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k-type hyperbolic framed slant helices in hyperbolic 3-space

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Abstract. In this paper, we give the existence and uniqueness theorems for hyperbolic framed curves and define the k-type hyperbolic framed slant helices in three-dimensional hyperbolic space. Using the hyperbolic curvature, we investigate the k-type hyperbolic framed slant helices and the connection between them. Hyperbolic framed slant helices might have singular points, they are a generalization of hyperbolic slant helices. Moreover, as their applications, we give some examples of k-type hyperbolic framed slant helices.

1. Introduction

Special curves, such as helices, Bertrand curves and Mannheim curves, have captivated the attention of researchers due to their intriguing properties and have a wide range of applications in the fields of physics and engineering. Among them, the study of helices is particularly important and extensive. A helix in Euclidean 3-space is a regular curve whose tangent vector forms a constant angle with a fixed line at any point. And such a mesmerizing phenomenon finds profound applications in the realms of biology and physics. For example, in molecular biology, we can regard the double helix structure of DNA as a typical example [20].

The properties of curves in Euclidean 3-space or Minkowski 3-space are generally characterized by the algebraic equations concerning their curvature and torsion functions. This holds true for the study of helices as well. In 1802, Lancret firstly proved that a curve is a helix if and only if the ratio of its curvature and torsion is constant. In 1997, Barros extended the definition of helices to 3-dimensional real-space-form and generalized the theorem of Lancret [2]. Subsequently, an extensive body of research unfolded, delving into the intricate properties of helices in Euclidean space and other spaces [3, 10, 24]. By applying a transformation to the moving frame of a helix, a new curve emerges – slant helix, whose principal normal vector makes a constant angle with a fixed line [1, 14, 17, 27]. Further, the authors studied k-type slant helices in different spaces [5, 22, 25, 26]. Similar to the investigation of curves in other spaces, the necessary and sufficient conditions for a hyperbolic curve in hyperbolic 3-space to be a k-type slant helix have also been established, defining these conditions in relation to its hyperbolic curvature functions [25].

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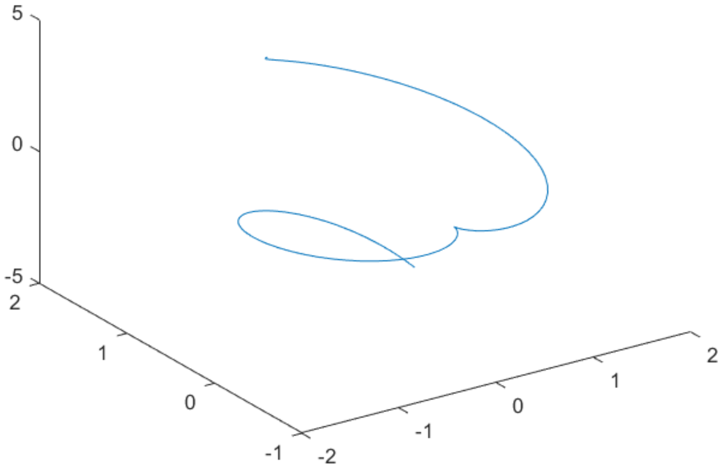

Figure 1: A slant helix has a singular point.

On the other hand, Honda and Takahashi investigated framed curves in Euclidean space [7]. This is an excellent promotion of Frenet curves and Legendre curves. The theory of framed curves provides more possibilities for the study of singularities, inspiring many researchers to apply this idea to study geometric objects in diverse spaces [6, 11–13, 18, 23, 28]. In [21], the authors gave an example of a slant helix containing a singularity in Euclidean space (Figure 1). In hyperbolic 3-space, regular geometric objects have been studied in many papers [8, 9, 15, 16, 19]. Using the theory of framed curves as a tool, we can also generalize regular curves in hyperbolic 3-space to curves which may have singularities.

In the present paper, we give the existence and uniqueness theorems for hyperbolic framed curves and define the k-type hyperbolic framed slant helices in three-dimensional hyperbolic space. In Section 2, we review the basic notions and concepts of hyperbolic framed curve. In Section 3, we prove that a hyperbolic framed curve is uniquely determined by hyperbolic curvature through a Lorentz motion in hyperbolic 3-space. In Section 4, we investigate the k-type hyperbolic framed slant helices and the connection between them using the hyperbolic curvature. Hyperbolic framed slant helices might have singular points, they are a generalization of hyperbolic slant helices. As the applications of the results of Section 4, we give some examples of k-type hyperbolic framed slant helices in Section 5.

All maps and manifolds considered here are of class C^∞ unless otherwise stated.

2. Preliminaries

Let $\mathbb{R}^4$ be the 4-dimensional real vector space. For any vectors $\boldsymbol{x} = (x_0, x_1, x_2, x_3)$, $\boldsymbol{y} = (y_0, y_1, y_2, y_3)$ in $\mathbb{R}^4$, the pseudo scalar product of $\boldsymbol{x}$ and $\boldsymbol{y}$ is defined by

$$\langle \boldsymbol{x}, \boldsymbol{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3,$$

we say that $\boldsymbol{x}$ and $\boldsymbol{y}$ is pseudo-orthogonal if $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0$, then the space $\mathbb{R}^4_1 = (\mathbb{R}^4, \langle, \rangle)$ is callled the *Minkowski 4-space*. We say that a non-zero vector $\boldsymbol{x} \in \mathbb{R}^4_1$ is *spacelike*, *lightlike* or *timelike* if $\langle \boldsymbol{x}, \boldsymbol{x} \rangle > 0$, $\langle \boldsymbol{x}, \boldsymbol{x} \rangle = 0$ or $\langle \boldsymbol{x}, \boldsymbol{x} \rangle < 0$, respectively. The norm of vector $\boldsymbol{x}$ is given by $\|\boldsymbol{x}\| = \sqrt{|\langle \boldsymbol{x}, \boldsymbol{x} \rangle|}$. For any vectors $\boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3} \in \mathbb{R}^4_1$, we define a vector $\boldsymbol{x_1} \wedge \boldsymbol{x_2} \wedge \boldsymbol{x_3}$ by

$$\boldsymbol{x_1} \wedge \boldsymbol{x_2} \wedge \boldsymbol{x_3} = \begin{vmatrix} -\boldsymbol{e}_0 & \boldsymbol{e}_1 & \boldsymbol{e}_2 & \boldsymbol{e}_3 \\ x_0^1 & x_1^1 & x_2^1 & x_3^1 \\ x_0^2 & x_1^2 & x_2^2 & x_3^2 \\ x_0^3 & x_1^3 & x_2^3 & x_3^3 \end{vmatrix},$$

where $\boldsymbol{e}_0, \boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$ are the canonical basis of $\mathbb{R}^4_1$, $\boldsymbol{x}_i = (x_0^i, x_1^i, x_2^i, x_3^i)$.

