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From the hyperbolic distance to the hyperbolic length

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Abstract. In this paper we show one way to define the hyperbolic length of a curve in the unit disc. We start from the formula for hyperbolic distance in the unit disc and via the hyperbolic lengths of the inscribed hyperbolic polygonal lines we arrive at the formula for calculating the hyperbolic length of the C^1 curve in a natural way.

1. Introduction

It is well known (see for example, [16, p. 136-137]) that in Euclidean plane (as a model of that plane in this paper we consider the set of all complex numbers $\mathbb{C}$) Euclidean length of the curve $\gamma : [a, b] \to \mathbb{C}$ is introduced as follows:

1) first, the length of a line segment with the endpoints z_1 and z_2 is defined as the distance between that points, i.e. that length is equal to $d_e(z_1, z_2)$, where $d_e(z_1, z_2) = |z_1 - z_2|$;

2) then, the length of a polygonal line is defined as the sum of lengths of the segments which form that polygonal line;

3) further, if $P : a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ is a partition of the interval $[a, b]$, then we assign to the partition P and the curve γ the number

$$\ell_e(P, \gamma) = \sum_{j=1}^{n} d_e(\gamma(t_j), \gamma(t_{j-1})),$$

i.e. we assign them the length of polygonal line consisting of the segments

$$[\gamma(t_0), \gamma(t_1)], \quad \ldots, \quad [\gamma(t_{n-1}), \gamma(t_n)].$$

4) finally, Euclidean length of the curve γ (which we denote by $\ell_e(\gamma)$) is defined by

$$\ell_e(\gamma) = \sup_{P \in \Pi} \ell_e(P, \gamma),$$

where Π is the set of all partitions of the interval $[a, b]$.

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If $\ell_{\mathrm{e}}(\gamma) < +\infty$ then we say that the curve γ is rectifiable. One could show (see [16, Theorem 6.27]) that any C^1 curve $\gamma : [a, b] \to \mathbb{C}$ is rectifiable as well as $\ell_{\mathrm{e}}(\gamma) = \int_\gamma |dz|$.

On the other hand in the literature (see for example [1–3, 11]) it is usual that in hyperbolic plane (as a model of that plane in this paper we consider the unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$), hyperbolic length of a C^1 curve $\gamma : [a, b] \to \mathbb{U}$ is defined in the following way

$$\int_\gamma \rho_{\mathbb{U}}(z)|dz|, \tag{1}$$

where $\rho_{\mathbb{U}} : \mathbb{U} \to (0, +\infty)$ is hyperbolic density on the unit disc defined by

$$\rho_{\mathbb{U}}(z) = \frac{2}{1 - |z|^2}.$$

In this paper we introduce hyperbolic length of C^1 curve in the hyperbolic plane (i.e. in the unit disc $\mathbb{U}$) in an analogous way as it is introduce in the Euclidean plane (i.e. in $\mathbb{C}$, which we identify as a metric space with $\mathbb{R}^2$) for Euclidean length of C^1 curve. More precisely, starting from the formula for hyperbolic distance of two points in the unit disc $\mathbb{U}$ we firstly define the hyperbolic length of a hyperbolic segment, then the hyperbolic length of a hyperbolic polygonal line and at the end we define hyperbolic length of arbitrary C^1 curve as the supremum of hyperbolic lengths of hyperbolic polygonal lines which is inscribed in that curve. It turns out that the hyperbolic length of a C^1 curve $\gamma : [a, b] \to \mathbb{U}$ that was introduced in this way is equal to the hyperbolic length of that curve, which is given by the formula (1).

2. The hyperbolic length of curve in the unit disc

Let $\gamma : [a, b] \to \mathbb{U}$ be an arbitrary curve and let $P : a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ be partition of the interval $[a, b]$. Analogous to the Euclidean case to the partition P and the curve γ we assign the number

$$\ell_{\mathrm{h}}(P, \gamma) = \sum_{j=1}^{n} d_{\mathrm{h}}(\gamma(t_j), \gamma(t_{j-1})),$$

where $d_{\mathrm{h}} : \mathbb{U} \times \mathbb{U} \to [0, +\infty)$ is defined by

$$d_{\mathrm{h}}(z_1, z_2) = \log \frac{1 + \left|\dfrac{z_1 - z_2}{1 - \overline{z_2} z_1}\right|}{1 - \left|\dfrac{z_1 - z_2}{1 - \overline{z_2} z_1}\right|}. \tag{2}$$

In other words, d_{h} is the hyperbolic distance in the unit disc. In the paper [13] the method of deriving the equality (2) without using the notion of hyperbolic density in the unit disc $\mathbb{U}$ is described in details.

