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On almost generalized gradient Ricci-Yamabe soliton

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Abstract. In this paper, we study the geometric characterizations and classify of the Riemannian manifold with generalized gradient Ricci-Yamabe soliton or almost generalized gradient Ricci-Yamabe soliton. In addition, theorems were obtained to construct a model space with gradient Ricci-Yamabe soliton, generalized gradient Ricci-Yamabe soliton, almost gradient Ricci-Yamabe soliton and almost generalized gradient Ricci-Yamabe soliton.

1. Introduction

The notion of the Ricci soliton was introduced by Hamilton [6], which is a natural generalization of Einstein's metrics and self-similar solutions to the Ricci flow. The Ricci soliton is defined on a Riemannian manifold (M, g) as follows:

$$S + \frac{1}{2}\mathfrak{L}_X g = \rho g, \tag{1}$$

where X is a smooth vector field on M, $\mathfrak{L}_X$ is the Lie derivative with respect to X, S is a Ricci tensor of g and ρ is a constant (see [2], [3], [4] and [9]). The Ricci soliton is said to be expanding, steady, or shrinking according to $\rho < 0$, $\rho = 0$ or $\rho > 0$, respectively. If X is a gradient of some smooth function h, that is $X = \nabla h$, then M is called a gradient Ricci soliton with a potential function h or (h, ρ). In this case, the equation (1) reduces to

$$S + \nabla^2 h = \rho g, \tag{2}$$

where $\nabla^2 h$ is the Hessian form of h. If ρ is a function on M, then M is called an almost gradient Ricci soliton with (h, ρ) (see [7]).

A Riemannian metric g on a Riemannian manifold M is called a Yamabe soliton if there exists a smooth vector field X and a constant ρ such that

$$(r - \rho)g = \frac{1}{2}\mathfrak{L}_X g, \tag{3}$$

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where r is the scalar curvature of g (see [2], [3], [4], [9] and [12]). When $X = \nabla h$ for some function h on M, we say that M is a gradient Yamabe soliton with a potential function h or (h, ρ). In this case, the equation (3) becomes

$$(r - \rho)g = \nabla^2 h. \tag{4}$$

If ρ is a function on M, then M is called an almost gradient Yamabe soliton with (h, ρ).

There are many attempts and papers for Ricci and Yamabe solitons and generalizations with their geometric characterizations. In 2019, S. Guler and M. Crasmareanu [5] introduced the notion of Ricci-Yamabe flow on a Riemannian manifold by considering a scalar combination of the Ricci flow and the Yamabe flow. In [3], the authors introduced the notion of Ricci-Yamabe soliton $(g, X, \lambda, \alpha, \beta)$ from the Ricci-Yamabe flow on a Riemannian manifold as there exists a smooth vector field X on M and constants λ, α, β such that

$$\mathfrak{L}_X g + 2\alpha S = (2\lambda - \beta r)g. \tag{5}$$

If there exists a smooth function λ on M satisfying the equation (5), then M is said to admit almost Ricci-Yamabe soliton[7]. In [7], the authors examine the isometries of almost Ricci-Yamabe solitons and showed that the potential function of a compact gradient almost Ricci-Yamabe soliton agrees with the Hodge-de Rham potential function. In particular, if $X = \nabla h$ for some smooth function h on M, we say that M is a gradient Ricci-Yamabe soliton with $(g, h, \lambda, \alpha, \beta)$. In this case, the equation (5) becomes

$$\nabla^2 h + \alpha S = (\lambda - \frac{1}{2}\beta r)g. \tag{6}$$

If λ is a function on M, then M or g is said to be an almost gradient Ricci-Yamabe soliton [7]. The notion of gradient Ricci-Yamabe soliton generalizes a large class of solitons like equations. Ricci-Yamabe soliton (or gradient Ricci-Yamabe soliton) is said to be a (see [2] and [5])
(a) Ricci soliton (or gradient Ricci soliton) if $\alpha = 1, \beta = 0$
(b) Yamabe soliton (or gradient Yamabe soliton) if $\alpha = 0, \beta = 1$
(c) Einstein soliton (or gradient Einstein soliton) if $\alpha = 1, \beta = -1$
(d) ρ-Einstein soliton (or gradient ρ-Einstein soliton) if $\alpha = 1, \beta = -2\rho$.

The Ricci-Yamabe soliton or gradient Ricci-Yamabe soliton is said to be proper if $\alpha \neq 0, 1$ (see [2] and [8]). In [4], the authors extended these solitons to a more generalized version, that is a generalized Ricci-Yamabe soliton.

An n-dimensional Riemannian manifold (M, g), $n > 2$ is said to admit a generalized Ricci-Yamabe soliton $(g, X, \lambda, \alpha, \beta, w)$ if

$$\mathfrak{L}_X g + 2\alpha S = (2\lambda - \beta r)g + 2w V^\# V^\#, \tag{7}$$

where $\lambda, \alpha, \beta, w \in R$ and $V^\#$ is the 1-form dual to V. If V is a gradient of some function h on M, then M or g is called a generalized gradient Ricci-Yamabe soliton $(g, h, \lambda, \alpha, \beta, w)$. In this case, the equation (7) becomes

$$\nabla^2 h + \alpha S = (\lambda - \frac{1}{2}\beta r)g + w dh \otimes dh. \tag{8}$$

If λ is a function on M, then M or g is said to be an almost generalized gradient Ricci-Yamabe soliton. The generalized gradient Ricci-Yamabe soliton is said to be proper if $w \neq 0$.

A generalized Ricci-Yamabe soliton is said to be a (see [4])
(a) proper Ricci-Yamabe soliton if $w = 0$ and $\alpha \neq 0, 1$.
(b) Ricci soliton if $\alpha = 1, \beta = w = 0$.
(c) Yamabe soliton if $\alpha = w = 0, \beta = 2$.
(d) quasi-Yamabe soliton if $\alpha = 0$ and $\beta = 2$.

(e) Einstein soliton if $\alpha = 1, \beta = -1$ and $w = 0$.
(f) ρ-Einstein soliton if $\alpha = 1, \beta = -2\rho$ and $w = 0$.

In [4], the authors investigated a Sasakian 3-metric as a generalized gradient Ricci-Yamabe soliton and some related results and they suggested a 3-dimensional unit sphere S^3 as an example of a generalized gradient Ricci-Yamabe soliton by taking an orthonormal basis for the tangent space at any point of S^3. In [3], they studied Ricci-Yamabe solitons and a 3-dimensional Riemannian manifold with a model space of gradient Ricci-Yamabe soliton. In [8], the present authors studied a gradient Ricci-Yamabe soliton and gave theorems for constructing a gradient Ricci-Yamabe soliton on the product manifold. In this paper, we studied geometric characterizations of the manifold with almost gradient Ricci-Yamabe soliton and almost generalized gradient Ricci-Yamabe soliton in sections 2, 6, and 7. Moreover, we obtained theorems for the construction of the model space admitting a gradient Ricci-Yamabe soliton, a generalized gradient Ricci-Yamabe soliton, an almost gradient Ricci-Yamabe soliton, and an almost generalized gradient Ricci-Yamabe soliton on the product manifold in Theorems 2.3, 2.4, 6.3, and 7.2 respectively.

We prepare the following theorem for later use.