$$\langle \boldsymbol{x}, \boldsymbol{x_1} \wedge \boldsymbol{x_2} \wedge \boldsymbol{x_3} \rangle = \det(\boldsymbol{x}, \boldsymbol{x_1}, \boldsymbol{x_2}, \boldsymbol{x_3}),$$

so that $\boldsymbol{x_1} \wedge \boldsymbol{x_2} \wedge \boldsymbol{x_3}$ is pseudo-orthogonal to any $\boldsymbol{x_i}$ $(i = 1, 2, 3)$. We now define *hyperbolic 3-space* by

$$H^3 = \left\{ \boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = -1 \right\},$$

and *de Sitter 3-space* by

$$S^3_1 = \left\{ \boldsymbol{x} \in \mathbb{R}^4_1 \mid \langle \boldsymbol{x}, \boldsymbol{x} \rangle = 1 \right\}.$$

The set Δ_5 in [4] is defined by

$$\Delta_5 = \left\{ (\boldsymbol{\nu_1}, \boldsymbol{\nu_2}) \in S^3_1 \times S^3_1 \mid \langle \boldsymbol{\nu_1}, \boldsymbol{\nu_2} \rangle = 0 \right\}.$$

Framed curves may have singularities. The fundamental theorem of framed curves in the Euclidean space has been given in [7]. Now we have the definition of hyperbolic framed curve in hyperbolic 3-space.

Definition 2.1. We call $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ a *hyperbolic framed curve* if $\langle \gamma(t), \boldsymbol{\nu_i}(t) \rangle = \langle \gamma'(t), \boldsymbol{\nu_i}(t) \rangle = 0$ for any $t \in I$, $i = 1, 2$. $\gamma : I \to H^3$ is called a *hyperbolic framed base curve* if there exists $(\boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to \Delta_5$ such that $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ is a hyperbolic framed curve.

Define $\boldsymbol{\mu}(t) = \gamma(t) \wedge \boldsymbol{\nu_1}(t) \wedge \boldsymbol{\nu_2}(t)$, then $\{\gamma(t), \boldsymbol{\nu_1}(t), \boldsymbol{\nu_2}(t), \boldsymbol{\mu}(t)\}$ is a moving frame along γ. We have the Frenet type formulas

$$\begin{pmatrix} \gamma'(t) \\ \boldsymbol{\nu_1'}(t) \\ \boldsymbol{\nu_2'}(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & m(t) \\ 0 & 0 & n(t) & a(t) \\ 0 & -n(t) & 0 & b(t) \\ m(t) & -a(t) & -b(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{\nu_1}(t) \\ \boldsymbol{\nu_2}(t) \\ \boldsymbol{\mu}(t) \end{pmatrix},$$

where $m(t) = \langle \gamma'(t), \boldsymbol{\mu}(t) \rangle$, $n(t) = \langle \boldsymbol{\nu_1'}(t), \boldsymbol{\nu_2}(t) \rangle$, $a(t) = \langle \boldsymbol{\nu_1'}(t), \boldsymbol{\mu}(t) \rangle$, $b(t) = \langle \boldsymbol{\nu_2'}(t), \boldsymbol{\mu}(t) \rangle$. The map $(m, n, a, b) : I \to \mathbb{R}^4$ is called the curvature of $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2})$. Clearly, t_0 is a singular point of γ if and only if $m(t_0) = 0$.

3. Existence and uniqueness theorems of hyperbolic framed curves

The curvatures of the framed curve are quite useful to analyse the framed curves and their singularities. Now we give the local theory for hyperbolic framed curves by using their curvature.

Theorem 3.1. (The Existence Theorem) Given a smooth map $(m, n, a, b) : I \to \mathbb{R}^4$, then there exists a hyperbolic framed curve $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ with the curvature $(m(t), n(t), a(t), b(t))$.

Proof. Fix $t_0 \in I$. Consider the equation system

$$\begin{pmatrix} \gamma'(t) \\ \boldsymbol{\nu_1'}(t) \\ \boldsymbol{\nu_2'}(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & m(t) \\ 0 & 0 & n(t) & a(t) \\ 0 & -n(t) & 0 & b(t) \\ m(t) & -a(t) & -b(t) & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{\nu_1}(t) \\ \boldsymbol{\nu_2}(t) \\ \boldsymbol{\mu}(t) \end{pmatrix} \tag{1}$$

with the initial value

$$\langle\gamma(t_0),\gamma(t_0)\rangle = -1,\ \langle\boldsymbol{\nu_1}(t_0),\boldsymbol{\nu_1}(t_0)\rangle = \langle\boldsymbol{\nu_2}(t_0),\boldsymbol{\nu_2}(t_0)\rangle = \langle\boldsymbol{\mu}(t_0),\boldsymbol{\mu}(t_0)\rangle = 1,$$
$$\langle\gamma(t_0),\boldsymbol{\nu_1}(t_0)\rangle = \langle\gamma(t_0),\boldsymbol{\nu_2}(t_0)\rangle = \langle\gamma(t_0),\boldsymbol{\mu}(t_0)\rangle = \langle\boldsymbol{\nu_1}(t_0),\boldsymbol{\nu_2}(t_0)\rangle = \langle\boldsymbol{\nu_1}(t_0),\boldsymbol{\mu}(t_0)\rangle = \langle\boldsymbol{\nu_2}(t_0),\boldsymbol{\mu}(t_0)\rangle = 0.$$

Then we have a solution $(\gamma,\boldsymbol{\nu_1},\boldsymbol{\nu_2},\boldsymbol{\mu})$ of the equation (1).