Finally, the hyperbolic length of the curve γ (which we denote by $\ell_{\mathrm{h}}(\gamma)$) we define by

$$\ell_{\mathrm{h}}(\gamma) = \sup_{P \in \Pi} \ell_{\mathrm{h}}(P, \gamma),$$

where Π is the set of all partitions of the interval $[a, b]$. Moreover, if $\ell_{\mathrm{h}}(\gamma) < +\infty$, then we say that the curve γ is rectifiable in the hyperbolic sense.

As the main result of this paper, we state the following theorem.

Theorem 2.1. *Let $\gamma : [0, 1] \to \mathbb{U}$ be an arbitrary C^1 curve. Then*

$$\ell_{\mathrm{h}}(\gamma) = \int_0^1 \frac{2|\gamma'(t)|}{1 - |\gamma(t)|^2} dt = \int_\gamma \frac{2}{1 - |z|^2} |dz|.$$

3. Proof of the main result

We will prove Theorem 2.1 by formulating and proving two propositions (see Proposition 3.1 and Proposition 3.8) and several lemmas.

Note that by $\operatorname{tr}(\gamma)$ we will denote the trace of a curve $\gamma : [a,b] \to \mathbb{U}$, i.e. the set $\{\gamma(t) : t \in [a,b]\}$.

Proposition 3.1. *Let $\gamma : [0,1] \to \mathbb{U}$ be an arbitrary C^1 curve. Then*

$$\ell_{\mathrm{h}}(\gamma) \leqslant \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}dt. \tag{3}$$

Proof. Let $P : 0 = t_0 < t_1 < \ldots < t_{n-1} < t_n = 1$ be a partition of the interval $[0,1]$, let f_j ($j \in \{1,\ldots,n\}$) be conformal automorphism of the unit disc $\mathbb{U}$ such that $f_j(\gamma(t_{j-1})) = 0$ and $f_j(\gamma(t_j)) \geqslant 0$ and let $\Gamma_j = f_j \circ \gamma$. From classical complex analysis it is well known that f_j exist and preserves the distance d_{h}. So,

$$\begin{aligned} d_{\mathrm{h}}(\gamma(t_j),\gamma(t_{j-1})) &= d_{\mathrm{h}}(f_j(\gamma(t_j)), f_j(\gamma(t_{j-1}))) \\ &= d_{\mathrm{h}}(\Gamma_j(t_j), 0) \\ &= \log\frac{1+|\Gamma_j(t_j)|}{1-|\Gamma_j(t_j)|} \\ &= \log\frac{1+\operatorname{Re}\Gamma_j(t_j)}{1-\operatorname{Re}\Gamma_j(t_j)} \\ &= \int_{t_{j-1}}^{t_j}\frac{2\operatorname{Re}\Gamma_j'(t)}{1-(\operatorname{Re}\Gamma_j(t))^2}dt. \end{aligned} \tag{4}$$

It can immediately be shown that

$$\int_{t_{j-1}}^{t_j}\frac{2\operatorname{Re}\Gamma_j'(t)}{1-(\operatorname{Re}\Gamma_j(t))^2}dt \leqslant \int_{t_{j-1}}^{t_j}\frac{2|\Gamma_j'(t)|}{1-|\Gamma_j(t)|^2}dt \tag{5}$$

is valid and by the famous Schwarz-Pick lemma (see [3, Theorem 3.2]) we obtain

$$\int_{t_{j-1}}^{t_j}\frac{2|\Gamma_j'(t)|}{1-|\Gamma_j(t)|^2}dt = \int_{t_{j-1}}^{t_j}\frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}dt. \tag{6}$$

Hence, from (4), (5) and (6) we get

$$\sum_{j=1}^n d_{\mathrm{h}}(\gamma(t_j),\gamma(t_{j-1})) \leqslant \sum_{j=1}^n \int_{t_{j-1}}^{t_j}\frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}dt = \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}dt,$$

i.e.

$$\ell_{\mathrm{h}}(P,\gamma) \leqslant \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2}dt. \tag{7}$$

Finally, by taking the supremum over all partitions P of the interval $[0,1]$, from (7) we obtain (3). □

Note that from Proposition 3.1 it immediately follows that each C^1 curve $\gamma : [0,1] \to \mathbb{U}$ is rectifiable in the hyperbolic sense.