Theorem 1.1. *([11]) Let M be a complete Riemannian manifold of dimension $n \geq 2$ and suppose it admits a special concircular field ρ satisfying the equation*

$$\nabla_\mu \nabla_\lambda \rho = (-k\rho + b) g_{\mu\lambda}$$

Then M is one of the following manifolds :
(I, A) if $k = b = 0$, the direct product $V \times I$ of an (n-1)-dimensional complete Riemannian manifold V with a straight line I,
(I, B) if $k = 0$ but $b \neq 0$, a Euclidean space,
(II, A) if $k = -c^2 < 0$ and $N = 0$, a pseudo-hyperbolic space of zero or negative type,
(II, B) if $k = -c^2 < 0$ and $N = 1$, a hyperbolic space of curvature $-c^2$, and
(III) if $k = c^2 > 0$, a spherical space of curvature c^2,
where c is a positive constant and N is the number of isolated stationary points of a concircular scalar field ρ.

This paper is organized as follows: After an introduction, section 2 introduces the results of geometric characterization of generalized gradient Ricci-Yamabe soliton in the product manifold and gave methods for construction of model space of gradient Ricci-Yamabe soliton in Theorems 2.3, and 2.4. In section 3, we study the geometric characterization of the warped product manifold $R^n \times_f F$ with generalized gradient Ricci-Yamabe soliton. In sections 4 and 5, we study the geometric characterizations of generalized gradient Ricci-Yamabe soliton in the warped product manifold $B \times_f F$ and the twisted product manifold $B \times_f F$ respectively. In sections 6 and 7, we study the geometric characterizations of almost gradient Ricci-Yamabe soliton and almost generalized gradient Ricci-Yamabe soliton in the product manifold and obtain Theorem 6.3 and Theorem 7.2 that how we can construct the model space.

2. Generalized gradient Ricci-Yamabe soliton in the product manifold

Let M be the product manifold $B \times F$ of an n-dimensional Riemannian manifold (B, g) and a p-dimensional Riemannian manifold $(F, \bar{g})$. If $M = B \times F$ be a generalized gradient Ricci-Yamabe soliton *with*$(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$, then we obtain

$$\begin{aligned} \nabla_b \tilde{h}_a + \tilde{\alpha} S_{ba} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r}) g_{ba} + \tilde{w}\tilde{h}_b\tilde{h}_a, \\ \partial_b \tilde{h}_x &= \tilde{w}\tilde{h}_b\tilde{h}_x, \\ \bar{\nabla}_y \tilde{h}_x + \tilde{\alpha}\bar{S}_{yx} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r}) \bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x, \\ \tilde{r} &= r + \bar{r}, \end{aligned} \tag{9}$$

where ∇ and $\bar{\nabla}$ are Levi-Civita connections for g and $\bar{g}$ respectively, and S and $\bar{S}$ are Ricci curvatures on B and F respectively, and $\tilde{r}$, r and $\bar{r}$ are scalar curvatures on M, B and F respectively. The ranges of the indices $a, b, ...,$ and $x, y, ...$ are $1, 2, ..., n$, and $n+1, n+2, ..., n+p$, respectively. Moreover $\tilde{h}_a = \frac{\partial \tilde{h}}{\partial u_a}$, $\tilde{h}_x = \frac{\partial \tilde{h}}{\partial \bar{u}_x}$ for the coordinate neighborhoods $\{\tilde{U} = (U, \bar{U}); \tilde{U}^h = (u^a, \bar{u}^x)\}$ in $B \times F$.

Assume that the potential function $\tilde{h}$ is repressed by $\tilde{h} = k + l$ for some function k on B and l on F respectively. Then, from (9), we get

$$\begin{aligned} \nabla_b k_a + \tilde{\alpha} S_{ba} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r}) g_{ba} + \tilde{w} k_b k_a, \\ \partial_b l_x &= \tilde{w} k_b l_x, \\ \bar{\nabla}_y l_x + \tilde{\alpha} \bar{S}_{yx} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r}) \bar{g}_{yx} + \tilde{w} l_y l_x. \end{aligned} \tag{10}$$

Then we see that $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ becomes a constant from the first and third equation of (10), and that $\tilde{r}$ becomes a constant. From the second equation of (10), we see that $\tilde{w} = 0$ or $k_a = 0$ or $l_x = 0$.
In the first case where $\tilde{w} = 0$, M becomes a gradient Ricci-Yamabe soliton. In addition, B and F become gradient Ricci solitons with $(\frac{k}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ and $(\frac{l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ respectively if $\tilde{\alpha} \neq 0$. If $\tilde{\alpha} = 0$, then the first and third equations of (10) become $\nabla_b k_a = \tilde{K} g_{ba}$ and $\bar{\nabla}_y l_x = \tilde{K} \bar{g}_{yx}$. Hence if we consider these equations and Theorem 1.1, then B and F become Euclidean spaces, when $\tilde{K} \neq 0$, and M becomes the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V with a straight line I when $\tilde{K} = 0$.
In the second case where $k_a = 0$, that is k is a constant, then the base space B and F become Einstein if $\tilde{\alpha} \neq 0$. If $\tilde{\alpha} = 0$, then $\tilde{K} = 0$. Finally, let us consider the third case where $l_x = 0$, that is l is a constant. If we think about it in the same way as in the second case, we see that F becomes Einstein if $\tilde{\alpha} \neq 0$. If $\tilde{\alpha} = 0$, then $\tilde{K} = 0$. Thus we have

Theorem 2.1. *Let $M = B \times F$ be a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$. If $h = k + l$ for some functions l and m on B and F respectively, then $\tilde{K}$ and $\tilde{r}$ become constants and M becomes one of the following three cases :*
(a) M becomes a gradient Ricci soliton with $(\frac{k+l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$. Moreover, B and F become gradient Ricci solitons with $(\frac{k}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ and $(\frac{l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ respectively if $\tilde{\alpha} \neq 0$. And if $\tilde{\alpha} = 0$, then B and F become Euclidean spaces if $\tilde{K} \neq 0$, and M becomes the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V manifold a straight line I and $W \times J$ of a $(p-1)$-dimensional complete Riemannian manifold W manifold a straight line J respectively if $\tilde{K} = 0$.
(b) B becomes an Einstein or $\tilde{K} = 0$.
(c) F becomes an Einstein or $\tilde{K} = 0$.

Even if $\tilde{\lambda}$ is a function on $\tilde{M}$, that is $\tilde{M}$ is a generalized gradient almost Ricci-Yamabe soliton, then we have a similar result of Theorem 2.1 except $\tilde{r}$ becomes a constant. Thus we have

Theorem 2.2. *Let $M = B \times F$ be an almost generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$. If $h = k + l$ for some functions l and m on B and F respectively, then $\tilde{K}$ becomes a constant and M becomes one of the following three cases :*
(a) M becomes a gradient Ricci soliton with $(\frac{k+l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$. Moreover, B and F become gradient Ricci solitons with $(\frac{k}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ and $(\frac{l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ respectively if $\tilde{\alpha} \neq 0$. And if $\tilde{\alpha} = 0$, then B and F become Euclidean spaces if $\tilde{K} \neq 0$, and M becomes the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V manifold a straight line I and $W \times J$ of a $(p-1)$-dimensional complete Riemannian manifold W manifold a straight line J respectively if $\tilde{K} = 0$.
(b) B becomes an Einstein or $\tilde{K} = 0$.
(c) F becomes an Einstein or $\tilde{K} = 0$.

In [8], the present authors proved

Theorem 2.3. *Let B and F be gradient Ricci solitons with (h, ρ) and $(\bar{h}, \bar{\rho})$ respectively. Assume that $\rho = \bar{\rho}$ and $r + \bar{r}$ is a constant, then the product space $M = B \times F$ becomes a gradient Ricci-Yamabe soliton with $(g + \bar{g}, \alpha h + \alpha \bar{h}, \frac{\alpha(r+\bar{r})}{n+p}, \alpha, \frac{-2\alpha}{n+p})$.*

So, if we use Theorem 2.3, then we can construct the model space of gradient Ricci-Yamabe solitons on the product space.

Also if we refer to Theorems 2.1 and 2.2, then we can prove the following Theorem using the equations (8) and (10).