Define ten smooth functions $a_1,...,a_{10} : I \to \mathbb{R}$ by

$$a_1(t) = \langle\gamma(t),\gamma(t)\rangle, a_2(t) = \langle\gamma(t),\boldsymbol{\nu_1}(t)\rangle, a_3(t) = \langle\boldsymbol{\nu_1}(t),\boldsymbol{\nu_1}(t)\rangle, a_4(t) = \langle\gamma(t),\boldsymbol{\nu_2}(t)\rangle, a_5(t) = \langle\boldsymbol{\nu_1}(t),\boldsymbol{\nu_2}(t)\rangle,$$
$$a_6(t) = \langle\boldsymbol{\nu_2}(t),\boldsymbol{\nu_2}(t)\rangle, a_7(t) = \langle\gamma(t),\boldsymbol{\mu}(t)\rangle, a_8(t) = \langle\boldsymbol{\nu_1}(t),\boldsymbol{\mu}(t)\rangle, a_9(t) = \langle\boldsymbol{\nu_2}(t),\boldsymbol{\mu}(t)\rangle, a_{10}(t) = \langle\boldsymbol{\mu}(t),\boldsymbol{\mu}(t)\rangle.$$

Consider the equation

$$\begin{pmatrix} a_1'(t)\\ a_2'(t)\\ a_3'(t)\\ a_4'(t)\\ a_5'(t)\\ a_6'(t)\\ a_7'(t)\\ a_8'(t)\\ a_9'(t)\\ a_{10}'(t)\end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 2m(t) & 0 & 0 & 0\\ 0 & 0 & 0 & n(t) & 0 & 0 & a(t) & m(t) & 0 & 0\\ 0 & 0 & 0 & 0 & 2n(t) & 0 & 0 & 2a(t) & 0 & 0\\ 0 & -n(t) & 0 & 0 & 0 & 0 & b(t) & 0 & m(t) & 0\\ 0 & 0 & -n(t) & 0 & 0 & n(t) & 0 & b(t) & a(t) & 0\\ 0 & 0 & 0 & 0 & -2n(t) & 0 & 0 & 0 & 2b(t) & 0\\ m(t) & -a(t) & 0 & -b(t) & 0 & 0 & 0 & 0 & 0 & m(t)\\ 0 & m(t) & -a(t) & 0 & -b(t) & 0 & 0 & 0 & n(t) & a(t)\\ 0 & 0 & 0 & m(t) & -a(t) & -b(t) & 0 & -n(t) & 0 & b(t)\\ 0 & 0 & 0 & 0 & 0 & 0 & 2m(t) & -2a(t) & -2b(t) & 0 \end{pmatrix}\begin{pmatrix} a_1(t)\\ a_2(t)\\ a_3(t)\\ a_4(t)\\ a_5(t)\\ a_6(t)\\ a_7(t)\\ a_8(t)\\ a_9(t)\\ a_{10}(t)\end{pmatrix}$$

with the initial value

$$a_1(t_0) = -1, a_3(t_0) = a_6(t_0) = a_{10}(t_0) = 1,$$

$$a_2(t_0) = a_4(t_0) = a_5(t_0) = a_7(t_0) = a_8(t_0) = a_9(t_0) = 0.$$

Thus we have

$$a_1(t) = -1, a_3(t) = a_6(t) = a_{10}(t) = 1,$$

$$a_2(t) = a_4(t) = a_5(t) = a_7(t) = a_8(t) = a_9(t) = 0,$$

for any $t \in I$.

Hence, $(\gamma,\boldsymbol{\nu_1},\boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ is a hyperbolic framed curve with the curvature $(m(t), n(t), a(t), b(t))$. □

Definition 3.2. Let $(\gamma,\boldsymbol{\nu_1},\boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ and $(\bar{\gamma},\bar{\boldsymbol{\nu}}_1,\bar{\boldsymbol{\nu}}_2) : I \to H^3 \times \Delta_5$ be two hyperbolic framed curves. We say that $(\gamma,\boldsymbol{\nu_1},\boldsymbol{\nu_2})$ and $(\bar{\gamma},\bar{\boldsymbol{\nu}}_1,\bar{\boldsymbol{\nu}}_2)$ are *congruent* through a Lorentz motion if there exists a matrix $\boldsymbol{M}$ and a constant vector $\boldsymbol{c} \in \mathbb{R}^4_1$ such that

$$\bar{\gamma}(t) = \boldsymbol{M}(\gamma(t)) + \boldsymbol{c} \text{ and } \bar{\boldsymbol{\nu}}_{\boldsymbol{i}}(t) = \boldsymbol{M}(\boldsymbol{\nu_i}(t)),\ \ i = 1,2$$

for all $t \in I$, where $\boldsymbol{M}$ satisfies

$$\boldsymbol{M}^{T}\boldsymbol{G}\boldsymbol{M} = \boldsymbol{G},\ \det(\boldsymbol{M}) = 1,\ \boldsymbol{G} = \begin{pmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{pmatrix}.$$

Theorem 3.3. (The Uniqueness Theorem) Let $(\gamma,\boldsymbol{\nu_1},\boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ and $(\bar{\gamma},\bar{\boldsymbol{\nu}}_1,\bar{\boldsymbol{\nu}}_2) : I \to H^3 \times \Delta_5$ be two hyperbolic framed curves with the same curvature $(m(t), n(t), a(t), b(t))$. Then $(\gamma,\boldsymbol{\nu_1},\boldsymbol{\nu_2})$ and $(\bar{\gamma},\bar{\boldsymbol{\nu}}_1,\bar{\boldsymbol{\nu}}_2)$ are congruent through a Lorentz motion.

