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Nonlocal second order delay semilinear integro-differential time varying evolution equation with convex-power condensing operators

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Abstract. In this paper, we present findings concerning the existence of solutions to a particular type of nonlocal second-order delay semilinear integro-differential equation with time-varying evolution. We utilize the theory of the resolvent family, the Kuratowski measure of noncompactness, and fixed point theorems in conjunction with a convex-power condensing operator to establish novel and sufficient conditions for ensuring the existence of mild solutions. Finally, we provide an illustrative example to demonstrate the validity of the proposed results.

1. Introduction

The investigation of initial value problems for evolution equations with local or nonlocal conditions has various applications in physics and other fields of applied mathematics. It is widely recognized that time delays are commonly observed in numerous industrial and practical systems, including but not limited to chemical processing, bioengineering, fuzzy systems, automatic control, neural networks, circuits, and vehicle suspension systems [6, 8, 23, 36]. For more recent results on differential and integral equations, see [2, 3, 9, 10, 16–18, 22, 33, 34].

Currently, numerous researchers worldwide are investigating semilinear integrodifferential equations with nonlocal and delay properties. Recent articles on the topic can be found in [4, 5, 12–14, 21, 28–30, 32, 35, 41], and additional resources include the monographs of [1]. In many instances, it is more advantageous to directly treat second-order abstract differential equations, rather than converting them to first-order systems. The theory of abstract second-order equations in the autonomous case is associated with the concept of cosine families. Serizawa and Watanabe [37] studied non-autonomous abstract second-order equations and assumed that $\mathcal{A}(t) = A + \mathcal{B}(t)$, where A is the infinitesimal generator of a strongly continuous cosine family $C_0(t)$ in a Banach space X, and $\mathcal{B}(t)$ is a family of operators for which the

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mapping $t \to \mathcal{B}(t)x$ is continuously differentiable for all $x \in X$ and for which the function $C_0(\cdot)x$ is of class C^1.

The existence of solutions to the non-autonomous second-order abstract Cauchy problem associated with the family $\{\mathcal{A}(t) : t \in [0, d]\}$ is closely tied to the idea of an evolution operator created by this family. The first concept was introduced by Kozak [31] and expanded upon in subsequent works such as [11, 15, 20, 26, 27, 37, 39, 40, 42].

The aim of this paper is to discuss the following nonlocal second order delay semilinear integro-differential time varying evolution equation:

$$\begin{cases} \dfrac{d^2y(t)}{dt^2} = \mathcal{A}(t)y(t) + \mathcal{N}\left(t, y_t, \displaystyle\int_0^t \mathcal{M}(t,s,y_s)ds\right), \ t \in [0,d], \\ y(t) = \phi(t) + (\hbar y)(t), \ t \in [-\eta, 0], \quad y'(0) = y_1, \end{cases} \tag{1}$$

where E is a Banach space endowed with a norm $|\cdot|$, $\mathfrak{I} = [0, d]$, $\{\mathcal{A}(t)\}_{0\le t\le d} : D(\mathcal{A}) \subset E \to E$ is a linear closed operator, that generates an evolution system of linear bounded operators $\{\mathcal{G}(t, s)\}_{(t,s)\in\mathfrak{D}}$. $\mathcal{N} : \mathfrak{I} \times \Omega \times E \to E$ is a given function and $\mathcal{M} : \Delta \times \Omega \to E, \hbar : \mathfrak{C}(\mathfrak{I}, E) \to \Omega,\ \Omega = \mathfrak{C}([-\eta, 0], E),\ \Delta = \{(t, s) \in \mathfrak{I} \times \mathfrak{I} : s \le t\}$.

The layout of our work is as follows. In Section 2, we provide some essential preliminary information that is required for the subsequent sections. In Section 3, we prove the existence theorem of a mild solution of (1) under certain sufficient conditions, using the measure of noncompactness and fixed-point theorem with respect to a convex-power condensing operator. In Section 4, we present an example that serves to illustrate our results.