In order to prove Theorem 2.1, it is necessary to proved that in (3) the opposite inequality also holds. To prove that we will first formulate and prove several lemmas.

Lemma 3.2. *Let $\varepsilon > 0$ be arbitrary. Then there is $\delta > 0$ such that for all $z \in \mathbb{U}$, $|z| < \delta$ the following double inequality $0 \leqslant d_{\mathrm{h}}(0,z) - 2|z| < \varepsilon|z|$ holds.*

Proof. The proof trivially follows from the equalities

$$d_{\mathrm{h}}(0,z) = \log\frac{1+|z|}{1-|z|}$$

and

$$\lim_{|z|\to 0}\frac{1}{|z|}\left(\log\frac{1+|z|}{1-|z|} - 2|z|\right) = 0.$$

□

Before we formulate next lemma, for the sake of completeness we give definition of the parameter of a partition. Parameter of the partition $P : a = t_0 < t_1 < \ldots < t_{n-1} < t_n = b$ of the interval $[a,b]$ is

$$\lambda(P) = \max_{1\leqslant j\leqslant n}|t_j - t_{j-1}|.$$

Lemma 3.3. *Let $\gamma : [0,1] \to \mathbb{U}$ be a curve and $\varepsilon > 0$. Then there exists $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ and for all $j \in \{1,\ldots,n\}$ the inequality*

$$\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| < \varepsilon$$

holds.

Proof. Let m be the minimum of the function $f(z,w) = |1 - \bar{z}w|$ on the compact set $\operatorname{tr}(\gamma)\times\operatorname{tr}(\gamma)$. It is clear that $m > 0$. Since γ is uniformly continuous on $[0,1]$ there exists $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ and for all $j \in \{1,\ldots,n\}$ the inequality $|\gamma(t_j) - \gamma(t_{j-1})| < m\varepsilon$ holds. Hence for all $j \in \{1,\ldots,n\}$ we have

$$\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| \leqslant \frac{1}{m}|\gamma(t_j) - \gamma(t_{j-1})| < \varepsilon.$$

□

Lemma 3.4. *Let $\gamma : [0,1] \to \mathbb{U}$ be a curve and $\varepsilon > 0$. Then exists $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ and for all $j \in \{1,\ldots,n\}$ the double inequality*

$$0 \leqslant d_{\mathrm{h}}(\gamma(t_{j-1}),\gamma(t_j)) - 2\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| < \varepsilon|\gamma(t_j) - \gamma(t_{j-1})|$$

holds.

Proof. Let m be as in the proof of Lemma 3.3. By using Lemma 3.2 there exists $\delta_1 > 0$ such that

$$0 \leqslant d_{\mathrm{h}}(0,z) - 2|z| < m\varepsilon|z|,$$

for all $z \in \mathbb{U}$, $|z| < \delta_1$.

On other hand, by applying Lemma 3.3, there exists $\delta > 0$ such that for all partition

$$P : 0 = t_0 < t_1 < \ldots < t_n = 1,$$

of the interval $[0,1]$, for which $\lambda(P) < \delta$, and for all $j \in \{1,\ldots,n\}$ the following inequality

$$\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| < \delta_1$$

holds.

Hence, for any partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$, and for all $j \in \{1, \ldots, n\}$ we get

$$\begin{aligned} d_{\mathrm{h}}\left(0, \left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right|\right) - 2\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| &< m\varepsilon \left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| \\ &\leqslant \frac{m\varepsilon}{m}|\gamma(t_j) - \gamma(t_{j-1})| \\ &= \varepsilon|\gamma(t_j) - \gamma(t_{j-1})|. \end{aligned}$$

Finally, since

$$d_{\mathrm{h}}(\gamma(t_{j-1}), \gamma(t_j)) = d_{\mathrm{h}}\left(0, \left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right|\right),$$

we obtain the proof. $\square$

Lemma 3.5. *Let $\gamma : [0,1] \to \mathbb{U}$ be a curve. Then there exists constant $M > 0$ such that*

$$\left|\left|\frac{z - w}{1 - \overline{w}z}\right| - \left|\frac{z - w}{1 - \overline{w}w}\right|\right| \leq M|z - w|^2,$$

for all $z, w \in \mathrm{tr}\,(\gamma)$.