Proof. Let $\boldsymbol{\mu}(t) = \gamma(t) \wedge \boldsymbol{\nu_1}(t) \wedge \boldsymbol{\nu_2}(t)$ and $\bar{\boldsymbol{\mu}}(t) = \bar{\gamma}(t) \wedge \bar{\boldsymbol{\nu}}_{\boldsymbol{1}}(t) \wedge \bar{\boldsymbol{\nu}}_{\boldsymbol{2}}(t)$. For a fixed $t_0 \in I$, there exists a 4×4 matrix $\boldsymbol{M}$ such that $\bar{\boldsymbol{\nu}}_{\boldsymbol{1}}(t_0) = \boldsymbol{M}(\boldsymbol{\nu_1}(t_0))$, $\bar{\boldsymbol{\nu}}_{\boldsymbol{2}}(t_0) = \boldsymbol{M}(\boldsymbol{\nu_2}(t_0))$, $\bar{\boldsymbol{\mu}}(t_0) = \boldsymbol{M}(\boldsymbol{\mu}(t_0))$, where $\boldsymbol{M}$ satisfies

$$\boldsymbol{M}^T\boldsymbol{G}\boldsymbol{M} = \boldsymbol{G},\ \det(\boldsymbol{M}) = 1,\ \boldsymbol{G} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\bar{\gamma}(t_0) = \boldsymbol{M}(\gamma(t_0))$.

Consider $(\boldsymbol{M}(\gamma), \boldsymbol{M}(\boldsymbol{\nu_1}), \boldsymbol{M}(\boldsymbol{\nu_2}), \boldsymbol{M}(\boldsymbol{\mu})) : I \to H^3 \times \Delta_5 \times S_1^3$ and $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}, \boldsymbol{\mu}) : I \to H^3 \times \Delta_5 \times S_1^3$. They are both the solution of the equation system (1) with the same initial value. So we have $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}, \boldsymbol{\mu})$ $= (\boldsymbol{M}(\gamma), \boldsymbol{M}(\boldsymbol{\nu_1}), \boldsymbol{M}(\boldsymbol{\nu_2}), \boldsymbol{M}(\boldsymbol{\mu}))$. Thus $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2})$ and $(\bar{\gamma}, \bar{\boldsymbol{\nu}}_{\boldsymbol{1}}, \bar{\boldsymbol{\nu}}_{\boldsymbol{2}})$ are congruent through a Lorentz motion. □

Honda and Takahashi defined the adapted frame along a curve in $\mathbb{R}^3$ and gave Frenet type formulas in [7]. Similarly, we also consider a special moving frame of γ in H^3. $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ is a framed curve with the curvature $(m(t), n(t), a(t), b(t))$, which satisfies $a^2(t) + b^2(t) \neq 0$, for any $t \in I$. Let

$$\boldsymbol{n}_1(t) = \frac{1}{\sqrt{a^2(t) + b^2(t)}}(a(t)\boldsymbol{\nu}_1(t) + b(t)\boldsymbol{\nu}_2(t)),$$

$$\boldsymbol{n}_2(t) = \frac{1}{\sqrt{a^2(t) + b^2(t)}}(-b(t)\boldsymbol{\nu}_1(t) + a(t)\boldsymbol{\nu}_2(t)).$$

$(\gamma, \boldsymbol{n_1}, \boldsymbol{n_2})$ is also a hyperbolic framed curve and $\gamma(t) \wedge \boldsymbol{n_1}(t) \wedge \boldsymbol{n_2}(t) = \boldsymbol{\mu}(t)$, then we have a new frame $\{\gamma(t), \boldsymbol{n_1}(t), \boldsymbol{n_2}(t), \boldsymbol{\mu}(t)\}$ along γ, which is called the *Frenet type frame* along γ. And we have the following Frenet type formulas

$$\begin{pmatrix} \gamma'(t) \\ \boldsymbol{n}_{\boldsymbol{1}}'(t) \\ \boldsymbol{n}_{\boldsymbol{2}}'(t) \\ \boldsymbol{\mu}'(t) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & M(t) \\ 0 & 0 & N(t) & A(t) \\ 0 & -N(t) & 0 & 0 \\ M(t) & -A(t) & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma(t) \\ \boldsymbol{n_1}(t) \\ \boldsymbol{n_2}(t) \\ \boldsymbol{\mu}(t) \end{pmatrix}, \tag{2}$$

where

$$M(t) = m(t),\ A(t) = \sqrt{a^2(t) + b^2(t)},$$

$$N(t) = \frac{a(t)b'(t) - b(t)a'(t) + \left(a^2(t) + b^2(t)\right)n(t)}{a^2(t) + b^2(t)}.$$

Remark 3.4. A regular curve $\gamma : I \to H^3$ is also a hyperbolic framed base curve, we can easily show that the condition $a^2(t) + b^2(t) = 0$ is equivalent to the hyperbolic curvature $\kappa_h(t)$ of γ vanishes [9]. So we assume that $a^2(t) + b^2(t) \neq 0$ for any $t \in I$.

The vectors $\boldsymbol{\mu}(t), \boldsymbol{n_1}(t), \boldsymbol{n_2}(t)$ are called *the generalized tangent vector*, *the generalized principal normal vector* and *the generalized binormal vector* of the curve γ, respectively. The functions $(M(t), N(t), A(t), 0)$ are referred to as the framed curvature of γ, and called the *Frenet type curvature*. Then we have the definition of hyperbolic Frenet type framed curve.

Definition 3.5. Let $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to H^3 \times \Delta_5$ be a hyperbolic framed curve with curvature $(m(t), n(t), a(t), b(t))$. We call $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2})$ a *hyperbolic Frenet type framed curve* if $a^2(t) + b^2(t) \neq 0$, for any $t \in I$. $\gamma : I \to H^3$ is called a *hyperbolic Frenet type framed base curve* if there exists $(\boldsymbol{\nu_1}, \boldsymbol{\nu_2}) : I \to \Delta_5$ such that $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2})$ is a hyperbolic Frenet type framed curve.

