BJTD - BNQD \\ &\quad -CGYD + GQTD + ABNS - ABJY - AJNQ + BJMQ \\ &\quad +BJMQ - CFNS + CGMS + CFJY - AGST + BFST \\ &\quad +AGQY - BFQY + FNQ^2 - GMQ^2 - FJQT, \end{aligned}$$

$$\begin{aligned} s_{13} &= AJ^2V - CJ^2P - B^2PS + B^2ZD + CJTD - BJVD - CGZD \\ &\quad -BQTD + GQVD - ABJZ + ABST + BJOP + BJPQ \\ &\quad +CFJZ + CGPS - AJOT - CFST + AGQZ - AGSV \\ &\quad -BFQZ + BFSV - FJQV - GPQ^2 + FTQ^2, \end{aligned}$$

$$\begin{aligned}
s_{14} &= AJ^2Z - CJ^2X - B^2SX + B^2DI - BJZD + CJYD - BQYD \\
&\quad +GQZD - ABJI + CFJI + AGQI - BFQI - CGDI \\
&\quad +ABSY - AJQY + BJQX - AGSZ + BFSZ + BJQX \\
&\quad -CFSY + CGSX - FJQZ + FQ^2Y - GQ^2X,
\end{aligned}$$

$$\begin{aligned}
s_{21} &= -A^2MS + A^2JX - CMD^2 + BPD^2 + CJLD - ABXD \\
&\quad -AJPD + AMQD - BLQD + AMQD + CFXD + CMSE \\
&\quad -BPSE - CJXE - FPQD + BQXE + JPQE - MQ^2E \\
&\quad +ABLS - AJLQ - CFLS + AFPS - AFQX + FLQ^2,
\end{aligned}$$

$$\begin{aligned}
s_{22} &= A^2JY - A^2NS - CND^2 + BTD^2 + CJMD - ABYD \\
&\quad +ANQD - BMQD - AJTD + ANQD + CFYD + CNSE \\
&\quad -CJYE - BSTE + BQYE - FQTD - NQ^2E + JQTE \\
&\quad +ABMS - AJMQ - CFMS + AFST - AFQY + FMQ^2,
\end{aligned}$$

$$\begin{aligned}
s_{23} &= A^2JZ - A^2ST - CTD^2 + BVD^2 - ABZD + CJPD \\
&\quad -AJVD - BQPD + CFZD + AQTD + AQTD - CJZE \\
&\quad +CSTE + BQZE - BSVE - FQVD + JQVE - Q^2TE \\
&\quad +ABPS - AJPQ - CFPS - AFQZ + AFSV + FPQ^2,
\end{aligned}$$

$$\begin{aligned}
s_{24} &= -A^2SY + A^2JI - CYD^2 + BZD^2 - AJZD + CJXD \\
&\quad +AQYD - BQXD + AQYD - BSZE + CSYE - FQZD \\
&\quad +JQZE - Q^2YE - AFQI - ABDI + CFDI - CJEI \\
&\quad +BQEI + ABSX + AFSZ - AJQX - CFSX + FQ^2X,
\end{aligned}$$

$$\begin{aligned}
s_{31} &= AJ^2L - F^2PS + F^2QX + BMD^2 - GPD^2 - J^2PE - AJMD \\
&\quad -BJLD + AGXD - BFXD + 2FJPD - FMQD + GLQD \\
&\quad -BMSE + BJXE + JMQE + GPSE - GQXE + AFMS \\
&\quad -AGLS + BFLS - AFJX - FJLQ,
\end{aligned}$$

$$\begin{aligned}
s_{32} &= AJ^2M - F^2ST + F^2OY + BND^2 - GTD^2 - J^2TE - AJND \\
&\quad -BJMD + AGYD - BFYD - FNQD + GMQD - BNSE \\
&\quad +2FJTD + BJYE + JNQE + GSTE - GQYE + AFNS \\
&\quad -AGMS + BFMS - AFJY - FJMQ,
\end{aligned}$$

$$\begin{aligned}
s_{33} &= AJ^2P + F^2QZ - F^2SV + BTD^2 - GVD^2 - J^2VE - BJPD \\
&\quad -AJTD + AGZD - BFZD + 2FJVD + GQPD + BJZE \\
&\quad -FQTD - BSTE + JQTE - GQZE + GSVE - AFJZ \\
&\quad -AGPS + BFPS + AFST - FJQP,
\end{aligned}$$

$$\begin{aligned}
s_{34} &= AJ^2X - F^2SZ + BYD^2 - GZD^2 - J^2ZE + F^2QI - AJYD \\
&\quad -BJXD + 2FJZD - FQYD + GQXD - BSYE + JQYE \\
&\quad +GSZE - AFJI + AGDI - BFDI + BJEI - GQEI \\
&\quad +AFSY - AGSX + BFSX - FJQX,
\end{aligned}$$