Proof. Let $z, w \in \mathrm{tr}\,(\gamma)$ be arbitrary points. Then

$$\begin{aligned} \left|\left|\frac{z - w}{1 - \overline{w}z}\right| - \left|\frac{z - w}{1 - \overline{w}w}\right|\right| &= |z - w|\left|\left|\frac{1}{1 - \overline{w}z}\right| - \left|\frac{1}{1 - \overline{w}w}\right|\right| \\ &\leqslant |z - w|\left|\frac{1}{1 - \overline{w}z} - \frac{1}{1 - \overline{w}w}\right| \\ &= |z - w|^2\left|\frac{\overline{w}}{(1 - \overline{w}z)(1 - \overline{w}w)}\right| \\ &\leqslant M|z - w|^2, \end{aligned}$$

where M is the maximum of the function $f(z_1, z_2) = \left|\frac{\overline{z_2}}{(1 - \overline{z_2}z_1)(1 - \overline{z_2}z_2)}\right|$ on the compact set $\mathrm{tr}\,(\gamma) \times \mathrm{tr}\,(\gamma)$. $\square$

Lemma 3.6. *Let $\gamma : [0,1] \to \mathbb{U}$ be a curve and $\varepsilon > 0$. Then there is $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ and for all $j \in \{1, \ldots, n\}$ the following inequality*

$$\left|\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| - \left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_{j-1})}\right|\right| < \varepsilon|\gamma(t_j) - \gamma(t_{j-1})|$$

holds.

Proof. Let M be maximum of the function $f(z_1, z_2) = \left|\frac{\overline{z_2}}{(1 - \overline{z_2}z_1)(1 - \overline{z_2}z_2)}\right|$ on the compact set $\mathrm{tr}\,(\gamma) \times \mathrm{tr}\,(\gamma)$. Since γ uniformly continuous on $[0,1]$ there is $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ and for all $j \in \{1, \ldots, n\}$ the inequality $|\gamma(t_j) - \gamma(t_{j-1})| < \frac{\varepsilon}{M}$ holds. Hence, by using lemma 3.5, we have

$$\left|\left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_j)}\right| - \left|\frac{\gamma(t_j) - \gamma(t_{j-1})}{1 - \overline{\gamma(t_{j-1})}\gamma(t_{j-1})}\right|\right| \leqslant M|\gamma(t_j) - \gamma(t_{j-1})|^2$$

$$
\begin{aligned}
&< M\frac{\varepsilon}{M}|\gamma(t_j)-\gamma(t_{j-1})| \\
&= \varepsilon|\gamma(t_j)-\gamma(t_{j-1})|.
\end{aligned}
$$

□

Lemma 3.7. *Let $\gamma : [0,1] \to \mathbb{U}$ be a C^1 curve and $\varepsilon > 0$. Then there is $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ and for all $j \in \{1, \ldots, n\}$ the following inequality*

$$\left|\frac{|\gamma(t_j)-\gamma(t_{j-1})|}{1-|\gamma(t_{j-1})|^2} - \frac{|\gamma'(t_{j-1})|}{1-|\gamma(t_{j-1})|^2}(t_j - t_{j-1})\right| < \varepsilon(t_j - t_{j-1})$$

holds.

Proof. Let $P : 0 = t_0 < t_1 < \ldots < t_n = 1$. Then

$$\left|\frac{|\gamma(t_j)-\gamma(t_{j-1})|}{1-|\gamma(t_{j-1})|^2} - \frac{|\gamma'(t_{j-1})|\cdot(t_j - t_{j-1})}{1-|\gamma(t_{j-1})|^2}\right| < \frac{|\gamma(t_j)-\gamma(t_{j-1})-\gamma'(t_{j-1})(t_j-t_{j-1})|}{1-|\gamma(t_{j-1})|^2}. \tag{8}$$

Further, let $\alpha = \operatorname{Re}\gamma$ and $\beta = \operatorname{Im}\gamma$. By the Lagrange mean value theorem, there are $\xi_j, \eta_j \in (t_{j-1}, t_j)$ such that

$$\gamma(t_j) - \gamma(t_{j-1}) = (\alpha'(\xi_j) + i\beta'(\eta_j))(t_j - t_{j-1}). \tag{9}$$

Let us denote by M the maximum of the function $f(z) = \dfrac{1}{1-|z|^2}$ on the compact set $\operatorname{tr}(\gamma)$. Since α' and β' are uniformly continous on $[0,1]$, it follows that there is $\delta > 0$, such that

$$|\alpha'(t) - \alpha'(s)| < \frac{\varepsilon}{2M} \tag{10}$$

and

$$|\beta'(t) - \beta'(s)| < \frac{\varepsilon}{2M}, \tag{11}$$

for all $t, s \in [0,1]$, $|t-s| < \delta$.