Theorem 2.4. *Let B be a gradient Ricci-Yamabe soliton with* $(g, k, \lambda, \alpha, \beta)$ *and F be an Einstein space, that is* $\bar{S} = \mu\bar{g}$ *for a constant* μ. *Assume that r or* $\|k_a\|^2$ *are constant, then the product space* $M = B \times F$ *becomes a generalized gradient Ricci-Yamabe soliton with* $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w}) = (g + \bar{g}, k, \lambda, \alpha, \frac{2(\lambda-\mu\alpha)}{r+\bar{r}}, w)$, *where* $\tilde{w}$ *satisfies*

$$(\lambda - \frac{\beta r}{2} - \mu\alpha)g_{ab} = \tilde{w}k_a k_b. \tag{11}$$

Proof. If we take $(\tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$ as $\tilde{h} = k, \tilde{\lambda} = \lambda, \tilde{\alpha} = \alpha, \tilde{\beta} = \frac{2(\lambda-\mu\alpha)}{(r+\bar{r})}$ and $\tilde{w}$ satisfying the equation (11), then we can calculate the components of $\tilde{\nabla}\tilde{h} + \tilde{\alpha}\tilde{S}$ and $(\tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r})\tilde{g} + \tilde{w}d\tilde{h}d\tilde{h}$ as follows

$$\tilde{\nabla}\tilde{\nabla}\tilde{h} + \tilde{\alpha}\tilde{S} = \begin{pmatrix} \nabla_b k_a + \{(\lambda - \frac{\beta r}{2})g_{ba} - \nabla_b k_a\} & 0 \\ 0 & \tilde{\alpha}\bar{S}_{yx} \end{pmatrix} = \begin{pmatrix} (\lambda - \frac{\beta r}{2})g_{ba} & 0 \\ 0 & \alpha\mu\bar{g}_{yx} \end{pmatrix}, \tag{12}$$

$$(\tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r})\tilde{g} + \tilde{w}d\tilde{h}d\tilde{h} = \begin{pmatrix} (\lambda - \frac{\beta r}{2})g_{ba} & 0 \\ 0 & \alpha\mu\bar{g}_{yx} \end{pmatrix}, \tag{13}$$

where we have put $\bar{S} = \mu\bar{g}$.
Hence we can see that the following equation

$$\tilde{\nabla}\tilde{\nabla}\tilde{h} + \tilde{\alpha}\tilde{S} = (\tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r})\tilde{g} + \tilde{w}d\tilde{h}d\tilde{h} \tag{14}$$

is true on the product space $B \times F$, that is $B \times F$ becomes a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$.

Hence if we use Theorem 2.3, we can construct the model space of generalized Ricci-Yamabe solitons on the product space.

3. Generalized gradient Ricci-Yamabe soliton in the warped product manifold $R^n \times_f F$

Let the warped product manifold $R \times_f F$ be a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$. Then we get

$$\begin{aligned} \tilde{h}_{11} - \frac{\tilde{\alpha}p}{f}f_{11} &= \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r} + \tilde{w}\tilde{h}_1\tilde{h}_1, \\ \partial_1\tilde{h}_x - \frac{f_1}{f}\tilde{h}_x &= \tilde{w}\tilde{h}_1\tilde{h}_x, \\ (\tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r})f^2\bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x &= \bar{\nabla}_y\tilde{h}_x + ff^1\tilde{h}_1\bar{g}_{yx} + \tilde{\alpha}[\bar{S}yx - f(\triangle f)\bar{g}_{yx} - (p-1)\|f_1\|^2\bar{g}_{yx}], \\ \tilde{r} &= \frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{p(p-1)}{f^2}\|f_1\|^2, \end{aligned} \tag{15}$$

where $\bar{\nabla}$ is Levi-Civita connection for $\bar{g}$ on F, and the ranges of the indices x and y are $2, 3, \cdots, p+1$. Moreover $f_1 = \frac{\partial f}{\partial t}, \tilde{h}_1 = \frac{\partial\tilde{h}}{\partial t}, \tilde{h}_x = \frac{\partial\tilde{h}}{\partial\bar{u}_x}$ for the coordinate neighborhoods $\{\tilde{U} = (U, \bar{U}); \tilde{U}^h = (t, \bar{u}^x)\}$ in $R \times F$.
From (15), we have

Theorem 3.1. *If the warped product manifold $M = R \times_f F$ is a proper generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$, then we have the followings :*
(a) If $\tilde{h}_x = 0$ for all $x = 2, 3, ..., p+1$, then F becomes an Einstein, $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ and $\tilde{r}$ depend only on R. So $\bar{r}$ is a constant.
(b) If $\tilde{h}_1 = 0$, then either M is the Riemann product of R and a special generalized gradient Ricci-Yamabe soliton or F is an Einstein,
where p is the dimension of F.

Proof. (a) Assume that $\tilde{h}_x = 0$ for all x that is $\tilde{h}$ depends only on R. Then we see that $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ becomes a function on R from the first equation of (15), so that $\tilde{r} = \tilde{r}(t)$ and $\bar{r}$ become a constant, where t is a parameter of R.

On the other hand, from the third equation of (15), we obtain

$$\tilde{\alpha}\bar{S}yx = [\tilde{K}f^2 - ff^1\tilde{h}_1 + \tilde{\alpha}f(\triangle f) + \tilde{\alpha}(p-1)||f_1||^2]\bar{g}_{yx}. \tag{16}$$

Then the equation (16) is reduced to the form $\tilde{\alpha}\bar{S}_{yx} = \frac{\tilde{\alpha}\bar{r}}{p}\bar{g}_{yx}$, that is F is an Einstein when $\tilde{\alpha} \neq 0$.

(b) If $\tilde{h}_1 = 0$, then see that $\bar{r}$ becomes a constant from the first and fourth equation of (15). From the second equation of (15), we get $f_1\tilde{h}_x = 0$, and that $f_1 = 0$ or $\tilde{h}_x = 0$ because f and $\tilde{h}$ are functions that depend only on B and F respectively. In the first case where $f_1 = 0$, we get $\tilde{K} = 0$ from the first equation of (15). Moreover, we obtain $\tilde{\nabla}_j\tilde{\nabla}_i\tilde{h} + \tilde{\alpha}\tilde{S}_{ji} = \tilde{w}\tilde{h}_j\tilde{h}_i$ and $\bar{\nabla}_y\tilde{h}_x + \tilde{\alpha}\bar{S}_{yx} = \tilde{w}\tilde{h}_y\tilde{h}_x$ using (8) and the third equation of (15). Hence M is the Riemann product of R and a special generalized gradient Ricci-Yamabe soliton and M itself becomes a special generalized gradient Ricci-Yamabe soliton. The manifold is called a special generalized gradient Ricci-Yamabe soliton if the Ricci tensor S satisfies $Ric + Hessh = wdhdh$ for some function h and a constant w. For the second case $\tilde{h}_x = 0$, $\tilde{h}$ becomes a constant. Then we get $\tilde{\alpha}\bar{S}_{yx} = \tilde{\alpha}(f\triangle f + (p-1)||f_1||^2 - pff_{11})\bar{g}_{yx}$ from the first and third equation of (15). Hence F becomes an Einstein.