2. Preliminaries

In this section, we present the notations, definitions, lemmas and preliminary facts needed to obtain our main results.

Denote by $\mathfrak{C}(\mathfrak{I}, E)$, the space of all continuous E-valued functions on interval $\mathfrak{I}$ which is a Banach space with the norm

$$\|y\| = \sup\{|y(t)|, t \in \mathfrak{I}\}.$$

$L^p(\mathfrak{I}, E)$ denotes the space of E-valued Bochner functions on $\mathfrak{I}$ with the norm

$$\|y\|_{L^p} = \left(\int_0^d |y(t)|^p dt\right)^{\frac{1}{p}}, \qquad p \ge 1.$$

$\mathfrak{B}(E)$ is the Banach space of bounded linear operators from E into E.

In what follows, let $\{\mathcal{A}(t),\ t \ge 0\}$ be a family of closed linear operators on the Banach space E with domain $D(\mathcal{A}(t))$ which is dense in E and independent of t.

First we recall the concept of the evolution operator $\mathcal{G}(t, s)$ for problem:

$$\frac{d^2y(t)}{dt^2} = \mathcal{A}(t)y(t), \text{ for } t \ge 0, \tag{2}$$

introduced by Kozak in [31] and recently used by Henríquez, Poblete and Pozo in [26].

Definition 2.1 ([26]). *A family of bounded linear operators $\mathcal{G}(t, s) : \mathfrak{I} \times \mathfrak{I} \to E$, $(t, s) \in \Delta = \{(t, s) \in \mathfrak{I} \times \mathfrak{I} : s \le t\}$, is called an evolution operator of the equation (2) if the following conditions are satisfied:*

(e_1) *For any $y \in E$ the function $\mathcal{G}(\cdot, \cdot)y : \mathfrak{I} \times \mathfrak{I} \to E$ is of class C^1 and*

(a) *for any* $t \in \Im$, $\mathcal{G}(t,t) = 0$,

(b) *for all* $(t,s) \in \mathfrak{D}$ *and for any* $y \in E$, $\frac{\partial}{\partial t}\mathcal{G}(t,s)y\Big|_{t=s} = y$ *and* $\frac{\partial}{\partial s}\mathcal{G}(t,s)y\Big|_{t=s} = -y$.

(e_2) *For all* $(t,s) \in \Delta$, *if* $y \in D(\mathcal{A}(t))$, *then* $\frac{\partial}{\partial s}\mathcal{G}(t,s)y \in D(\mathcal{A}(t))$, *the map* $(t,s) \longmapsto \mathcal{G}(t,s)y$ *is of class* C^2 *and*

(a) $\frac{\partial^2}{\partial t^2}\mathcal{G}(t,s)y = \mathcal{A}(t)\mathcal{G}(t,s)y$,

(b) $\frac{\partial^2}{\partial s^2}\mathcal{G}(t,s)y = \mathcal{G}(t,s)\mathcal{A}(s)y$,

(c) $\frac{\partial^2}{\partial s \partial t}\mathcal{G}(t,s)y\Big|_{t=s} = 0$.

(e_3) *For all* $(t,s) \in \Delta$, *then* $\frac{\partial}{\partial s}\mathcal{G}(t,s)y \in D(\mathcal{A}(t))$, *there exist*

$$\frac{\partial^3}{\partial t^2 \partial s}\mathcal{G}(t,s)y, \quad \frac{\partial^3}{\partial s^2 \partial t}\mathcal{G}(t,s)y,$$

and

(a) $\frac{\partial^3}{\partial t^2 \partial s}\mathcal{G}(t,s)y = \mathcal{A}(t)\frac{\partial}{\partial s}(t)\mathcal{G}(t,s)y$. *Moreover, the map*

$$(t,s) \longmapsto \mathcal{A}(t)\frac{\partial}{\partial s}(t)\mathcal{G}(t,s)y$$

is continuous,

(b) $\frac{\partial^3}{\partial s^2 \partial t}\mathcal{G}(t,s)y = \frac{\partial}{\partial t}\mathcal{G}(t,s)\mathcal{A}(s)y$.