$$\begin{aligned} s_{41} &= A^2JM - A^2GX - CF^2X + F^2PQ + B^2LD - B^2XE - ABMD \\ &+CFMD - CGLD + AGPD - BFPD - CJME + BJPE \\ &+BMQE + CGXE - GPQE - ABJL + CFJL + 2ABFX \\ &-AFJP - AFMQ + AGLQ - BFLQ, \end{aligned}$$

$$\begin{aligned} s_{42} &= A^2JN - A^2GY - CF^2Y + F^2QT + B^2MD - B^2YE - ABND \\ &+CFND - CGMD - CJNE + AGTD - BFTD + BJTE \\ &+BNQE + CGYE - GQTE - ABJM + CFJM + 2ABFY \\ &-AFJT - AFNQ + AGMQ - BFMQ, \end{aligned}$$

$$\begin{aligned} s_{43} &= A^2JT - A^2GZ - CF^2Z + F^2QV + B^2PD - B^2ZE - ABTD \\ &-CGPD + CFTD + AGVD - BFVD - CJTE + BJVE \\ &+CGZE + BQTE - GQVE - ABJP + 2ABFZ + CFJP \\ &-AFJV + AGPQ - BFPQ - AFQT, \end{aligned}$$

$$\begin{aligned} s_{44} &= A^2JY + F^2QZ + B^2XD - A^2GI - CF^2I - B^2EI - ABYD \\ &+AGZD - BFZD + CFYD - CGXD + BJZE - CJYE \\ &+BQYE - GQZE + 2ABFI + CGEI - ABJX - AFJZ \\ &+CFJX - AFQY + AGQX - BFQX. \end{aligned}$$

Definition 2.5. *The mathematical expressions for the mean curvature and the Gauss−Kronecker curvature of a hypersurface $\mathfrak{x}$ in 5-space are provided by the following formulas*

$$\mathcal{K}_1 = \frac{1}{4} tr(\mathbb{S}), \tag{1}$$

$$\mathcal{K}_4 = \det(\mathbb{S}) = \frac{\det \mathbb{III}}{\det \mathbb{I}}, \tag{2}$$

where

$$\begin{aligned} tr(\mathbb{S}) &= [(EN + GL - 2FM)\left(CS - Q^2\right) + \left(EG - F^2\right)(SV + IC) \\ &- (GI + NS)A^2 - (LS + EI)B^2 - (CN + GV)D^2 - (EV + CL)J^2 \\ &+2(A^2JY + B^2XD + D^2BT + J^2AP + F^2QZ + CJMD - ABYD \\ &-BJPD + ANQD - AJTD - BMQD + AGZD - BFZD + CFYD \\ &-AGPS - CGXD + FJVD + GQPD + BJZE - CJYE + BFPS \\ &-BSTE - FQTD + BQYE + JQTE + AGQX - BFQX - GQZE \\ &+ABFI - FJPQ + AFST - AFQY + ABMS - ABJX - AJMQ \\ &+BJLQ + CFJX - AFJZ)]/\det \mathbb{I}. \end{aligned}$$

A hypersurface $\mathfrak{x}$ is j-minimal if $\mathcal{K}_j = 0$ identically on $\mathfrak{x}$.

Definition 2.6. *In $\mathbb{E}^5$, the curvature formulas $\mathcal{K}_i$, where $i = 0, ..., 4$, are obtained by the characteristic polynomial of $\mathbb{S}$:*

$$P_{\mathbb{S}}(\lambda) = \sum_{k=0}^{4} (-1)^k \mathfrak{s}_k \lambda^{n-k} = \det(\mathbb{S} - \lambda \mathcal{I}_4) = 0, \tag{3}$$

$\mathcal{I}_4$ describes the identity matrix of order 4. Hence, we reveal the curvature formulas $\binom{n}{i}\mathcal{K}_i = \mathfrak{s}_i$. Here, $\binom{4}{0}\mathcal{K}_0 = \mathfrak{s}_0 = 1$ (by definition), $\binom{4}{1}\mathcal{K}_1 = \mathfrak{s}_1, \ldots, \binom{4}{4}\mathcal{K}_4 = \mathfrak{s}_4$, and $\mathcal{K}_1$ is the mean curvature, $\mathcal{K}_4$ is the Gauss-Kronecker curvature, and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

See [23] for details. See also [16, 17, 21, 22] for details of dimension 4.

3. A Helicoidal Hypersurfaces Family in $\mathbb{E}^5$

In this section, we establish the definition of a helicoidal hypersurfaces family in five-dimensional Euclidean space $\mathbb{E}^5$. Consider an open interval I, which is a subset of $\mathbb{R}^2$, and let $\gamma : I \subset \mathbb{R}^2 \longrightarrow \Pi \subset \mathbb{R}^5$ represent a surface in $\mathbb{E}^5$. Furthermore, let ℓ denote a straight line in Π.