4. Hyperbolic Framed Slant Helices

The necessary and sufficient conditions for hyperbolic curves in H^3 to be k-type slant helices have been given in [25]. As a generalization of them, we show the theorems of k-type hyperbolic framed slant helices which may have singular points in this section. Let us set that

$$\boldsymbol{V_0}(t) = \gamma(t),\ \boldsymbol{V_1}(t) = \boldsymbol{\mu}(t),\ \boldsymbol{V_2}(t) = \boldsymbol{n_1}(t),\ \boldsymbol{V_3}(t) = \boldsymbol{n_2}(t).$$

Then, the k-type hyperbolic framed slant helices in H^3 are defined as follows.

Definition 4.1. A hyperbolic Frenet type framed base curve $\boldsymbol{\gamma} : I \to H^3$ with the Frenet type frame $\{\boldsymbol{V_0}(t), \boldsymbol{V_1}(t), \boldsymbol{V_2}(t), \boldsymbol{V_3}(t)\}$ is called a *k-type hyperbolic framed slant helix* for $k \in \{0, 1, 2, 3\}$ if there exists a non-zero fixed vector $\boldsymbol{p} \in \mathbb{R}^4_1$ such that $\langle \boldsymbol{V_k}(t), \boldsymbol{p}\rangle$ is a constant.

4.1. 0-type and 1-type hyperbolic framed slant helices

Now we consider 0-type and 1-type hyperbolic framed slant helices in H^3. We can get necessary and sufficient conditions for a hyperbolic curve to be a 0-type slant helix or 1-type slant helix, and these algebraic formulas can be converted to each other under certain conditions, enabling us to elucidate the relationship between these two types of curves through algebraic representations.

Theorem 4.2. Let $\boldsymbol{\gamma} : I \to H^3$ be a hyperbolic Frenet type framed base curve with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. Then $\boldsymbol{\gamma}$ is a 0-type hyperbolic framed slant helix if and only if

$$\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)} = 0. \tag{3}$$

Proof. Assume $\boldsymbol{\gamma}$ is a 0-type hyperbolic framed slant helix in H^3 parametrized by t with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. Then there exists a non-zero fixed vector $\boldsymbol{p} \in \mathbb{R}^4_1$ such that

$$\langle \gamma(t), \boldsymbol{p}\rangle = c,\quad c \in \mathbb{R}. \tag{4}$$

If t are regular points of $\boldsymbol{\gamma}$, taking derivative of the equation (4) and using the equation (2), we get

$$\langle \boldsymbol{\mu}(t), \boldsymbol{p}\rangle = 0,\quad \langle \boldsymbol{n_1}(t), \boldsymbol{p}\rangle = c\,\frac{M(t)}{A(t)}. \tag{5}$$

By using (5), we can write $\boldsymbol{p}$ with respect to the frame $\{\gamma(t), \boldsymbol{\mu}(t), \boldsymbol{n_1}(t), \boldsymbol{n_2}(t)\}$ as follows

$$\boldsymbol{p} = -c\gamma(t) + c\,\frac{M(t)}{A(t)}\boldsymbol{n_1}(t) + \lambda(t)\boldsymbol{n_2}(t), \tag{6}$$

where $\lambda(t)$ is a smooth function. Taking derivative of the equation (6), we have

$$\Big(c\,\Big(\frac{M(t)}{A(t)}\Big)' - \lambda(t)N(t)\Big)\boldsymbol{n_1}(t) + \Big(c\,\frac{M(t)N(t)}{A(t)} + \lambda'(t)\Big)\boldsymbol{n_2}(t) = \boldsymbol{0}.$$

If $c = 0$, it can be known that $\boldsymbol{p} = \boldsymbol{0}$, so we assume that $c \in \mathbb{R}\backslash\{0\}$. Then we get

$$\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)} = 0.$$

Especially, when t_0 is a singular point of $\boldsymbol{\gamma}$, by the continuity of curvature $(M(t), N(t), A(t), 0)$, t_0 also satisfies the equation (3).

Conversely, assume that (3) holds, choosing the vector $\boldsymbol{p}(t)$ as

$$\boldsymbol{p}(t) = -c\Big(\gamma(t) - \frac{M(t)}{A(t)}\boldsymbol{n_1}(t) - \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'\boldsymbol{n_2}(t)\Big),\quad c \in \mathbb{R}\backslash\{0\},$$

we get $\boldsymbol{p}'(t) = \boldsymbol{0}$ and $\langle \gamma(t), \boldsymbol{p}(t)\rangle = c$ (constant). Then $\boldsymbol{\gamma}$ is a 0-type hyperbolic framed slant helix. □

Corollary 4.3. The axis of a 0-type hyperbolic framed slant helix parametrized by t is given by

$$\boldsymbol{p} = -c\Big(\gamma(t) - \frac{M(t)}{A(t)}\boldsymbol{n_1}(t) - \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'\boldsymbol{n_2}(t)\Big), \tag{7}$$

where $c \in \mathbb{R}\backslash\{0\}$.

Corollary 4.4. Let γ be a hyperbolic Frenet type framed base curve parametrized by t with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. Then γ is a 0-type hyperbolic framed slant helix if and only if

$$\Big(\frac{M(t)}{A(t)}\Big)^2 + \Big(\frac{1}{N(t)}\Big)^2\Big(\Big(\frac{M(t)}{A(t)}\Big)'\Big)^2 = c, \quad c \in \mathbb{R}. \tag{8}$$

Proof. Assume that γ is a 0-type hyperbolic framed slant helix parametrized by t with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. From (7), we have

$$\Big(\frac{M(t)}{A(t)}\Big)^2 + \Big(\frac{1}{N(t)}\Big)^2\Big(\Big(\frac{M(t)}{A(t)}\Big)'\Big)^2 = c.$$

Conversely, assume that the equation (8) holds. Then taking derivative of the equation (8) with respect to t, we get

$$\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)} = 0,$$

which means that γ is a 0-type hyperbolic framed slant helix. □

If $\gamma : I \to H^3$ is a 1-type hyperbolic framed slant helix, then there exists a non-zero fixed vector $\boldsymbol{p} \in \mathbb{R}^4_1$ such that

$$\langle \boldsymbol{\mu}(t), \boldsymbol{p}\rangle = c, \quad c \in \mathbb{R}. \tag{9}$$

We have the following theorem.