If $\lambda(P) < \delta$, then from (9), (10) and (11), we get

$$
\begin{aligned}
&|\gamma(t_j)-\gamma(t_{j-1})-\gamma'(t_{j-1})(t_j-t_{j-1})| \\
&= |(\alpha'(\xi_j)-\alpha'(t_{j-1}) + i(\beta'(\eta_j)-\beta'(t_{j-1}))|(t_j-t_{j-1}) \\
&< \left(\frac{\varepsilon}{2M}+\frac{\varepsilon}{2M}\right)(t_j-t_{j-1}) = \frac{\varepsilon}{M}(t_j-t_{j-1}).
\end{aligned} \tag{12}
$$

Finally, from (8) and (12) it follows that

$$\left|\frac{|\gamma(t_j)-\gamma(t_{j-1})|}{1-|\gamma(t_{j-1})|^2} - \frac{|\gamma'(t_{j-1})|}{1-|\gamma(t_{j-1})|^2}(t_j - t_{j-1})\right| < M\frac{\varepsilon}{M}(t_j - t_{j-1}) = \varepsilon(t_j - t_{j-1}).$$

□

Note that from next proposition the opposite inequality in (3) also holds, from which we obtain the Theorem 2.1.

Proposition 3.8. *Let $\gamma : [0,1] \to \mathbb{U}$ be an arbitrary C^1 curve and $\varepsilon > 0$. Then there is $\delta > 0$ such that for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$ of the interval $[0,1]$ for which $\lambda(P) < \delta$ the following inequality*

$$\left|\sum_{j=1}^{n} d_{\mathrm{h}}(\gamma(t_j), \gamma(t_{j-1})) - \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt\right| < \varepsilon$$

holds.

Proof. Let $P : 0 = t_0 < t_1 < \ldots < t_n = 1$ be an arbitrary partition of the interval $[0,1]$. Then

$$\left|\sum_{j=1}^{n} d_{\mathrm{h}}(\gamma(t_j), \gamma(t_{j-1})) - \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt\right| \leqslant |I_1| + |I_2| + |I_3| + |I_4|, \tag{13}$$

where

$$\begin{aligned}
I_1 &= \sum_{j=1}^{n} d_{\mathrm{h}}(\gamma(t_j), \gamma(t_{j-1})) - \sum_{j=1}^{n} \frac{2|\gamma(t_j) - \gamma(t_{j-1})|}{|1 - \overline{\gamma(t_{j-1})}\gamma(t_j)|},\\
I_2 &= \sum_{j=1}^{n} \frac{2|\gamma(t_j) - \gamma(t_{j-1})|}{|1 - \overline{\gamma(t_{j-1})}\gamma(t_j)|} - \sum_{j=1}^{n} \frac{2|\gamma(t_j) - \gamma(t_{j-1})|}{1 - |\gamma(t_{j-1})|^2},\\
I_3 &= \sum_{j=1}^{n} \frac{2|\gamma(t_j) - \gamma(t_{j-1})|}{1 - |\gamma(t_{j-1})|^2} - \sum_{j=1}^{n} \frac{2|\gamma'(t_{j-1})|}{1 - |\gamma(t_{j-1})|^2}(t_j - t_{j-1}),\\
I_4 &= \sum_{j=1}^{n} \frac{2|\gamma'(t_{j-1})|}{1 - |\gamma(t_{j-1})|^2}(t_j - t_{j-1}) - \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt.
\end{aligned}$$