Definition 3.1. *A rotational hypersurface in $\mathbb{E}^5$ is characterized as a hypersurface formed by rotating a profile surface around an axis $\mathbb{E}^5$. The rotation is accompanied by the parallel displacement of lines orthogonal to the axis ℓ, where the displacement speed is proportionate to the rotational speed. Consequently, the resulting hypersurface is identified as the helicoidal hypersurface with the axis ℓ and pitches a and b, both of which are real numbers excluding zero. When both a and b are equal to zero, the resulting hypersurface is simply a rotational hypersurface.*

Considering the line ℓ defined by the vector $(0,0,0,0,1)^T$, the rotation matrix $\mathcal{R} = \mathcal{R}(s,t)$ in five-dimensional Euclidean space can be mathematically represented by

$$\mathcal{R} = \begin{pmatrix} \cos s & -\sin s & 0 & 0 & 0 \\ \sin s & \cos s & 0 & 0 & 0 \\ 0 & 0 & \cos t & -\sin t & 0 \\ 0 & 0 & \sin t & \cos t & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \ s,t \in [0,2\pi), \tag{4}$$

where $\mathcal{R}\cdot\ell = \ell$, $\mathcal{R}^t\cdot\mathcal{R} = \mathcal{R}\cdot\mathcal{R}^t = \mathcal{I}_5$, $\det\mathcal{R} = 1$. When the axis of rotation is ℓ, a Euclidean transformation occurs, through which the axis is converted to the x_5-axis of $\mathbb{E}^5$. The parametrization of the profile surface is defined by $\gamma(u,v) = (f,0,g,0,h)$, where f, g, and h are differentiable functions that depend on u and v, both belonging to the open interval $I \subset \mathbb{R}^2$. In $\mathbb{E}^5$, the helicoidal hypersurfaces family $\mathfrak{x}$, spanned by the vector $(0,0,0,0,1)$, can be expressed as $\mathfrak{x} = \mathcal{R}\cdot\gamma^T + (as+bt)\,\ell^T$, where u and v are in I, and s and t are in the interval $[0,2\pi)$, while a and b are real numbers excluding zero.

Consequently, the helicoidal hypersurfaces family can be expressed in the following explicit form

$$\mathfrak{x}(u,v,s,t) = \begin{pmatrix} f(u,v)\cos s \\ f(u,v)\sin s \\ g(u,v)\cos t \\ g(u,v)\sin t \\ h(u,v) + as + bt \end{pmatrix}. \tag{5}$$

4. Curvatures in $\mathbb{E}^5$

In this section, we reveal the curvature formulas of any hypersurface $\mathfrak{x} = \mathfrak{x}(u,v,s,t)$ in $\mathbb{E}^5$.

Theorem 4.1. *For a hypersurface $\mathfrak{x}$ in $\mathbb{E}^8$, the curvature formulas are given by*

$$\mathcal{K}_0 = 1,\ 4\mathcal{K}_1 = -\frac{\mathfrak{b}}{\mathfrak{a}},\ 6\mathcal{K}_2 = \frac{\mathfrak{c}}{\mathfrak{a}},\ 4\mathcal{K}_3 = -\frac{\mathfrak{d}}{\mathfrak{a}},\ \mathcal{K}_4 = \frac{\mathfrak{e}}{\mathfrak{a}}. \tag{6}$$

Here, $P_{\mathbb{S}}(\lambda) = \mathfrak{a}\lambda^4 + \mathfrak{b}\lambda^3 + \mathfrak{c}\lambda^2 + \mathfrak{d}\lambda + \mathfrak{e} = 0$ represents the characteristic polynomial of shape operator matrix $\mathbb{S}$, $\mathfrak{a} = \det\mathbb{I}$, $\mathfrak{e} = \det\mathbb{III}$, and $\mathbb{I}$, $\mathbb{III}$ correspond to the first, and the second fundamental form matrices, respectively.

Proof. By computing the product matrix $\mathbb{I}^{-1}\cdot\mathbb{III}$, we obtain the shape operator matrix $\mathbb{S}$ for the hypersurface $\mathfrak{x}$ in five-dimensional space. Next, we compute the curvature formulas, denoted as $\mathfrak{x}$, where i takes values from 0 to 4. This computation reveals the characteristic polynomial $P_{\mathbb{S}}(\lambda) = \det(\mathbb{S} - \lambda\mathcal{I}_4) = 0$ associated with $\mathbb{S}$. Consequently, we determine the following curvatures in five-dimensional space