Theorem 4.5. Let $\gamma : I \to H^3$ be a hyperbolic Frenet type framed base curve with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. Under the above notion, γ is a 1-type hyperbolic framed slant helix if and only if either
(i) $c \neq 0$ and

$$\Big(\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)}\Big)\int M(t)dt - \Big(\frac{A(t)}{N(t)}\Big)' + \Big(\frac{M(t)}{N(t)}\Big)'\frac{M(t)}{A(t)} + 2\,\frac{M(t)}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)' = 0, \tag{10}$$

or
(ii) $c = 0$ and

$$\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)} = 0. \tag{11}$$

Proof. Assume that γ is a 1-type hyperbolic framed slant helix parametrized by t with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. By equation (9), we can write $\boldsymbol{p}$ with respect to the frame $\{\gamma(t), \boldsymbol{\mu}(t), \boldsymbol{n_1}(t), \boldsymbol{n_2}(t)\}$ as follows

$$\boldsymbol{p} = \lambda(t)\gamma(t) + c\boldsymbol{\mu}(t) + \lambda_1(t)\boldsymbol{n_1}(t) + \lambda_2(t)\boldsymbol{n_2}(t), \tag{12}$$

where $\lambda(t), \lambda_1(t), \lambda_2(t)$ are smooth functions of t. Taking derivative of the equation (12) with respect to t, we have

$$\begin{aligned}&(\lambda'(t) + cM(t))\gamma(t) + (\lambda(t)M(t) + \lambda_1(t)A(t))\boldsymbol{\mu}(t)\\ &+(\lambda_1'(t) - cA(t) - \lambda_2(t)N(t))\boldsymbol{n_1}(t) + (\lambda_1(t)N(t) + \lambda_2'(t))\boldsymbol{n_2}(t) = \boldsymbol{0},\end{aligned} \tag{13}$$

solving (13), if $c \neq 0$, we get

$$\Big(\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)}\Big)\int M(t)dt - \Big(\frac{A(t)}{N(t)}\Big)' + \Big(\frac{M(t)}{N(t)}\Big)'\frac{M(t)}{A(t)} + 2\,\frac{M(t)}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)' = 0.$$

When $c = 0$, we can know that

$$\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)} = 0.$$

Conversely, assume that (10) or (11) holds, choosing the vector $\boldsymbol{p}(t)$ as

$$\boldsymbol{p}(t) = \Big(-\int M(t)dt\Big)\gamma(t) + \boldsymbol{\mu}(t) + \Big(\frac{M(t)}{A(t)}\int M(t)dt\Big)\boldsymbol{n_1}(t) + \Big(\frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'\int M(t)dt + \frac{M^2(t)}{N(t)A(t)} - \frac{A(t)}{N(t)}\Big)\boldsymbol{n_2}(t)$$

or

$$\boldsymbol{p}(t) = \gamma(t) - \frac{M(t)}{A(t)}\boldsymbol{n_1}(t) - \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'\boldsymbol{n_2}(t).$$

We can get $\boldsymbol{p}'(t) = \mathbf{0}$ and $\langle \boldsymbol{\mu}(t), \boldsymbol{p}(t)\rangle = 1$ or 0. Then γ is a 1-type hyperbolic framed slant helix. □

Corollary 4.6. The axis of a 1-type hyperbolic framed slant helix is given by

$$\boldsymbol{p} = \Big(-\int cM(t)dt\Big)\gamma(t) + c\boldsymbol{\mu}(t) + c\Big(\frac{M(t)}{A(t)}\int M(t)dt\Big)\boldsymbol{n_1}(t) + c\Big(\frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'\int M(t)dt + \frac{M^2(t)}{N(t)A(t)} - \frac{A(t)}{N(t)}\Big)\boldsymbol{n_2}(t)$$

or

$$\boldsymbol{p} = c\gamma(t) - \frac{cM(t)}{A(t)}\boldsymbol{n_1}(t) - \frac{c}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'\boldsymbol{n_2}(t),$$

where $c \in \mathbb{R}\backslash\{0\}$.

According to the above theorem, if γ is a 1-type hyperbolic framed slant helix and satisfies $\langle \boldsymbol{\mu}(t), \boldsymbol{p}\rangle = 0$, we have

$$\Big(\frac{1}{N(t)}\Big)'\Big(\frac{M(t)}{A(t)}\Big)' + \frac{1}{N(t)}\Big(\frac{M(t)}{A(t)}\Big)'' + \frac{M(t)N(t)}{A(t)} = 0,$$

which means that γ is also a 0-type hyperbolic framed slant helices. Thus we have the following corollary.

Corollary 4.7. Let γ be a hyperbolic Frenet type framed base curve with Frenet type curvature $(M(t), N(t), A(t), 0)$, where $N(t) \neq 0$. Then γ is a 0-type hyperbolic framed slant helix if and only if γ is a 1-type hyperbolic framed slant helix whose axis $\boldsymbol{p}$ satisfies $\langle \boldsymbol{\mu}(t), \boldsymbol{p}\rangle = 0$.

4.2. 2-type and 3-type hyperbolic framed slant helices

A similar discussion can lead to sufficient and necessary conditions for curves to be 2-type or 3-type hyperbolic framed slant helices and connection between the two types of curves. If there exists a non-zero fixed vector $\boldsymbol{p} \in \mathbb{R}^4_1$ such that

$$\langle \boldsymbol{n_1}(t), \boldsymbol{p}\rangle = c, \quad c \in \mathbb{R}, \tag{14}$$

we say that $\gamma : I \to H^3$ is a 2-type hyperbolic framed slant helix, then we have the following theorem.