Theorem 3.2. *Let the warped product manifold $M = R \times_f F$ be a generalized gradient Ricci -Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$ and $\tilde{h} = k + l$ for some functions k and l in R and F respectively. Then $\bar{r}$ is a constant and either F becomes an Einstein or $\tilde{w}k = ln\frac{A}{f}$ for some constant A.*

Proof. Since $h = k + l$, we get

$$\begin{aligned} k_{11} - \frac{\tilde{\alpha}p}{f}f_{11} &= \tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r} + \tilde{w}||k_1||^2, \\ \partial_1 l_x - \frac{f_1}{f}l_x &= \tilde{w}k_1 l_x, \\ (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})f^2\bar{g}_{yx} + \tilde{w}l_y l_x &= \bar{\nabla}_y l_x + ff^1k_1\bar{g}_{yx} + \tilde{\alpha}[\bar{S}yx - f(\triangle f)\bar{g}_{yx} - (p-1)||f_1||^2\bar{g}_{yx}], \\ \tilde{r} &= \frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{p(p-1)}{f^2}||f_1||^2. \end{aligned} \tag{17}$$

From the first equation of (17), we see that $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ and $\tilde{r}$ depend only on R. If we combine these facts and the fourth equation of (17), we get $\partial_x\bar{r} = 0$, that is $\bar{r}$ becomes a constant. Since l is a function on F, $\partial_1 l_x = 0$. Hence we get $l_x(\tilde{w}k_1 + \frac{f_1}{f}) = 0$ and that $l_x = 0$ or $\tilde{w}k_1 + \frac{f_1}{f} = 0$ from the second equation of (17). In the first case $l_x = 0$, that is l is a constant, we obtain

$$(\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})f^2\bar{g}_{yx} = ff^1\tilde{k}_1\bar{g}_{yx} + \tilde{\alpha}[\bar{S}yx - f(\triangle f)\bar{g}_{yx} - (p-1)||f_1||^2\bar{g}_{yx}],$$

equivalently,

$$\bar{S}_{yx} = [\tfrac{1}{\alpha}(\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})f^2 - \tfrac{1}{\tilde{\alpha}}ff^1k_1 + f(\triangle f) + (p-1)||f_1||^1]\bar{g}_{yx} = \tfrac{\bar{r}}{p}\bar{g}_{yx}.$$

Hence F becomes an Einstein. For the second case, we can put $\tilde{w}k + lnf = lnA$ for some constant A, because $\tilde{w}k_1 + \frac{f_1}{f} = \partial_1(\tilde{w}k + lnf)$, that is $\tilde{w}k = ln\frac{A}{f}$.

If we consider the warped product manifold $M = R^n \times_f F$ with a generalized gradient Ricci-Yamabe soliton $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$, then we obtain

$$\begin{aligned} \nabla_b\tilde{h}_a - \frac{\tilde{\alpha}p}{f}\nabla_c f_b &= \tilde{K}g_{ba} + \tilde{w}\tilde{h}_b\tilde{h}_a, \\ \partial_b\tilde{h}_x - \frac{f_b}{f}\tilde{h}_x &= \tilde{w}\tilde{h}_b\tilde{h}_x, \\ \bar{\nabla}_y\tilde{h}_x + \tilde{\alpha}[\bar{S}yx - f(\triangle f)\bar{g}_{yx} - (p-1)||f_e||^2\bar{g}_{yx}] + ff^d h_d\bar{g}_{yx} &= \tilde{K}f^2\bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x, \end{aligned} \tag{18}$$

where we have put $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$.

By the similar method of the proof of Theorem 3.1, we have

Theorem 3.3. *If the warped product manifold $M = R^n \times_f F$ is a proper generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$, then we have the followings :*
(a) If $\tilde{h}_x = 0$ for all $x = n+1, \cdots, n+p$, then F becomes an Einstein, $\tilde{K}$ and $\tilde{r}$ depend only on R^n. So $\bar{r}$ becomes a constant.
(b) If $\tilde{h}_a = 0$ for all $a = 1, 2, \cdots, n$, then either M becomes an Einstein or a product manifold, that is f becomes a constant, where $\tilde{h}_a = \frac{\partial\tilde{h}}{\partial u_a}$, $\tilde{h}_x = \frac{\partial\tilde{h}}{\partial\bar{u}_x}$ for the coordinate neighborhoods $\{\tilde{U} = (U, \bar{U}); \tilde{U}^h = (u^a, \bar{u}^x)\}$ in $R^n \times F$.

Proof. (a) Assume $\tilde{h}_x = 0$ for all x, that is $\tilde{h}$ depends only on R^n. Then we see that $\tilde{K}$ and $\tilde{r}$ become functions on R^n from the first equation of (18). So $\bar{r}$ becomes a constant. By the similar method of the proof of theorem 3.1, we see that F becomes an Einstein.

(b) If $\tilde{h}_a = 0$ for all a, then we get $f_1\tilde{h}_x = 0$ from the second equation of (18). Since f and $\tilde{h}_x$ are functions on R^n and F respectively, we get either $f_1 = 0$ or $\tilde{h}_x = 0$, that is either f is a constant or $\tilde{h}$ is a constant. If $\tilde{h}$ is a constant, we easily see that M becomes an Einstein manifold. Hence we complete the proof.

4. Generalized gradient Ricci-Yamabe soliton in the warped product manifold $B \times_f F$

In this section, we consider the warped product manifold $M = B \times_f F$ of Riemannian manifolds (B, g) and $(F, \bar{g})$. If M is a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$, then we have

$$\begin{aligned} \nabla_b\tilde{h}_a + \tilde{\alpha}(S_{ab} - \frac{p}{f}\nabla_b f_a) &= \tilde{K}g_{ab} + \tilde{w}\tilde{h}_b\tilde{h}_a, \\ \partial_b\tilde{h}_x - \frac{f_b}{f}\tilde{h}_x &= \tilde{w}\tilde{h}_b\tilde{h}_x, \\ \bar{\nabla}_y\tilde{h}_x + ff^c\tilde{h}_c\bar{g}_{yx} + \tilde{\alpha}[\bar{S}yx - f(\triangle f)\bar{g}_{yx} - (p-1)||f_e||^2\bar{g}_{yx}] &= \tilde{K}f^2\bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x, \end{aligned} \tag{19}$$

where $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$, $\tilde{r} = r + \frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{p(p-1)}{f^2}||f_e||^2$, $f_a = \frac{\partial f}{\partial u^a}$, $\tilde{h}_a = \frac{\partial\tilde{h}}{\partial u_a}$, $\tilde{h}_x = \frac{\partial\tilde{h}}{\partial\bar{u}_x}$ for the coordinate neighborhoods $\{\tilde{U} = (U, \bar{U}); \tilde{U}^h = (u^a, \bar{u}^x)\}$ in $B \times F$.

From (19), we have

Theorem 4.1. *If the warped product space $M = B \times_f F$ is a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$, then we have the following :*
(a) If $\tilde{h}_a = 0$ for all $a = 1, 2, \cdots, n$, then M is either a Riemann product of the Einstein space B with $n > 2$ and F or F becomes Einstein.
(b) If $\tilde{h}_x = 0$ for all $x = n+1, \cdots, n+p$, then $\bar{r}$ is a constant and F becomes an Einstein.

Proof. (*a*) Since $\tilde{h}_a = 0$ for all a, we see that $f_b\tilde{h}_x = 0$ from the second equation of (19) and $\tilde{K}$ depends only on B from the first equation of (19). Moreover, we can easily see that $\bar{r}$ is a constant. The relation $f_b\tilde{h}_x = 0$ means that f is a constant or h is a constant. If $\tilde{h}$ is a constant, we can reduce the relation $\bar{S} = A\bar{g}$ from the third equation of (19) and A depends only on the base space B. Hence A is a constant on F and F becomes an Einstein. If f is a constant, we get $\tilde{\alpha}S_{ab} = \tilde{K}g_{ab}$ and $\bar{\nabla}_y\tilde{h}_x + \tilde{\alpha}\bar{S}_{yx} = \tilde{K}f^2\bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x$, that is B becomes an Einstein space if $n > 2$. Moreover, r becomes a constant because $\tilde{K}$ is a constant if $n > 2$, and F becomes a generalized gradient Ricci-Yamabe soliton with $(\bar{g}, \tilde{h}, f^2\tilde{\lambda} - \frac{\tilde{\beta}}{2}f^2 r, \tilde{\alpha}, \frac{\tilde{\beta}}{2})$. Hence M becomes the Riemannian product of the Einstein space and a generalized gradient Ricci-Yamabe soliton.
(b) If $\tilde{h}_x = 0$, then we see that $\tilde{K}$ depends only on the base space B from the first equation of (19) and that $\bar{r}$ becomes a constant. From this fact and the third equation of (19), we see that F becomes an Einstein manifold.