$$\begin{aligned}
\binom{4}{0}\mathcal{K}_0 &= 1, \\
\binom{4}{1}\mathcal{K}_1 &= \sum_{i=1}^{4} k_i = -\frac{\mathfrak{b}}{\mathfrak{a}}, \\
\binom{4}{2}\mathcal{K}_2 &= \sum_{1=i_1<i_2}^{4} k_{i_1}k_{i_2} = \frac{\mathfrak{c}}{\mathfrak{a}}, \\
\binom{4}{3}\mathcal{K}_3 &= \sum_{1=i_1<i_2<i_3}^{4} k_{i_1}k_{i_2}k_{i_3} = -\frac{\mathfrak{d}}{\mathfrak{a}}, \\
\binom{4}{4}\mathcal{K}_4 &= \prod_{i=1}^{4} k_i = \frac{\mathfrak{e}}{\mathfrak{a}}.
\end{aligned}$$

Here, k_i, where i ranges from 1 to 4, represents the principal curvatures of the hypersurface $\mathfrak{x}$. □

For the case of $\mathbb{E}^4$, refer to the works by Güler et al. [12, 16, 17, 21, 22] for more details.

Theorem 4.2. *A hypersurface $\mathfrak{x} = \mathfrak{x}(u, v, s, t)$ in $\mathbb{E}^5$ satisfies the following relation*

$$\mathcal{K}_0\mathbb{V} - 4\mathcal{K}_1\mathbb{IV} + 6\mathcal{K}_2\mathbb{IIII} - 4\mathcal{K}_3\mathbb{III} + \mathcal{K}_4\mathbb{I} = \mathbb{O},$$

where $\mathbb{I}, \mathbb{III}, \mathbb{IIII}, \mathbb{IV}, \mathbb{V}$ denote the fundamental form matrices of the hypersurface, $\mathbb{O}$ describes the zero matrix, each having a order of 4×4.

Proof. By considering $n = 4$ in the determinant expression given (3), the result is evident. □

By taking the first derivatives of the family defined by Eq. (5) with respect to u, v, s, and t, we obtain the following quantities

$$\mathbb{I} = \begin{pmatrix} f_u^2 + g_u^2 + h_u^2 & f_uf_v + g_ug_v + h_uh_v & ah_u & bh_u \\ f_uf_v + g_ug_v + h_uh_v & f_v^2 + g_v^2 + h_v^2 & ah_v & bh_v \\ ah_u & ah_v & f^2 + a^2 & ab \\ bh_u & bh_v & ab & g^2 + b^2 \end{pmatrix}. \tag{7}$$

Moreover, we can express the determinant of the first fundamental form matrix given by (7) as follows

$$\det \mathbb{I} = \left(a^2g^2 + b^2f^2\right)\mathfrak{B}^2 + f^2g^2\left(\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2\right),$$

where $\mathfrak{A} = g_uh_v - g_vh_u$, $\mathfrak{B} = f_ug_v - f_vg_u$, $\mathfrak{C} = f_uh_v - f_vh_u$.

The Gauss map of the helicoidal hypersurfaces family given by Eq. (5) is characterized by the following parametrization

$$\mathbb{G} = \frac{1}{(\det \mathbb{I})^{1/2}} \begin{pmatrix} -fg\mathfrak{A}\cos s - ag\mathfrak{B}\sin s \\ ag\mathfrak{B}\cos s - fg\mathfrak{A}\sin s \\ fg\mathfrak{C}\cos t - bf\mathfrak{B}\sin t \\ bf\mathfrak{B}\cos t + fg\mathfrak{C}\sin t \\ -fg\mathfrak{B} \end{pmatrix}. \tag{8}$$

By calculating the second derivatives of $\mathfrak{x}$ with respect to u, v, s, and t, and utilizing them in conjunction with Eq. (8), we obtain the following second fundamental form matrix

$$\mathbb{II} = \frac{1}{(\det \mathbb{I})^{1/2}} \begin{pmatrix} fg\left(-\mathfrak{A}f_{uu} + \mathfrak{C}g_{uu} - \mathfrak{B}h_{uu}\right) & fg\left(-\mathfrak{A}f_{uv} + \mathfrak{C}g_{uv} - \mathfrak{B}h_{uv}\right) & af_ug\mathfrak{B} & bfg_u\mathfrak{B} \\ fg\left(-\mathfrak{A}f_{uv} + \mathfrak{C}g_{uv} - \mathfrak{B}h_{uv}\right) & fg\left(-\mathfrak{A}f_{vv} + \mathfrak{C}g_{vv} - \mathfrak{B}h_{vv}\right) & af_vg\mathfrak{B} & bfg_v\mathfrak{B} \\ af_ug\mathfrak{B} & af_vg\mathfrak{B} & f^2g\mathfrak{A} & 0 \\ bfg_u\mathfrak{B} & bfg_v\mathfrak{B} & 0 & fg^2\mathfrak{C} \end{pmatrix}.$$

The product matrix $\mathbb{I}^{-1}\cdot\mathbb{III}$ yields the shape operator matrix $\mathbb{S}$ of the hypersurface $\mathfrak{x}$. Subsequently, we calculate the mean curvature $\mathcal{K}_1$ and the Gauss−Kronecker curvature $\mathcal{K}_4$. Therefore, the following relationship holds.