Next, we introduce the definition for Kuratowski measure of noncompactness, which will be used in the proof of our main results.

Definition 2.2 ([7]). *The Kuratowski measure of noncompactness* $\beta_E(\cdot)$ *defined on bounded set* $\mathfrak{V}$ *of Banach space* E *is*

$$\beta_E(\mathfrak{V}) = \inf\{\ell > 0 : \mathfrak{V} \text{ has a finite cover by sets of diameter} \leq \ell\},$$

We now recall some useful properties of Kuratowski measures of noncompactness. For more details, please refer [7].

Lemma 2.3 ([7]). *Let* $\mathfrak{V}$ *be bounded set of* E. *The Kuratowski measure of noncompactness satisfies some properties:*

(V_1) $\beta_E(\mathfrak{V}) = 0$, *if and only if* $\overline{\mathfrak{V}}$ *is compact, where* $\overline{\mathfrak{V}}$ *means the closure of* $\mathfrak{V}$,

(V_2) $\beta_E(\overline{\mathfrak{V}}) = \beta_E(\mathfrak{V}) = \beta_E(conv\mathfrak{V})$, *where* $conv\mathfrak{V}$ *means the convex hull of* $\mathfrak{V}$,

(V_3) $\beta_E(\mathfrak{V}_1) \leq \beta_E(\mathfrak{V}_2)$ *when* $\mathfrak{V}_1 \subset \mathfrak{V}_2$,

(V_4) $\beta_E(\mathfrak{V}_1 + \mathfrak{V}_2) \leq \beta_E(\mathfrak{V}_1) + \beta_E(\mathfrak{V}_2)$, *where*

$$\mathfrak{V}_1 + \mathfrak{V}_2 = \{x \mid x = y + z, y \in \mathfrak{V}_1, z \in \mathfrak{V}_2\},$$

(V_5) $\beta_E(c\mathfrak{V}) = |c|\beta_E(\mathfrak{V})$ *for any* $c \in \mathbb{R}$,

(V_6) $\beta_E(\mathfrak{V} + y) = \beta_E(\mathfrak{V})$ *for any* $y \in E$, *where* $\mathfrak{V} + y = \{x \mid x = y + z, z \in \mathfrak{V}\}$.

The map $Q : X \subset E \to E$ is said to be a β_E-contraction if there exists a positive constant $k < 1$ such that $\beta_E(Q(\mathfrak{V})) \leq k\beta_E(\mathfrak{V})$ for any bounded closed subset $\mathfrak{V} \subset E$, where E is a Banach space.

Lemma 2.4 ([19]). *Let E be a Banach space,* $\mathfrak{V} \subset E$ *be bounded. Then, there exists a countable set* $W_0 \subset \mathfrak{V}$, *such that*

$$\beta_E(\mathfrak{V}) \leq 2\beta_E(W_0).$$

Lemma 2.5 ([24]). *Let E be a Banach space, and let* $\mathfrak{V} = \{y_n\}_{n\in\mathbb{N}} \subset \mathfrak{C}([0, d], E)$ *be a bounded and countable. Then,* $\beta_E(\mathfrak{V}(t))$ *is Lebesgue integral on* $[0, d]$, *and*

$$\beta_E\Big(\Big\{\int_0^d y_n(t)dt : n \in \mathbb{N}\Big\}\Big) \leq 2\int_0^d \beta_E(\mathfrak{V}(t))dt.$$

Denote $\beta_{\mathfrak{C}}$ by the Kuratowski measure of noncompactness of $\mathfrak{C}(\mathfrak{I}, E)$. Before we prove the existence results, we need the following results.