If the potential function $\tilde{h} = k + l$ for some functions k and l on B and F respectively, then we see that $\bar{r}$ is a constant from (19) and that $\tilde{K}$ depends only on B. Moreover we get $(\tilde{w}k_b + \frac{f_b}{f})l_x = 0$ from the second equation of (19). Hence $\tilde{w}k_b + \frac{f_b}{f} = 0$ or $l_x = 0$. If $\tilde{w}k_b + \frac{f_b}{f} = 0$, then we get $\partial_b(\tilde{w}k + lnf) = 0$. Since $\tilde{w}k + lnf$ depends only on B, we can put $\tilde{w}k + lnf = lnA$ for some constant A, that is $\tilde{w}k = ln\frac{A}{f}$.

If l is a constant, then we obtain

$$\begin{aligned} ff^c\tilde{k}_c\bar{g}_{yx} + \tilde{\alpha}[\bar{S}_{yx} - f(\triangle f)\bar{g}_{yx} - (p-1)\|f_e\|^2\bar{g}_{yx}] &= \tilde{K}f^2\bar{g}_{yx}, \\ pff^c\tilde{k}_c + \tilde{\alpha}[\bar{r} - pf(\triangle f) - p(p-1)\|f_e\|^2] &= p\tilde{K}f^2 \end{aligned} \tag{20}$$

from the third equation of (19), and we see that $\tilde{K}$ depends only on B from the first equation of (19). Moreover, $\bar{r}$ is a constant due to the second equation of (20). From these facts and the first equation of (19), we see that F becomes an Einstein. Thus we have

Theorem 4.2. *Let the warped product space $M = B \times_f F$ be a gradient Ricci-Yamabe soliton with $(\bar{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ and $\tilde{h} = k + l$ for some functions k and l in R and F respectively, then $\bar{r}$ is a constant and either F becomes an Einstein or $\tilde{w}k = ln\frac{A}{f}$ for some constant A.*

5. Generalized gradient Ricci-Yamabe soliton in the twisted product manifold $B \times_f F$

In [1], the authors proved that

Theorem 5.1. *If the twisted product manifold $M = B \times_f F$ of the Riemannian manifolds (B, g) and $(F, \bar{g})$ are conformally flat and $p \neq 1$, $n \neq 1$, then M is the warped product space $B \times_{f^*} F^*$ of B and F^*, where F^* is F with the metric $g^* = (\bar{f})^2\bar{g}$.*

In the process of proving Theorem 5.1, the authors knew that the warping function f can be expressed as a product of two functions f^* and $\bar{f}$ in B and F respectively, that is $f = f^*\bar{f}$. Now, let us study the geometric characterization of the conformally flat twisted product manifold with a generalized gradient Ricci-Yamabe soliton. If the twisted product manifold $M = B \times_f F$ is a generalized gradient Ricci-Yamabe soliton with $(\bar{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$. then we get

$$\begin{aligned} \nabla_b\tilde{h}_a + \tilde{\alpha}(S_{ab} - \tfrac{p}{f}\nabla_b f_a) &= \tilde{K}g_{ab} + \tilde{w}\tilde{h}_b\tilde{h}_a, \\ \partial_b\tilde{h}_x - \tfrac{f_b}{f}\tilde{h}_x - \tilde{\alpha}(p-1)(\tfrac{1}{f}\partial_b f_x - \tfrac{1}{f^2}f_b f_x) &= \tilde{w}\tilde{h}_b\tilde{h}_x, \\ \bar{\nabla}_y\tilde{h}_x - \tfrac{1}{f}(f_y\tilde{h}_x + f_x\tilde{h}_y - f^z\tilde{h}_z\bar{g}_{yx}) + ff^c\tilde{h}_c\bar{g}_{yx} + \tilde{\alpha}[\bar{S}_{yx} - f(\triangle f)\bar{g}_{yx} - (p-1)\|f_e\|^2\bar{g}_{yx} & \\ -\tfrac{\bar{\triangle}f}{f}\bar{g}_{yx} - \tfrac{(p-2)}{f}\bar{\nabla}_y f_x + \tfrac{2(p-2)}{f^2}f_y f_x - \tfrac{(p-3)}{f^2}\|f_w\|^2\bar{g}_{yx}] &= \tilde{K}f^2\bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x, \end{aligned} \tag{21}$$

where $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ and $\tilde{r} = r + \frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}||f_e||^2 - \frac{(p-1)(p-4)}{f^4}||f_x||^2$.

If the potential function $\tilde{h} = k + l$ for some functions k and l on B and F respectively and M is conformally flat, then the warping function f is expressed by $f = f^*\bar{f}$ according to the proof process of Theorem 5.1 and the first equation of (21) becomes

$$\nabla_b k_a + \tilde{\alpha}(S_{ab} - \frac{p}{f^*}\nabla_b f^*_a) = \tilde{K} g_{ba} + \tilde{w} k_b k_a. \quad (22)$$

Hence we see that $\tilde{K}$ depends only on B. Since $f = f^*\bar{f}$, we get $\frac{1}{f}\partial_b f_x - \frac{1}{f^2} f_b f_x = 0$. Hence from the second equation of (21), we get $(\frac{f^*_b}{f^*} + \tilde{w}k_b)l_x = 0$, that is either $\frac{f^*_b}{f^*} + \tilde{w}k_b = 0$ or $l_x = 0$.

In the first case, we get $\partial_b(lnf^* + \tilde{w}k) = \frac{f^*_b}{f^*} + \tilde{w}k_b = 0$. Thus we can put $lnf^* + \tilde{w}k = lnA$ for some constant A, that is $\tilde{w}k = ln\frac{A}{f^*}$. Since $\partial_a(lnf^*) = \frac{f^*_a}{f^*}$, we get

$$\nabla_b \nabla_a lnf^* = \frac{\nabla_b f^*_a}{f^*} - \frac{f^*_a f^*_b}{(f^*)^2}. \quad (23)$$

Since $\tilde{K}$ depends on B, we see that

$$\frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}||f_e||^2 - \frac{(p-1)(p-4)}{f^4}||f_x||^2 \quad (24)$$

depends on B. From these facts and the equations (22),(23), and (24), we obtain

$$\begin{aligned} \nabla_b\nabla_a(k - p\tilde{\alpha}lnf^*) + \tilde{\alpha}S_{ba} = [\{\tilde{\lambda} - \frac{\tilde{\beta}}{2}(\frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}||f_e||^2 - \frac{(p-1)(p-4)}{f^4}||f_x||^2)\} \\ -\frac{1}{2}\tilde{\beta}r]g_{ab} + \tilde{w}k_b k_a + \frac{p\tilde{\alpha}f^*_a f^*_b}{(f^*)^2}. \end{aligned} \quad (25)$$