Theorem 4.3. *The mean curvature and Gauss−Kronecker curvature of the helicoidal hypersurfaces family determined by Eq.* (5) *are given by, respectively,*

$$\begin{aligned}
\mathcal{K}_1 &= \frac{fg}{4(\det \mathbb{I})^{3/2}}[\left(\left(b^2f^2+a^2g^2\right)\left(f_v^2+g_v^2\right)+f^2g^2\left(f_v^2+g_v^2+h_v^2\right)\right)(-\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}-\mathfrak{C}g_{uu}) \\
&\quad +\left(a^2g\mathfrak{C}+b^2f\mathfrak{A}\right)\mathfrak{B}^2+fg\left(f\mathfrak{C}+\mathfrak{A}g\right)\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right) \\
&\quad -\left(\left(b^2f^2+a^2g^2\right)\left(f_u^2+g_u^2\right)+f^2g^2\left(f_u^2+g_u^2+h_u^2\right)\right)(\mathfrak{A}f_{vv}+\mathfrak{C}g_{vv}+\mathfrak{B}h_{vv})], \\
\mathcal{K}_4 &= \frac{f^2g^2\left(b^2\mathfrak{B}\left(g_v^2\Psi_1+g_u^2\Psi_2\right)+g^3\mathfrak{C}\left(a^2\mathfrak{B}\left(f_v^2\Psi_3-f_u^2\Psi_4\right)+f^3\mathfrak{A}\Psi_5\right)\right)}{(\det \mathbb{I})^3},
\end{aligned}$$

where

$$\begin{aligned}
\det \mathbb{I} &= \left(a^2g^2+b^2f^2\right)\mathfrak{B}^2+f^2g^2\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right), \\
\Psi_1 &= a^2f_u^2\mathfrak{B}^3+\mathfrak{A}^2f^3\left(g_uf_{uv}-g_vf_{uu}\right)+\mathfrak{A}\mathfrak{C}f^3\left(g_ug_{uv}-g_vg_{uu}\right) \\
&\quad +\mathfrak{A}\mathfrak{B}f^3\left(g_uh_{uv}-g_vh_{uu}\right)-\mathfrak{A}f^3f_ug_v^2h_{uu}, \\
\Psi_2 &= -f_v\left(a^2\mathfrak{B}^3+f^3\mathfrak{A}\left(h_u\left(g_ug_{vv}-g_vg_{uv}\right)+g_u\left(g_uh_{vv}-g_vh_{uv}\right)\right)\right) \\
&\quad +\mathfrak{A}\left(g_uf_{vv}-g_vf_{uv}\right)+f_ug_v\left(g_uh_{vv}-g_vh_{uv}\right)+f_uh_v\left(g_ug_{vv}-g_vg_{uv}\right), \\
\Psi_3 &= \mathfrak{A}\left(f_uf_{uv}-f_vf_{uu}\right)+\mathfrak{B}\left(f_uh_{uv}-f_vh_{uu}\right)+\mathfrak{C}\left(f_ug_{uv}-f_vg_{uu}\right), \\
\Psi_4 &= \mathfrak{A}\left(f_uf_{vv}-f_vf_{uv}\right)+\mathfrak{B}\left(f_uh_{vv}-f_vh_{uv}\right)+\mathfrak{C}\left(f_ug_{vv}-f_vg_{uv}\right), \\
\Psi_5 &= \left(\mathfrak{A}f_{uu}+\mathfrak{B}h_{uu}+\mathfrak{C}g_{uu}\right)\left(\mathfrak{A}f_{vv}+\mathfrak{B}h_{vv}+\mathfrak{C}g_{vv}\right)-\left(\mathfrak{A}f_{uv}+\mathfrak{B}h_{uv}+\mathfrak{C}g_{uv}\right)^2, \\
\mathfrak{A} &= g_uh_v-g_vh_u,\ \ \mathfrak{B}=f_ug_v-f_vg_u,\ \ \mathfrak{C}=f_uh_v-f_vh_u.
\end{aligned}$$

Here, $f=f(u,v)$, $g=g(u,v)$, $h=h(u,v)$, $f_u=\frac{\partial f}{\partial u}$, $f_{uv}=\frac{\partial^2 f}{\partial u\partial v}$, *etc.,* $a,b\in\mathbb{R}-\{0\}$.