Theorem 4.8. Let $\gamma : I \to H^3$ be a hyperbolic Frenet type framed base curve with Frenet type curvature $(M(t), N(t), A(t), 0)$. Under the above notion, γ is a 2-type hyperbolic framed slant helix if and only if either (i) $c \neq 0$ and

$$M(t)\int\Big(\frac{M(t)N(t)}{A(t)}\int N(t)dt\Big)dt - \Big(\frac{N(t)}{A(t)}\Big)'\int N(t)dt = \frac{N^2(t) + A^2(t)}{A(t)}, \tag{15}$$

or
(ii) $c = 0$ and

$$M(t)\int \frac{M(t)N(t)}{A(t)}dt - \Big(\frac{N(t)}{A(t)}\Big)' = 0. \tag{16}$$

Proof. Assume that γ is a 2-type hyperbolic framed slant helix parametrized by t with Frenet type curvature $(M(t), N(t), A(t), 0)$. By equation (14) we can write $\boldsymbol{p}$ with respect to the frame $\{\gamma(t), \boldsymbol{\mu}(t), \boldsymbol{n_1}(t), \boldsymbol{n_2}(t)\}$ as follows

$$\boldsymbol{p} = \lambda_1(t)\gamma(t) + \lambda_2(t)\boldsymbol{\mu}(t) + c\boldsymbol{n_1}(t) + \lambda_3(t)\boldsymbol{n_2}(t), \tag{17}$$

where $\lambda_1(t), \lambda_2(t), \lambda_3(t)$ are smooth functions of t and $c \in \mathbb{R}$. Taking derivative of the equation (17), we have

$$\begin{aligned}&(\lambda_1'(t) + \lambda_2(t)M(t))\gamma(t) + (\lambda_1(t)M(t) + \lambda_2'(t) + cA(t))\boldsymbol{\mu}(t)\\ &-(\lambda_2(t)A(t) + \lambda_3(t)N(t))\boldsymbol{n_1}(t) + (\lambda_3'(t) + cN(t))\boldsymbol{n_2}(t) = \mathbf{0},\end{aligned}$$

which implies that

$$\begin{cases} \lambda_1'(t) + \lambda_2(t)M(t) = 0,\\ \lambda_1(t)M(t) + \lambda_2'(t) + cA(t) = 0,\\ \lambda_2(t)A(t) + \lambda_3(t)N(t) = 0,\\ \lambda_3'(t) + cN(t) = 0. \end{cases} \tag{18}$$

If $c \neq 0$, solving (18), we get

$$M(t)\int \Big(\frac{M(t)N(t)}{A(t)}\int N(t)dt\Big)dt - \Big(\frac{N(t)}{A(t)}\Big)'\int N(t)dt = \frac{N^2(t) + A^2(t)}{A(t)}.$$

Assume that $c = 0$ in (18), then we have

$$\begin{cases} \lambda_1'(t) + \lambda_2(t)M(t) = 0,\\ \lambda_1(t)M(t) + \lambda_2'(t) = 0,\\ \lambda_2(t)A(t) + \lambda_3(t)N(t) = 0,\\ \lambda_3'(t) = 0, \end{cases}$$

which implies that

$$M(t)\int \frac{M(t)N(t)}{A(t)}dt - \Big(\frac{N(t)}{A(t)}\Big)' = 0.$$

Conversely, assume that (15) or (16) holds, choosing the vector $\boldsymbol{p}(t)$ as

$$\boldsymbol{p}(t) = -\Big(\int \frac{M(t)N(t)}{A(t)}\Big(\int N(t)dt\Big)dt\Big)\gamma(t) + \frac{N(t)}{A(t)}\Big(\int N(t)dt\Big)\boldsymbol{\mu}(t) + \boldsymbol{n_1}(t) - \Big(\int N(t)dt\Big)\boldsymbol{n_2}(t)$$

or

$$\boldsymbol{p}(t) = \Big(\int \frac{M(t)N(t)}{A(t)}dt\Big)\gamma(t) - \frac{N(t)}{A(t)}\boldsymbol{\mu}(t) + \boldsymbol{n_2}(t).$$

We can get $\boldsymbol{p}'(t) = \mathbf{0}$ and $\langle \boldsymbol{n_1}(t), \boldsymbol{p}(t)\rangle = 1$ or 0. Then γ is a 2-type hyperbolic framed slant helix. □

Corollary 4.9. The axis of a 2-type hyperbolic framed slant helix is given by

$$\boldsymbol{p} = -c\Big(\int \frac{M(t)N(t)}{A(t)}\Big(\int N(t)dt\Big)dt\Big)\gamma(t) + \frac{cN(t)}{A(t)}\Big(\int N(t)dt\Big)\boldsymbol{\mu}(t) + c\boldsymbol{n_1}(t) - \Big(\int cN(t)dt\Big)\boldsymbol{n_2}(t)$$

or

$$\boldsymbol{p} = \Big(\int \frac{cM(t)N(t)}{A(t)}dt\Big)\gamma(t) - \frac{cN(t)}{A(t)}\boldsymbol{\mu}(t) + c\boldsymbol{n_2}(t),$$

where $c \in \mathbb{R}\backslash\{0\}$.