Lemma 2.6 ([7]). *If* $\mathfrak{V} \subset \mathfrak{C}(\mathfrak{I}, E)$ *is bounded, then* $\beta_E(\mathfrak{V}(t)) \leq \beta_{\mathfrak{C}}(\mathfrak{V})$, *for all* $t \in \mathfrak{I}$, *where* $\mathfrak{V}(t) = \{y(t); y \in \mathfrak{V}\}$. *Furthermore if* $\mathfrak{V}$ *is equicontinuous on* $\mathfrak{I}$, *then* $\beta_E(\mathfrak{V}(t))$ *is continuous on* $\mathfrak{I}$ *and*

$$\beta_{\mathfrak{C}}(\mathfrak{V}) = \sup_{t\in\mathfrak{I}} \beta_E(\mathfrak{V}(t)).$$

Theorem 2.7 ([38]). *Let E be a Banach space, let* $\Theta \subset E$ *be bounded, closed and convex. Suppose that* $\Pi : \Theta \to \Theta$ *is a continuous operator and* $\Pi(\Omega)$ *is bounded. For any* $D \subset \Theta$ *and* $y_0 \in D$, *set*

$$\Pi^{(1,y_0)}(D) = \Pi(D),$$
$$\Pi^{(m,y_0)}(D) = \Pi\left(\overline{conv}\,\Pi^{(m-1,y_0)}(D), y_0\right).$$

If there exist $y_0 \in \Theta$ *and a positive integer* m_0 *such that for any bounded and noncompact subset* $D \subset \Theta$,

$$\beta_E(\Pi^{(m,y_0)}(D)) \leq \beta_E(D),$$

then Π *has at least one fixed point in* Θ.

3. Existence Results

In this section, we will establish the existence results for the problem (1) by using the Kuratowski measure of noncompactness. Firstly, let us propose the definition of the mild solution of system (1).

Definition 3.1. *A function* $y \in \mathfrak{C}([-\eta, d], E)$ *is said to be a mild solution of system (1) if*

(1) *for* $t \in [-\eta, 0]$, $y(t) = \phi(t) + (\hbar y)(t)$,
(2) *for each* $0 \leq t \leq d$, $y(t)$ *satisfies the following integral equation:*

$$(\mathbb{Q}y)(t) = -\frac{\partial}{\partial s}\mathcal{G}(t, 0)\,(\varphi(0) + \hbar(y)(0)) + \mathcal{G}(t, 0)y_1 + \int_0^t \mathcal{G}(t, s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau)d\tau\right)ds.$$

Before stating and proving the main results, we introduce the following hypotheses:

$(\mathbb{H}_1)$ There exist constants $N_1 > 0, N_2 > 0$ and $\lambda > 0$, such that

$$\|\mathcal{G}(t, s)\|_{B(E)} \leq N_1 e^{-\lambda(t-s)}, \qquad \left\|\frac{\partial}{\partial s}\mathcal{G}(t, s)\right\|_{B(E)} \leq N_2, (t, s) \in \Delta.$$

($\mathbb{H}_2$) The function $\mathcal{N} : \Im \times \Omega \times E \to E$ satisfies:

(i) For every $t \in \Im$, the function $\mathcal{N}(t, \cdot, \cdot) : \Omega \times E \to E$ is continuous, and for each $(y_1, y_2) \in \Omega \times E$, the function $\mathcal{N}(\cdot, y_1, y_2) : \Im \to E$ is strongly measurable.

(ii) There exist $\Phi_{\mathcal{N}} \in L^{\frac{1}{p}}(\Im, \mathbb{R}^+)$, $p \in [1, \infty)$ and a continuous nondecreasing function $\Psi_{\mathcal{N}} : [0, \infty) \to (0, \infty)$ such that:

$$\left|\mathcal{N}(t, y_1, y_2)\right| \leq \Phi_{\mathcal{N}}(t)\Psi_{\mathcal{N}}(\|y\|_\Omega + \left|y_2\right|) \text{ for a.e } t \in \Im \text{ and each } y_1 \in \Omega, y_2 \in E,$$

where $\Psi_{\mathcal{N}}$ satisfies $\lim_{r\to+\infty} \inf \dfrac{\Psi_{\mathcal{N}}(r)}{r} = \Lambda_{\mathcal{N}}$.