Through the utilization of the Cayley–Hamilton theorem, we uncover the characteristic polynomial of $\mathbb{S}$ as follows

$$\mathcal{K}_0\lambda^4-4\mathcal{K}_1\lambda^3+6\mathcal{K}_2\lambda^2-4\mathcal{K}_3\lambda+\mathcal{K}_4=0.$$

The curvatures $\mathcal{K}_i$ of the helicoidal hypersurfaces family $\mathfrak{x}$ can also be determined by solving the aforementioned equation.

An umbilical point on a hypersurface is a point where all principal curvatures are equal, implying that the surface exhibits uniform curvature in all directions at that particular point. Subsequently, we give the following.

Theorem 4.4. *The helicoidal hypersurfaces family $\mathfrak{x}$ in $\mathbb{E}^5$ possesses an umbilical point if and only if the following condition is satisfied*

$$
\begin{aligned}
& f^2 g^2 [\left(\left(b^2 f^2 + a^2 g^2\right)\left(f_v^2 + g_v^2\right) + f^2 g^2 \left(f_v^2 + g_v^2 + h_v^2\right)\right)\left(-\mathfrak{A} f_{uu} + \mathfrak{B} h_{uu} - \mathfrak{C} g_{uu}\right) \\
& + \left(a^2 g \mathfrak{C} + b^2 f \mathfrak{A}\right)\mathfrak{B}^2 + fg\left(f\mathfrak{C} + \mathfrak{A} g\right)\left(\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2\right) \\
& - \left(\left(b^2 f^2 + a^2 g^2\right)\left(f_u^2 + g_u^2\right) + f^2 g^2 \left(f_u^2 + g_u^2 + h_u^2\right)\right)\left(\mathfrak{A} f_{vv} + \mathfrak{C} g_{vv} + \mathfrak{B} h_{vv}\right)]^4 \\
& -256\left[\left(a^2 g^2 + b^2 f^2\right)\mathfrak{B}^2 + f^2 g^2\left(\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2\right)\right]^3 \\
& . \left[b^2 \mathfrak{B}\left(g_v^2 \Psi_1 + g_u^2 \Psi_2\right) + g^3 \mathfrak{C}\left(a^2 \mathfrak{B}\left(f_v^2 \Psi_3 - f_u^2 \Psi_4\right) + f^3 \mathfrak{A} \Psi_5\right)\right] = 0.
\end{aligned}
$$

Proof. Given that the hypersurface $\mathfrak{x}$ possesses an umbilical point, it can be rigorously shown that it satisfies the equation $(\mathcal{K}_1)^4 = \mathcal{K}_4$. □

Problem 4.5. *Determine the solutions $h = h(u, v)$ of the aforementioned second-order partial differential Eq.*

Corollary 4.6. *Consider $\mathfrak{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5$, an immersion defined by Eq.* (5). *The immersion $\mathfrak{x}$ exhibits zero mean curvature if and only if the subsequent condition is fulfilled*

$$
\begin{aligned}
& \left(a^2 g \mathfrak{C} + b^2 f \mathfrak{A}\right)\mathfrak{B}^2 + fg\left(f\mathfrak{C} + \mathfrak{A} g\right)\left(\mathfrak{A}^2 + \mathfrak{B}^2 + \mathfrak{C}^2\right) \\
& + \left(\left(b^2 f^2 + a^2 g^2\right)\left(f_v^2 + g_v^2\right) + f^2 g^2\left(f_v^2 + g_v^2 + h_v^2\right)\right)\left(-\mathfrak{A} f_{uu} + \mathfrak{B} h_{uu} - \mathfrak{C} g_{uu}\right) \\
& - \left(\left(b^2 f^2 + a^2 g^2\right)\left(f_u^2 + g_u^2\right) + f^2 g^2\left(f_u^2 + g_u^2 + h_u^2\right)\right)\left(\mathfrak{A} f_{vv} + \mathfrak{C} g_{vv} + \mathfrak{B} h_{vv}\right) = 0,
\end{aligned}
$$

where $f, g, h \neq 0$.

Problem 4.7. *Investigate and obtain the solutions $h = h(u, v)$ of the second-order partial differential Eq. mentioned above.*

Corollary 4.8. *Consider $\mathfrak{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5$, an immersion given by Eq.* (5). *The immersion $\mathfrak{x}$ exhibits zero Gauss−Kronecker curvature if and only if the following equation is satisfied*

$$
b^2 \mathfrak{B}\left(g_v^2 \Psi_1 + g_u^2 \Psi_2\right) + g^3 \mathfrak{C}\left(a^2 \mathfrak{B}\left(f_v^2 \Psi_3 - f_u^2 \Psi_4\right) + f^3 \mathfrak{A} \Psi_5\right) = 0,
$$

where $f, g, h \neq 0$.