Since γ is C^1 curve, it follows that γ is Euclidean rectifiable, i.e. $\ell_{\mathrm{e}}(\gamma) < +\infty$. Suppose, in addition, that γ is not a constant function. Then $\ell_{\mathrm{e}}(\gamma) > 0$. By using the Lemmas 3.4, 3.6 and 3.7 and definition of definite integral there is $\delta > 0$ such that for all partition

$$P : 0 = t_0 < t_1 < \ldots < t_n = 1$$

of the interval $[0,1]$ for which $\lambda(P) < \delta$, the following inequalities

$$\begin{aligned}
&|I_1| < \frac{\varepsilon}{4\ell_{\mathrm{e}}(\gamma)} \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| \leqslant \frac{\varepsilon}{4},\\
&|I_2| < \frac{\varepsilon}{4\ell_{\mathrm{e}}(\gamma)} \sum_{j=1}^{n} |\gamma(t_j) - \gamma(t_{j-1})| \leqslant \frac{\varepsilon}{4},\\
&|I_3| < \frac{\varepsilon}{4} \sum_{j=1}^{n} (t_j - t_{j-1}) = \frac{\varepsilon}{4}\\
&|I_4| < \frac{\varepsilon}{4}
\end{aligned} \tag{14}$$

hold.

Finally, from (13) and (14) for all partition $P : 0 = t_0 < t_1 < \ldots < t_n = 1$, $\lambda(P) < \delta$ we get

$$\left|\sum_{j=1}^{n} d_{\mathrm{h}}(\gamma(t_j), \gamma(t_{j-1})) - \int_0^1 \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt\right| < \varepsilon.$$

□

Proof. [Proof of the Theorem 2.1] Directly follows from Proposition 3.1 and Proposition 3.8. □

4. Appendix

In this section we give a short review of some applications related to the notion of hyperbolic density, as well as some recent results in which the hyperbolic density plays crucial role.

First of all, the classical result, usually called the Schwarz-Pick lemma for the unit disc, is well known (for example, see [3, Theorem 3.2]). This assertion can be stated in the following form: If $f : \mathbb{U} \to \mathbb{U}$ be a holomorphic function then

$$\rho_{\mathbb{U}}(f(z))|f'(z)| \leqslant \rho_{\mathbb{U}}(z), \quad \text{for all } z \in \mathbb{U}, \tag{15}$$

and

$$d_{\mathrm{h}}(f(z_1), f(z_2)) \leqslant d_{\mathrm{h}}(z_1, z_2), \quad \text{for all } z_1, z_2 \in \mathbb{U}. \tag{16}$$

Thereby, the equalities hold in (15) and (16) if and only if f is bijection, i.e. conformal automorphism of $\mathbb{U}$.

Further, if $\Omega \subset \mathbb{C}$ is simply connected domain, different from $\mathbb{C}$, then Ω is conformally equivalent to the unit disc $\mathbb{U}$ by the Riemann mapping theorem. In other words, there exist a conformal isomorphism $\phi : \Omega \to \mathbb{U}$, i.e. ϕ is univalent holomorphic mapping form Ω onto $\mathbb{U}$. This fact allows us to define the hyperbolic density on Ω as follows

$$\rho_{\Omega}(z) = \rho_{\mathbb{U}}(\phi(z))|\phi'(z)|, \quad z \in \Omega,$$

and hyperbolic distance on Ω as follows

$$d_{\Omega}(z_1, z_2) = d_{\mathrm{h}}(\phi(z_1), \phi(z_2)), \quad z_1, z_2 \in \Omega,$$

It can be easily shown that these definitions do not depend on the choice of mapping ϕ. Moreover, a conformal isomorphism $\phi : \Omega \to \mathbb{U}$ allows us to consider the domain Ω as a model of hyperbolic plane.

Example 4.1. *Let $\mathbb{S} = \{z \in \mathbb{C} : -1 < \operatorname{Re} z < 1\}$ be the vertical strip. Then $\phi : \mathbb{S} \to \mathbb{U}$ defined by $\phi(z) = \tan\left(\frac{\pi}{4} z\right)$ is a conformal isomorphism. By direct calculation, we obtain*

$$\rho_{\mathbb{S}}(z) = \frac{\pi}{2} \frac{1}{\cos\left(\frac{\pi}{2} \operatorname{Re} z\right)}. \tag{17}$$

Note that

$$\rho_{\mathbb{S}}(z) = \rho_{\mathbb{S}}(\operatorname{Re} z), \quad \textit{for all } z \in \mathbb{S}. \tag{18}$$