If we take $h = k - p\tilde{\alpha}lnf^*$ and $w = \frac{1}{\tilde{w}} + p\tilde{\alpha}$, then $h_a h_b = (\frac{1}{\tilde{w}} + p\tilde{\alpha})^2\frac{f^*_a f^*_b}{(f^*)^2}$, and the equation (25) can be rewritten as

$$\begin{aligned} \nabla_b\nabla_a h + \tilde{\alpha}S_{ba} = [\{\tilde{\lambda} - \frac{\tilde{\beta}}{2}(\frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}||f_e||^2 \\ -\frac{(p-1)(p-4)}{f^4}||f_x||^2)\} - \frac{1}{2}\tilde{\beta}r]g_{ab} + wh_b h_a. \end{aligned} \quad (26)$$

Therefore B admits a generalized gradient almost Ricci-Yamabe soliton $(g, h = k - p\tilde{\alpha}lnf^*, \lambda = \tilde{\lambda} - \frac{\tilde{\beta}}{2}(\frac{\bar{r}}{f^2} - \frac{2p(\triangle f)}{f} - \frac{2(p-1)}{f^3}(\bar{\triangle} f) - \frac{p(p-1)}{f^2}||f_e||^2 - \frac{(p-1)(p-4)}{f^4}||f_x||^2), \alpha = \tilde{\alpha}, \beta = \tilde{\beta}, w = \frac{1}{\tilde{w}} + p\tilde{\alpha})$ due to the fact λ is a function on B.

In the second case, that is $l_x = 0$, the third equation of (21) becomes

$$\begin{aligned} \tilde{\alpha}\bar{S}_{yx} - \tilde{\alpha}(\frac{(p-2)}{f}\bar{\nabla}_y f_x - \frac{2(p-2)}{f^2}f_y f_x) = [-ff^c k_c + \tilde{\alpha}\{f(\triangle f) + (p-1)||f_e||^2 \\ + \frac{\bar{\triangle} f}{f} + \frac{(p-3)}{f^2}||f_w||^2\} + \tilde{K}f^2]\bar{g}_{yx}. \end{aligned} \quad (27)$$

Since

$$\frac{\bar{\nabla}_y f_x}{f} = \frac{\bar{\nabla}_y \bar{f}_x}{\bar{f}} = \bar{\nabla}_y\bar{\nabla}_x lnf + \frac{\bar{f}_x\bar{f}_y}{\bar{f}^2}, \quad (28)$$

we get

$$\begin{aligned}\tilde{\alpha}\bar{S}_{yx} - \tilde{\alpha}(p-2)\bar{\nabla}_y\bar{\nabla}_x(lnf) = ([\tilde{\lambda}f^2 - \tfrac{\tilde{\beta}f^2}{2}\{r - \tfrac{2p(\triangle f)}{f} - \tfrac{2(p-1)}{f^3}(\bar{\triangle}f) - \tfrac{p(p-1)}{f^2}\|f_e\|^2 \\ -\tfrac{(p-1)(p-4)}{f^4}\|f_x\|^2\} - ff^ck_c + \tilde{\alpha}\{f(\triangle f) + (p-1)\|f_e\|^2 \\ +\tfrac{\bar{\triangle}f}{f} + \tfrac{(p-3)}{f^2}\|f_w\|^2\}] - \tfrac{1}{2}\tilde{\beta}\bar{r})\bar{g}_{yx} + \tilde{\alpha}\tfrac{(p-2)}{f^2}\bar{f}_y\bar{f}_x,\end{aligned} \tag{29}$$

that is F admits a generalized gradient almost Ricci-Yamabe soliton with
$(\bar{g}, \bar{h} = -\tilde{\alpha}(p-2)lnf, \bar{\lambda} = \tilde{\lambda}f^2 - \frac{\tilde{\beta}f^2}{2}\{r - \frac{2p(\triangle f)}{f} - \frac{2(p-1)}{f^3}(\bar{\triangle}f) - \frac{p(p-1)}{f^2}\|f_e\|^2$
$- \frac{(p-1)(p-4)}{f^4}\|f_x\|^2\} - ff^ck_c + \tilde{\alpha}\{f(\triangle f) + (p-1)\|f_e\|^2 + \frac{\bar{\triangle}f}{f} + \frac{(p-3)}{f^2}\|f_w\|^2\}, \bar{\alpha} = \tilde{\alpha}, \bar{\beta} = \tilde{\beta}, \bar{\varpi} = (p-2)\tilde{\alpha})$ and we can see that $\bar{\lambda}$ is a function on F from the equations (28) and (29).

Theorem 5.2. *Let the twisted product manifold $M = B \times_f F$ be a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$ and conformally flat. If $\tilde{h} = k + l$ for some functions k and l on B and F respectively, then the quantity $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ depends only on B and either B admits a generalized gradient almost Ricci-Yamabe soliton $(g, h, \lambda, \alpha, \beta, \varpi)$ or F admit a generalized gradient almost Ricci-Yamabe soliton $(\bar{g}, \bar{h}, \bar{\lambda}, \bar{\alpha}, \bar{\beta}, \bar{\varpi})$.*

6. Almost gradient Ricci-Yamabe soliton in the product manifold

Let M be the product manifold $B \times F$ of the n-dimensional Riemannian manifold (B, g) and p-dimensional Riemannian manifold $(F, \bar{g})$. If $M = B \times F$ be a generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{\varpi})$, then we obtain

$$\begin{aligned} \nabla_b\tilde{h}_a + \tilde{\alpha}S_{ba} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})g_{ba}, \\ \partial_b\tilde{h}_x &= 0, \\ \bar{\nabla}_y\tilde{h}_x + \tilde{\alpha}\bar{S}_{yx} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})\bar{g}_{yx}, \\ \tilde{r} &= r + \bar{r}, \end{aligned} \tag{30}$$

where ∇ and $\bar{\nabla}$ are Levi-Civita connections for g and $\bar{g}$ respectively and S and $\bar{S}$ are Ricci curvatures on B and F respectively. The ranges of the indices $a, b, ...,$ and $x, y, ...$ are $1, 2, ..., n,$ and $n+1, n+2, ..., n+p$, respectively.

From the second equation of (30), we see that the potential function $\tilde{h}$ is represented by $\tilde{h} = k + l$ for some function k on B and l on F respectively. Hence the equation of (30) can be rewritten as

$$\begin{aligned} \nabla_b k_a + \tilde{\alpha}S_{ba} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})g_{ba}, \\ \partial_b l_x &= 0, \\ \bar{\nabla}_y l_x + \tilde{\alpha}\bar{S}_{yx} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})\bar{g}_{yx}. \end{aligned} \tag{31}$$

Then we see that $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ becomes constant from the first and third equations of (31). From the first and third equations of (31), B and F become gradient Ricci soliton with $(\frac{k}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ and $(\frac{l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ respectively if $\tilde{\alpha} \neq 0$. If $\tilde{\alpha} = 0$, then the first and third equations of (31) become $\nabla_b k_a = \tilde{K}g_{ba}$ and $\bar{\nabla}_y l_x = \tilde{K}\bar{g}_{yx}$. Hence if we consider this equation and Theorem 1.1, then B and F become Euclidean spaces, when $\tilde{K} \neq 0$, and the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V with a straight line I when $\tilde{K} = 0$. Thus we have

Theorem 6.1. *Let $M = B \times F$ be an almost gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta})$. then we have*
(a) *$\tilde{K}$ becomes a constant.*
(b) *$\tilde{h} = k + l$ for some functions k on B and l on F respectively,*
(c) *$\tilde{M}$ becomes a gradient Ricci-soliton with $(\frac{k+l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$.*
(d) *B and F become gradient Ricci-Yamabe solitons with $(g, k, \lambda = \tilde{\lambda} - \frac{\beta}{2}\bar{r}, \tilde{\alpha}, \tilde{\beta})$ and $(\bar{g}, l, \bar{\lambda} = \tilde{\lambda} - \frac{\beta}{2}r, \tilde{\alpha}, \tilde{\beta})$ respectively.*
(e) *B and F become gradient Ricci-solitons with $(\frac{k}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ and $(\frac{l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ respectively if $\tilde{\alpha} \neq 0$.*
(f) *If $\tilde{\alpha} = 0$, then B and F become either Euclidean spaces if $\tilde{K} \neq 0$ or $B = V \times I$ of an $(n-1)$-dim. complete Riemannian manifold V and a straight-line I, and $F = W \times J$ of a $(p-1)$-dim. complete Riemannian manifold W and a straight-line J if $\tilde{K} = 0$.*