Problem 4.9. *Find the solutions $h = h(u, v)$ of the aforementioned second-order partial differential Eq.*

5. Helical Hypersurfaces Family Supplying $\Delta \mathfrak{x} = \mathcal{A} \mathfrak{x}$ in $\mathbb{E}^5$

In this section, we give the Laplace–Beltrami operator of a smooth function in $\mathbb{E}^5$. Then we calculate it by using the helicoidal hypersurfaces family defined by Eq. (5).

Definition 5.1. *The Laplace–Beltrami operator of a smooth function $\phi = \phi(x^1, x^2, x^3, x^4) \mid_{\mathbf{D}}$ (where $\mathbf{D} \subset \mathbb{R}^4$) belonging to the class C^4 and depending on the first fundamental form, is the differential operator defined by*

$$
\Delta\phi = \frac{1}{\mathbf{g}^{1/2}} \sum_{i,j=1}^{4} \frac{\partial}{\partial x^i}\left(\mathbf{g}^{1/2} \mathbf{g}^{ij} \frac{\partial \phi}{\partial x^j}\right), \tag{9}
$$

where $\left(\mathbf{g}^{ij}\right) = \left(\mathbf{g}_{kl}\right)^{-1}$ and $\mathbf{g} = \det\left(\mathbf{g}_{ij}\right)$.

Hence, the Laplace–Beltrami operator associated with the first fundamental form of the helicoidal hypersurfaces family $\mathfrak{x}$ defined by Eq. (5), is given by

$$
\begin{aligned}
\Delta\mathfrak{x} = \tfrac{1}{\mathbf{g}^{1/2}} \Big[& \tfrac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{11}\tfrac{\partial\mathfrak{x}}{\partial u}\right) + \tfrac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{12}\tfrac{\partial\mathfrak{x}}{\partial v}\right) + \tfrac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{13}\tfrac{\partial\mathfrak{x}}{\partial s}\right) + \tfrac{\partial}{\partial u}\left(\mathbf{g}^{1/2}\mathbf{g}^{14}\tfrac{\partial\mathfrak{x}}{\partial t}\right) \\
& + \tfrac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{21}\tfrac{\partial\mathfrak{x}}{\partial u}\right) + \tfrac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{22}\tfrac{\partial\mathfrak{x}}{\partial v}\right) + \tfrac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{23}\tfrac{\partial\mathfrak{x}}{\partial s}\right) + \tfrac{\partial}{\partial v}\left(\mathbf{g}^{1/2}\mathbf{g}^{24}\tfrac{\partial\mathfrak{x}}{\partial t}\right) \\
& + \tfrac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{31}\tfrac{\partial\mathfrak{x}}{\partial u}\right) + \tfrac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{32}\tfrac{\partial\mathfrak{x}}{\partial v}\right) + \tfrac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{33}\tfrac{\partial\mathfrak{x}}{\partial s}\right) + \tfrac{\partial}{\partial s}\left(\mathbf{g}^{1/2}\mathbf{g}^{34}\tfrac{\partial\mathfrak{x}}{\partial t}\right) \\
& + \tfrac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{41}\tfrac{\partial\mathfrak{x}}{\partial u}\right) + \tfrac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{42}\tfrac{\partial\mathfrak{x}}{\partial v}\right) + \tfrac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{43}\tfrac{\partial\mathfrak{x}}{\partial s}\right) + \tfrac{\partial}{\partial t}\left(\mathbf{g}^{1/2}\mathbf{g}^{44}\tfrac{\partial\mathfrak{x}}{\partial t}\right)\Big],
\end{aligned}
\tag{10}
$$

where

$$
\begin{aligned}
\mathbf{g}^{11} &= \frac{\left(f_v^2+g_v^2\right)\left(a^2g^2+b^2f^2\right)+f^2g^2\left(f_v^2+g_v^2+h_v^2\right)}{\det\mathbb{I}}, \\
\mathbf{g}^{12} &= -\frac{\left(a^2g^2+b^2f^2\right)\left(f_uf_v+g_ug_v\right)+f^2g^2\left(f_uf_v+g_ug_v+h_uh_v\right)}{\det\mathbb{I}} = \mathbf{g}^{21}, \\
\mathbf{g}^{13} &= \frac{ag^2\left(\left(f_uf_v+g_ug_v\right)h_v-\left(f_v^2+g_v^2\right)h_u\right)}{\det\mathbb{I}} = \mathbf{g}^{31}, \\
\mathbf{g}^{14} &= \frac{bf^2\left(\left(f_uf_v+g_ug_v\right)h_v-\left(f_v^2+g_v^2\right)h_u\right)}{\det\mathbb{I}} = \mathbf{g}^{41}, \\
\mathbf{g}^{22} &= \frac{\left(f_u^2+g_u^2\right)\left(a^2g^2+b^2f^2\right)+f^2g^2\left(f_u^2+g_u^2+h_u^2\right)}{\det\mathbb{I}}, \\
\mathbf{g}^{23} &= \frac{ag^2\left(\left(f_uf_v+g_ug_v\right)h_u-\left(f_u^2+g_u^2\right)h_v\right)}{\det\mathbb{I}} = \mathbf{g}^{32}, \\
\mathbf{g}^{24} &= \frac{\left(f_uf_v+g_ug_v\right)h_u-bf^2\left(\left(f_u^2+g_u^2\right)h_v\right)}{\det\mathbb{I}} = \mathbf{g}^{42}, \\
\mathbf{g}^{33} &= \frac{b^2\mathfrak{B}^2+g^2\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right)}{\det\mathbb{I}}, \\
\mathbf{g}^{34} &= -\frac{ab\left(f_ug_v-f_vg_u\right)^2}{\det\mathbb{I}} = \mathbf{g}^{43}, \\
\mathbf{g}^{44} &= \frac{a^2\mathfrak{B}^2+f^2\left(\mathfrak{A}^2+\mathfrak{B}^2+\mathfrak{C}^2\right)}{\det\mathbb{I}}.
\end{aligned}
$$