Theorem 4.10. Let $\gamma : I \to H^3$ be a hyperbolic Frenet type framed base curve with Frenet type curvature $(M(t), N(t), A(t), 0)$. Then γ is a 3-type hyperbolic framed slant helix if and only if

$$M(t)\int \frac{M(t)N(t)}{A(t)}dt - \Big(\frac{N(t)}{A(t)}\Big)' = 0. \tag{19}$$

Proof. Assume that γ is a 3-type hyperbolic framed slant helix parametrized by t with Frenet type curvature $(M(t), N(t), A(t), 0)$. Then there exists a non-zero fixed vector $\boldsymbol{p} \in \mathbb{R}^4_1$ such that

$$\langle \boldsymbol{n_2}(t), \boldsymbol{p}\rangle = c, \quad c \in \mathbb{R}. \tag{20}$$

Taking derivative of the equation (20), we get

$$\langle \boldsymbol{n_1}(t), \boldsymbol{p}\rangle = 0, \ \ \langle \boldsymbol{\mu}(t), \boldsymbol{p}\rangle = -c\,\frac{N(t)}{A(t)},$$

then we can write $\boldsymbol{p}$ with respect to the frame $\{\gamma(t), \boldsymbol{\mu}(t), \boldsymbol{n_1}(t), \boldsymbol{n_2}(t)\}$ as follows

$$\boldsymbol{p} = \lambda(t)\gamma(t) - c\,\frac{N(t)}{A(t)}\boldsymbol{\mu}(t) + c\boldsymbol{n_2}(t), \tag{21}$$

where $\lambda(t)$ is a smooth function. Taking derivative of the equation (21), we have

$$\Big(\lambda'(t) - c\frac{M(t)N(t)}{A(t)}\Big)\gamma(t) + \Big(\lambda(t)M(t) - c\,\Big(\frac{N(t)}{A(t)}\Big)'\Big)\boldsymbol{\mu}(t) = \boldsymbol{0}. \tag{22}$$

If $c = 0$, it can be known that $\boldsymbol{p} = \boldsymbol{0}$, so we assume that $c \in \mathbb{R}\backslash\{0\}$. Solving (22), we get

$$M(t)\int \frac{M(t)N(t)}{A(t)}dt - \Big(\frac{N(t)}{A(t)}\Big)' = 0.$$

Conversely, assume that (19) holds, choosing the vector $\boldsymbol{p}(t)$ as

$$\boldsymbol{p}(t) = c\Big(\Big(\int \frac{M(t)N(t)}{A(t)}dt\Big)\gamma(t) - \frac{N(t)}{A(t)}\boldsymbol{\mu}(t) + \boldsymbol{n_2}(t)\Big),$$

where $c \in \mathbb{R}\backslash\{0\}$. We can get $\boldsymbol{p}'(t) = \boldsymbol{0}$ and $\langle \boldsymbol{n_2}(t), \boldsymbol{p}(t)\rangle = c$ (constant). Then γ is a 3-type hyperbolic framed slant helix. □

Corollary 4.11. The axis of a 3-type hyperbolic framed slant helix is given by

$$\boldsymbol{p} = c\Big(\Big(\int \frac{M(t)N(t)}{A(t)}dt\Big)\gamma(t) - \frac{N(t)}{A(t)}\boldsymbol{\mu}(t) + \boldsymbol{n_2}(t)\Big),$$

where $c \in \mathbb{R}\backslash\{0\}$.

According to the Theorem 4.8 and Theorem 4.10, if γ is a 2-type hyperbolic framed slant helix and satisfies $\langle \boldsymbol{n_1}(t), \boldsymbol{p}\rangle = 0$, we have

$$M(t)\int \frac{M(t)N(t)}{A(t)}dt - \Big(\frac{N(t)}{A(t)}\Big)' = 0,$$

which means that γ is also a 3-type hyperbolic framed slant helix. Thus we have the following corollary.

Corollary 4.12. Let γ be a hyperbolic Frenet type framed base curve with Frenet type curvature $(M(t), N(t), A(t), 0)$. Then γ is a 3-type hyperbolic framed slant helix if and only if γ is a 2-type hyperbolic framed slant helix whose axis $\boldsymbol{p}$ satisfies $\langle \boldsymbol{n_1}(t), \boldsymbol{p}\rangle = 0$.

Remark 4.13. For a regular curve in H^3, the results of k-type hyperbolic slant helices in [25] are equivalent to those of k-type hyperbolic framed slant helices in this section. Being different from k-type hyperbolic slant helices, k-type hyperbolic framed slant helices might have singular points, which is a generalization of hyperbolic salnt helices.

5. Examples

In Section 4, we have investigated the k-type hyperbolic framed slant helices and the connection between them. Hyperbolic framed slant helices might have singular points, we will give two examples.

Example 5.1. (1) Let $\gamma : [0, 2\pi) \to H^3$ be a hyperbolic Frenet type framed base curve with curvature $(\cos t, 1, 1, 0)$.

By Theorem 4.2, we know that the curvature functions of γ satisfy equation (3) and γ is a 0-type hyperbolic framed slant helix. Note that γ is singular at $t = \pi/2,\ 3\pi/2$.

(2) The hyperbolic Frenet type framed base curve with curvature $(\cos(\ln t), 1/t, 1, 0),\ t \in (0, +\infty)$ can also be proved to be a 0-type hyperbolic framed slant helix.

Example 5.2. Consider the hyperbolic curvature $(\sin t, 1, 1, 0)$.

According to the existence and the uniqueness of hyperbolic framed curves, through a Lorentz motion, there exists a unique framed curve $(\gamma, \boldsymbol{\nu_1}, \boldsymbol{\nu_2})$ whose curvature is $(\sin t, 1, 1, 0)$, and the generalized tangent vector of γ is $\boldsymbol{\mu}(t) = \gamma(t) \wedge \boldsymbol{\nu_1}(t) \wedge \boldsymbol{\nu_2}(t)$. We can take

$$\boldsymbol{p}(t) = \gamma(t) - \sin t \boldsymbol{\nu_1}(t) - \cos t \boldsymbol{\nu_2}(t).$$

Then we get $\boldsymbol{p}'(t) = \mathbf{0}$, $\langle \boldsymbol{\mu}(t), \boldsymbol{p}(t) \rangle = 0$, and the curvature funtions satisfy equation (11), which implies that γ is a 1-type hyperbolic framed slant helix.

Besides, by Corollary 4.7 we can know that γ is also a 0-type hyperbolic framed slant helix.

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