From Theorem 6.1, we get

Corollary 6.2. *For the assumptions of the Theorem 6.1 and r is a constant, then B and F become a gradient Ricci-Yamabe soliton with $(g, k, \tilde{\lambda} - \frac{\tilde{\beta}}{2}\bar{r}, \tilde{\alpha}, \tilde{\beta})$ and an almost gradient Ricci-Yamabe soliton with $(\bar{g}, l, \tilde{\lambda} - \frac{\tilde{\beta}}{2}r, \tilde{\alpha}, \tilde{\beta})$, respectively.*

Let us consider the converse of Theorem 6.1. Assume that B and F are gradient Ricci solitons with (k, ρ) and $(l, \bar{\rho})$ respectively, that is

$$S_{ab} + \nabla_a \nabla_b k = \rho g, \bar{S}_{yx} + \bar{\nabla}_y \bar{\nabla}_x l = \bar{\rho}\bar{g}. \tag{32}$$

Take $\tilde{h} = \tilde{\alpha}(k + l)$ and $\rho = \bar{\rho}$, then $\tilde{\nabla}_i \tilde{\nabla}_j \tilde{h} + \tilde{\alpha}\tilde{S} = \tilde{\alpha}(\tilde{\nabla}_i \tilde{\nabla}_j (k + l) + \tilde{S}) = \tilde{\alpha}\rho\tilde{g}$. Since $r + \triangle k = n\rho$ and $\bar{r} + \bar{\triangle} l = p\rho$, we obtain $r + \bar{r} + \triangle k + \bar{\triangle} l = (n + p)\rho$, that is $\tilde{r} + \triangle k + \bar{\triangle} l = (n + p)\rho$. Then we see that

$$\tilde{\alpha}\rho\tilde{g} = \tilde{\alpha}\frac{\triangle k + \bar{\triangle} l + \tilde{r}}{n+p}\tilde{g} = (\frac{\tilde{\alpha}(\triangle k + \bar{\triangle} l)}{n+p} - \frac{1}{2}(-\frac{2\tilde{\alpha}}{n+p})\tilde{r})(g + \bar{g}).$$

Since $\frac{\tilde{\alpha}(\triangle k + \bar{\triangle} l)}{n+p}$ is a function on $B \times F$, $B \times F$ become an almost gradient Ricci-Yamabe soliton with $(g + \bar{g}, \tilde{\alpha}(k + l), \frac{\tilde{\alpha}(\triangle k + \bar{\triangle} l)}{n+p}, \tilde{\alpha}, \frac{-2\tilde{\alpha}}{n+p})$. Thus we have

Theorem 6.3. *Let B be the gradient Ricci soliton with (k, ρ) and F be the gradient Ricci soliton with $(l, \bar{\rho})$. If we take $\tilde{h} = \tilde{\alpha}(k+l)$ and $\rho = \bar{\rho}$, then $B \times F$ becomes an almost gradient Ricci -Yamabe soliton with $(g+\bar{g}, \tilde{\alpha}(k+l), \frac{\tilde{\alpha}(\triangle k + \bar{\triangle} l)}{n+p}, \tilde{\alpha}, \frac{-2\tilde{\alpha}}{n+p})$.*

Hence if we use Theorem 6.3, then we can construct a model space of an almost gradient Ricci-Yamabe soliton.

7. Almost generalized gradient Ricci-Yamabe soliton in the product manifold

Let M be the product manifold $B \times F$ of the n-dimensional Riemannian manifold (B, g) and p-dimensional Riemannian manifold $(F, \bar{g})$. If $M = B \times F$ be an almost generalized gradient Ricci-Yamabe soliton *with*$(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$, that is λ is a function on M with generalized gradient Ricci-Yamabe soliton, then we obtain

$$\begin{aligned} \nabla_b \tilde{h}_a + \tilde{\alpha} S_{ba} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})g_{ba} + \tilde{w}\tilde{h}_b\tilde{h}_a, \\ \partial_b \tilde{h}_x &= \tilde{w}\tilde{h}_b\tilde{h}_x, \\ \bar{\nabla}_y \tilde{h}_x + \tilde{\alpha}\bar{S}_{yx} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r})\bar{g}_{yx} + \tilde{w}\tilde{h}_y\tilde{h}_x, \\ \tilde{r} &= r + \bar{r}, \end{aligned} \tag{33}$$

where ∇ and $\bar{\nabla}$ are Levi-Civita connections for g and $\bar{g}$ respectively and S and $\bar{S}$ are Ricci curvatures on B and F respectively. The ranges of the indices $a, b, ...,$ and $x, y, ...$ are $1, 2, ..., n,$ and $n+1, n+2, ..., n+p$, respectively.

Assume that the potential function $\tilde{h}$ is represented by $\tilde{h} = l + m$ for some functions l on B and m on F respectively. Then, from (33), we get

$$\begin{aligned} \nabla_b k_a + \tilde{\alpha} S_{ba} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r}) g_{ba} + \tilde{w} k_b k_a, \\ 0 &= \tilde{w} k_b l_x, \\ \bar{\nabla}_y l_x + \tilde{\alpha} \bar{S}_{yx} &= (\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\tilde{r}) \bar{g}_{yx} + \tilde{w} l_y l_x, . \end{aligned} \tag{34}$$

Then we see that $\tilde{K} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r}$ becomes a constant from the first and third equations of (34). From the second equation of (34), we see that $\tilde{w} = 0$ or $k_a = 0$ or $l_x = 0$.

In the first case where $\tilde{w} = 0$, the first and third equations of (34) become

$$\begin{aligned} \nabla_b k_a + \tilde{\alpha} S_{ba} &= \tilde{K} g_{ba} = ((\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta}\bar{r}) - \tfrac{1}{2}\tilde{\beta} r) g_{ba}, \\ \bar{\nabla}_y l_x + \tilde{\alpha} \bar{S}_{yx} &= \tilde{K} \bar{g}_{yx} = ((\tilde{\lambda} - \tfrac{1}{2}\tilde{\beta} r) - \tfrac{1}{2}\tilde{\beta}\bar{r}) \bar{g}_{yx}. \end{aligned} \tag{35}$$

Since $\tilde{K}$ is a constant, $\tilde{\lambda}$ is a function on M, and $\tilde{r} = r + \bar{r}$, the quantities $\tilde{\lambda} - \frac{1}{2}\tilde{\beta}\bar{r}$ and $\tilde{\lambda} - \frac{1}{2}\tilde{\beta} r$ are functions on B and F respectively. Hence B and F become almost gradient Ricci-Yamabe solitons with $(g, k, \lambda = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\bar{r}, \tilde{\alpha}, \tilde{\beta})$ and $(\bar{g}, l, \bar{\lambda} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta} r, \tilde{\alpha}, \tilde{\beta})$ respectively. On the other hand, it could be said that B and F become gradient Ricci-Yamabe solitons with $(\frac{k}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ and $(\frac{l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ respectively if $\tilde{\alpha} \neq 0$ from the equation (34) and the fact that $\tilde{K}$ is a constant. If $\tilde{\alpha} = 0$, then B and F become Euclidean spaces when $\tilde{K} \neq 0$, and the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V with a straight line I if $\tilde{K} = 0$.