By utilizing the inverse matrix mentioned above of $\mathbb{I}$ as referenced in (7) and differentiating the functions in (10) with respect to u, v, s, and t, respectively, we obtain the following results.

Theorem 5.2. *The Laplace–Beltrami operator of the helicoidal hypersurfaces family described by Eq.* (5) *can be expressed as* $\Delta\mathfrak{x} = 4\mathcal{K}_1\mathbb{G}$*, where* $\mathcal{K}_1$ *represents the mean curvature and* $\mathbb{G}$ *corresponds to the Gauss map of* $\mathfrak{x}$*.*

Proof. By performing direct computations using Eq. (10), we arrive at the expression for $\mathfrak{x}$. □

Theorem 5.3. *Let* $\mathfrak{x} : M^4 \subset \mathbb{E}^4 \longrightarrow \mathbb{E}^5$ *be an immersion given by Eq.* (5)*. Then,* $\Delta\mathfrak{x} = \mathcal{A}\mathfrak{x}$*, where* $\mathcal{A}$ *is the matrix of order* 5*, if and only if* $\mathfrak{x}$ *has* $\mathcal{K}_1 = 0$*, indicating that it is a minimal helical hypersurface.*

Proof. We establish the Eq. $4\mathcal{K}_1\mathbb{G} = \mathcal{A}\mathfrak{x}$, thereby leading us to the subsequent deductions

$$\begin{aligned} & a_{11}f\cos s + a_{12}f\sin s + a_{13}g\cos t + a_{14}g\sin t + a_{15}(h + as + bt) \\ = \; & \Phi(-fg\mathfrak{A}\cos s - ag\mathfrak{B}\sin s), \\ & a_{21}f\cos s + a_{22}f\sin s + a_{23}g\cos t + a_{44}g\sin t + a_{25}(h + as + bt) \\ = \; & \Phi(ag\mathfrak{B}\cos s - fg\mathfrak{A}\sin s), \\ & a_{31}f\cos s + a_{32}f\sin s + a_{33}g\cos t + a_{34}g\sin t + a_{35}(h + as + bt) \\ = \; & \Phi(fg\mathfrak{C}\cos t - bf\mathfrak{B}\sin t), \\ & a_{41}f\cos s + a_{42}f\sin s + a_{43}g\cos t + a_{44}g\sin t + a_{45}(h + as + bt) \\ = \; & \Phi(bf\mathfrak{B}\cos t + fg\mathfrak{C}\sin t), \\ & a_{51}f\cos s + a_{52}f\sin s + a_{53}g\cos t + a_{54}g\sin t + a_{55}(h + as + bt) \\ = \; & -\Phi fg\mathfrak{B}, \end{aligned}$$

where $\mathcal{A}$ is the 5×5 matrix $\Phi = 4\mathcal{K}_1(\det \mathbb{I})^{-1/2}$. Upon taking the second derivative of the aforementioned ODEs with respect to the variable s, the resulting outcomes are as follows

$$a_{15} = a_{25} = a_{35} = a_{45} = a_{55} = 0, \; \Phi = 0.$$

Consequently, the following expression emerges

$$a_{i1}f\cos s + a_{i2}f\sin s = 0,$$

where $i = 1, ..., 5$. By considering the linear independence of the functions sin and cos with respect to s, it can be deduced that all components of the matrix $\mathcal{A}$ are equal to 0. Since $\mathcal{A} = 4\mathcal{K}_1(\det \mathbb{I})^{-1/2}$, it follows that $\mathcal{K}_1$ must be 0. This crucially implies that $\mathfrak{x}$ represents a minimal helicoidal hypersurface. □

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