Example 4.2. *Let $\mathbb{K} = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}$ be the right half-plane. Then $\psi : \mathbb{K} \to \mathbb{U}$ defined by $\psi(z) = \dfrac{z-1}{z+1}$ is a conformal isomorphism. By direct calculation, we obtain*

$$\rho_{\mathbb{K}}(z) = \frac{1}{\operatorname{Re} z}. \tag{19}$$

Note that

$$\rho_{\mathbb{K}}(z) = \rho_{\mathbb{K}}(\operatorname{Re} z), \quad \textit{for all } z \in \mathbb{K}. \tag{20}$$

As we have a defined hyperbolic density on an arbitrary simply connected domain $\Omega \subset \mathbb{C}$, different from $\mathbb{C}$, we can formulate the Schwarz-Pick lemma for simply connected domains (for example, see [3, Theorem 6.4]): Let $\Omega_1, \Omega_2 \subset \mathbb{C}$ be simply connected domains and let $f : \Omega_1 \to \Omega_2$ be holomorphic function. Then

$$\rho_{\Omega_2}(f(z))|f'(z)| \leqslant \rho_{\Omega_1}(z), \quad \text{for all } z \in \Omega_1, \tag{21}$$

and

$$d_{\Omega_2}(f(z_1), f(z_2)) \leqslant d_{\Omega_1}(z_1, z_2), \quad \text{for all } z_1, z_2 \in \Omega_1. \tag{22}$$

Thereby, the equalities hold in (21) and (22) if and only if f is bijection, i.e. conformal isomorphism of the domain Ω_1 onto the domain Ω_2.

Recently, some versions of the Schwarz-Pick lemma and it's generalizations for harmonic functions and for harmonic quasiregular mappings, in particular for harmonic quasiconformal mappings, have been considered (see, for example, [4, 5, 8–10, 12, 14, 15]). Also, for further results we refer the interested reader to the [6, 7].

For example, using the Schwarz-Pick lemma for simply connected domains, facts (18) and (20), as well as the fundamental relationship between harmonic and holomorphic functions defined on $\mathbb{U}$ (or on a simply connected domains), i.e. the fact that every real harmonic function defined on a simply connected domain is the real part of a holomorphic function defined on such a domain, the following recent results can be elegantly proved (see [12, 15]):

1) If $u : \mathbb{U} \to (-1, 1)$ is a harmonic function, then

$$|\nabla u(z)| \leqslant \frac{\rho_{\mathbb{U}}(z)}{\rho_{\mathbb{S}}(u(z))} = \frac{4}{\pi} \frac{\cos\left(\dfrac{\pi}{2} u(z)\right)}{1 - |z|^2}, \quad z \in \mathbb{U},$$

and hence

$$|\nabla u(z)| \leqslant \frac{4}{\pi} \frac{\rho_{\mathbb{U}}(z)}{\rho_{\mathbb{U}}(u(z))} = \frac{4}{\pi} \frac{1 - |u(z)|^2}{1 - |z|^2}, \quad z \in \mathbb{U}.$$

2) If $u : \mathbb{U} \to (0, +\infty)$ is a harmonic function, then

$$|\nabla u(z)| \leqslant \frac{\rho_{\mathbb{U}}(z)}{\rho_{\mathbb{K}}(u(z))} = \frac{2u(z)}{1 - |z|^2}, \quad z \in \mathbb{U}.$$

Here by ∇u we denote the gradient (u_x, u_y) of the function u.

Assertions 1) and 2), together with their proofs, are also called the strip method and half-plane method, respectively. Let us note that the mentioned methods represent a new approach when obtaining results of this type that describe the distortion of harmonic functions.

As a consequence of assertions 1) and 2), the following statements can also be proved:

3) If $u : \mathbb{U} \to (-1, 1)$ is a harmonic function, then

$$d_{\mathbb{S}}(u(z_1), u(z_2)) \leqslant d_{\mathbb{U}}(z_1, z_2), \quad z_1, z_2 \in \mathbb{U}$$

and

$$d_{\mathbb{U}}(u(z_1), u(z_2)) \leqslant \frac{4}{\pi} d_{\mathbb{U}}(z_1, z_2), \quad z_1, z_2 \in \mathbb{U},$$

4) If $u : \mathbb{U} \to (0, +\infty)$ is a harmonic function, then

$$d_{\mathbb{K}}(u(z_1), u(z_2)) \leqslant d_{\mathbb{U}}(z_1, z_2), \quad z_1, z_2 \in \mathbb{U}.$$

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