For the second case where $k_a = 0$, that is k is a constant, then the base space B and F becomes Einstein if $\tilde{\alpha} \neq 0$. If $\tilde{\alpha} = 0$, then $\tilde{K} = 0$. Finally, let us consider the third case $l_x = 0$, that is l is a constant. If we think about it in the same way as in the second case, we see that F becomes an Einstein if $\tilde{\alpha} \neq 0$. If $\tilde{\alpha} = 0$, then $\tilde{K} = 0$. Thus we have

Theorem 7.1. *Let $M = B \times F$ be an almost generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$. If $h = k + l$ for some functions l and m on B and F respectively, then $\tilde{K}$ is a constant, and M becomes one of the following three cases :*
(a) M becomes a gradient Ricci soliton with $(\frac{k+l}{\tilde{\alpha}}, \frac{\tilde{K}}{\tilde{\alpha}})$ if $\tilde{\alpha} \neq 0$. Moreover B and F become almost gradient Ricci-Yamabe solitons with $(g, k, \lambda = \tilde{\lambda} - \frac{1}{2}\tilde{\beta}\bar{r}, \tilde{\alpha}, \tilde{\beta})$ and $(\bar{g}, l, \bar{\lambda} = \tilde{\lambda} - \frac{1}{2}\tilde{\beta} r, \tilde{\alpha}, \tilde{\beta})$ respectively if $\tilde{\alpha} \neq 0$. And if $\tilde{\alpha} = 0$, then B and F become Euclidean spaces if $\tilde{K} \neq 0$, and the direct product $V \times I$ of an $(n-1)$-dimensional complete Riemannian manifold V manifold a straight line I and $W \times J$ of a $(p-1)$-dimensional complete Riemannian manifold W manifold a straight line J respectively if $\tilde{K} = 0$.
(b) B becomes an Einstein or $\tilde{K} = 0$.
(c) F becomes an Einstein or $\tilde{K} = 0$.

Assume that B is an almost gradient Ricci-Yamabe soliton with $(g, k, \lambda, \alpha, \beta)$ and F is an Einstein space, where we choose λ satisfying the relation $\lambda = \frac{\beta r}{2} + \frac{p\beta + 2\alpha}{2p}\bar{r}$ and $||k_a||^2$ is a constant. Since $\bar{r}$ is a constant, λ is a function on B. Then we see that

$$\begin{aligned} \nabla_b k_a + \alpha S_{ba} &= (\lambda - \tfrac{1}{2}\beta r) g_{ba}, \\ \bar{S}_{yx} &= \tfrac{\bar{r}}{p} \bar{g}_{yx}. \end{aligned} \tag{36}$$

If we take $\tilde{g} = g + \bar{g}$, $\tilde{h} = k$, $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$, $\tilde{\lambda} = \lambda$, and take $\tilde{w}$ such that $\frac{\beta\bar{r}}{2} g_{ba} = \tilde{w} k_b k_a$, then we can calculate

$$\tilde{\nabla}_j\tilde{\nabla}_i\tilde{h} + \tilde{\alpha}\tilde{S} = \begin{pmatrix} \nabla_b k_a + \alpha S_{ba} & 0 \\ 0 & \alpha\bar{S}_{yx} \end{pmatrix} = \begin{pmatrix} (\frac{(p\beta+2\alpha)}{2p}\bar{r}g_{ba} & 0 \\ 0 & \frac{\alpha\bar{r}}{p}\bar{g}_{yx} \end{pmatrix} \tag{37}$$

$$(\tilde{\lambda} - \frac{1}{2}\tilde{\beta}\tilde{r})\tilde{g}_{ij} + \tilde{w}\tilde{h}_j\tilde{h}_i = \begin{pmatrix} (\lambda - \frac{\beta(r+\bar{r})}{2})g_{ab} + \tilde{w}k_a k_b & 0 \\ 0 & (\lambda - \frac{\beta(r+\bar{r})}{2})\bar{g}_{yx} \end{pmatrix} = \begin{pmatrix} (\frac{(p\beta+2\alpha)}{2p}\bar{r}g_{ba} & 0 \\ 0 & \frac{\alpha\bar{r}}{p}\bar{g}_{yx} \end{pmatrix}. \tag{38}$$

Hence the product space $B \times F$ becomes an almost generalized gradient Ricci-Yamabe soliton with $(\tilde{g}, \tilde{h}, \tilde{\lambda}, \tilde{\alpha}, \tilde{\beta}, \tilde{w})$. Thus we have

Theorem 7.2. *Let B be an almost generalized gradient Ricci-Yamabe soliton with $(g, k, \lambda, \alpha, \beta)$ and F be an Einstein space. If we take $\tilde{h} = k$, $\tilde{\lambda} = \lambda = \frac{\beta r}{2} + \frac{p\beta+2\alpha}{2p}\bar{r}$, $\tilde{\alpha} = \alpha$, $\tilde{\beta} = \beta$ and choose $\tilde{w}$ such that $\frac{\beta(r+\bar{r})}{2}g_{ba} = \tilde{w}k_b k_a$. Then $B \times F$ becomes an almost generalized gradient Ricci-Yamabe soliton with $(g + \bar{g}, h, \frac{\beta r}{2} + \frac{p\beta+2\alpha}{2p}\bar{r}, \alpha, \beta, \tilde{w})$.*

Hence if we use Theorem 7.2, then we can construct a model space of an almost generalized gradient Ricci-Yamabe soliton.

References

[1] Y. Agaoka and B.H.Kim, *On conformally flat twisted product manifolds*, Memoirs of the Faculty of Integrated Arts and Sciences, Hiroshima University, Ser IV, **23** (1997) 1-7.

[2] U.C. De, A. Sardar and K. De, *Ricci-Yamabe solitons and 3-dimensional Riemannian manifolds*, Turkish Journal of Mathematics, **46** (2022), 1078-1088.

[3] D. Dey, *Almost Kenmotsu metric as Ricci-Yambe soliton*, arXiv : 2005, 02322v1[math.DG], 5 May 2020.

[4] D. Dey and P. Majhi, *Sasakian 3-metric as a generalized Ricci-Yamabe soliton*, Quaestiones Mathematicae, (2021), 1-13.

[5] S. Guler and M. Crasmareanu, Ricci-Yamabe maps for Riemannian flows and their volume variation and volume entropy, Turkish Journal of Mathematics, **43** (2019),2361-2641.

[6] R. S. Hamilton, *The Ricci flow on surfaces*, Mathematics and general relativity (Santa Cruz CA, 1986), 237-262, Contemporary Mathematics, 71, American Mathematical Society, 1988.

[7] M. Khatri, C. Zosangzuala, and J. P. Singh, *Isometries on almost Ricci-Yamabe solitons*, Arabian Journal of Mathematics, **12** (2023), 127-138.

[8] B. H. Kim, J. H. Choi, S. D. Lee, and C. Y. Han, *Gradient Ricci-Yamabe soliton on twisted product manifolds*, Journal of Mathematics, (2022), 1-6.

[9] M.F. Lopez and E.G. Rio, *A note on locally conformally flat gradient Ricci solitons*, Geometriae Dedicata, **168** (2014), 1-7.

[10] T. Seco and S. Maeta, *Classification of almost Yamabe solitons in Euclidean spaces*, Journal of Geometry and Physics, **136** (2019), 97-103.

[11] Y. Tashiro, *Complete Riemannian manifolds and some vector fields*, Transactions of the American Mathematical Society, **117** (1965) 251-275.

[12] W. Tokura, L. Adriano, R. Pina, and M. Barboza, *On warped product gradient Yamabe solitons*, Journal of Mathematical Analysis and Applications **473** (2019) 201-217.