(iii) There exists a function $\rho \in L^\infty(\Im, \mathbb{R}^+)$, such that $\|\rho\|_{L^\infty} \in \left(0, \dfrac{1}{2N_1}\right]$ and for any bounded and countable set $V_1 \subset \Omega, V_2 \subset E$, we have

$$\beta_E(\mathcal{N}(t, V_1, V_2)) \leq \rho(t)\left(\sup_{\theta\in[-\eta,0]} \beta_E(V_1(\theta)) + \beta_E(V_2)\right), \text{ for a.e } t \in \Im.$$

($\mathbb{H}_3$) The function $\mathcal{M} : \Delta \times \Omega \to E$ satisfies the following:

(i) There exists a positive function $\xi(t, s) \in L^1(\Delta, \mathbb{R}^+)$ such that:

$$\left|\mathcal{M}(t, s, y)\right| \leq \xi(t, s)\|y\|_\Omega, \quad \xi^* = \sup_{t\in\Im} \int_0^t \xi(t, s)ds < +\infty,$$

for a.e $t \in \Im$ and each $y \in \Omega$.

(ii) There exists a positive number K^*, such that for any bounded and countable set $V \subset \Omega$, we have

$$\beta_E(\mathcal{M}(t, s, V) \leq K^* \sup_{\theta\in[-\eta,0]} \beta_E(V(\theta)) \text{ for a.e } t \in J.$$

($\mathbb{H}_4$) The operator $\hbar : \mathfrak{C}(\Im, E) \to \Omega$, satisfies the following:

(i) For each $t \in [-\eta, 0]$, the operator $\Gamma : \mathfrak{C}(\Im, E)$ defined by $\Gamma_t(y) = (\hbar y)(t)$ is continuous and the subset $\hbar(\Lambda) \subset \Omega$ is equicontinuous for each bounded set $\Lambda \subset \mathfrak{C}(\Im, E)$.

(ii) There exist a positive number $N_\hbar$ and a continuous nondecreasing function $\Psi_\hbar : [0, \infty) \to (0, \infty)$ such that:

$$\|\hbar(y)\|_\Omega \leq N_\hbar\Psi_\hbar(\|y\|),$$

where $\Psi_\hbar$ satisfies $\lim_{r\to+\infty} \inf \dfrac{\Psi_\hbar((r)}{r} = \Lambda_\hbar \in \left[0, \dfrac{1}{N_\hbar}\right)$.

(iii) There exists a positive number $K_\hbar \in \left[0, \dfrac{1}{\max(N_2, 2)}\right]$ such that for any bounded and countable set $V \subset \mathfrak{C}(\Im, E)$

$$\beta_E(\hbar(V(t))) \leq K_\hbar\beta_E(V(t)) \text{ for a.e } t \in [-\eta, 0].$$

Theorem 3.2. *Assume that the conditions* ($\mathbb{H}_1$) − ($\mathbb{H}_4$) *hold. If*

$$N_2N_\hbar\Lambda_\hbar + \frac{(1-p)^{1-p}\|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{p}}}}{\lambda^{1-p}}\Lambda_{\mathcal{N}} < 1, \tag{3}$$

then the nonlocal Cauchy problem (1) has at least a mild solution on $[-\eta, d]$.

Proof. Let us consider the operator $\mathbb{Q} : \mathfrak{C}([-\eta, d], E) \to \mathfrak{C}([-\eta, d], E)$, defined by:

$$(\mathbb{Q}y)(t) = \begin{cases} \phi(t) + \hbar(y)(t), \quad t \in [-\eta, 0], \\ -\frac{\partial}{\partial s}\mathcal{G}(t,0)\left(\varphi(0) + \hbar(y)(0)\right) + \mathcal{G}(t,0)y_1 \\ + \displaystyle\int_0^t \mathcal{G}(t,s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)ds, \ t \in \mathfrak{I}. \end{cases} \tag{4}$$

It is obvious that the fixed point of $\mathbb{Q}$ is the mild solution of (1). We now show that $\mathbb{Q}$ satisfies all the assumptions of Theorem 2.7. To simplify the result, we subdivide the proof into four steps. Let

$$\mathfrak{A}_r = \{y \in \mathfrak{C}([-\eta, d], E) : \|y\|_{\mathfrak{C}} \leq r\},$$

where r be any positive constant. Then, $\mathfrak{A}_r$ is clearly a bounded closed and convex subset in $\mathfrak{C}([-\eta, d], E)$.

Step 1. We prove that there exits r such that $\mathbb{Q}$ maps $\mathfrak{A}_r$ into $\mathfrak{A}_r$.
If we assume that $\mathbb{Q}(\mathfrak{A}_r) \not\subseteq \mathfrak{A}_r$, then for every positive constant r and $t \in \mathfrak{I}$, there exists a $y_r(\cdot) \in \mathfrak{A}_r$, such that $|y(t)| > r$. For $t \in [-\eta, 0]$, and from the hypotheses $(\mathbb{H}_1)$, $(\mathbb{H}_4)$, we get

$$\begin{aligned} \left|y(t)\right| &\leq |\phi(t)| + |\hbar(y)(t))| \\ &\leq \|\phi\|_\Omega + |\hbar(y)(t))| \\ &\leq \|\phi\|_\Omega + N_\hbar \Psi_\hbar(\|y\|_\Omega). \end{aligned}$$

Dividing both sides by r and, let $r \to +\infty$, we get

$$1 \leq N_\hbar \lim_{r\to+\infty} \frac{\Psi_\hbar((r)}{r}.$$

Then,

$$1 \leq N_\hbar \Lambda_\hbar,$$

which is a contradiction to $(\mathbb{H}_4)(ii)$.
For $t \in [0, d]$, from the hypotheses $(\mathbb{H}_1)$-$(\mathbb{H}_4)$, we get

$$\begin{aligned} \left|y(t)\right| \leq& \left\|\frac{\partial}{\partial s}\mathcal{G}(t,0)\right\|_{\mathfrak{B}(E)} (|\phi(0)| + \|\hbar(y)(0))\|_{\mathfrak{B}(E)} + \|\mathcal{G}(t,0)\|_{\mathfrak{B}(E)} |y_1| \\ &+ \int_0^t \|\mathcal{G}(t,s)\|_{\mathfrak{B}(E)} \Phi_{\mathcal{N}}(s)\Psi_{\mathbb{F}}\left(|y_s| + \int_0^s \xi(s,\tau)|y_\tau|d\tau\right)ds \\ \leq& N_2(|\phi(0)| + N_\hbar) + N_1\left|y_1\right| \\ &+ N_1 \int_0^t e^{-\lambda(t-s)}\Phi_{\mathcal{N}}(s)\Psi_{\mathcal{N}}\left(|y_s| + \int_0^s \xi(s,\tau)|y_\tau|d\tau\right)ds. \\ \leq& N_2(|\phi(0)| + N_\hbar\Psi_\hbar(\|y\|_C)) + N_1\left|y_1\right| \\ &+ \int_0^t N_1 e^{-\lambda(t-s)}\Phi_{\mathcal{N}}(s)\Psi_{\mathcal{N}}\left(|y_s| + \int_0^s \xi(s,\tau)|y_\tau|d\tau\right)ds. \\ \leq& N_2(|\phi(0)| + N_\hbar\Psi_\hbar(\|y\|_C)) + N_1\left|y_1\right| \\ &+ \int_0^t N_1 e^{-\lambda(t-s)}\Phi_{\mathcal{N}}(s)\Psi_{\mathcal{N}}\left(|y_s| + \int_0^s \xi(s,\tau)|y_\tau|d\tau\right)ds. \\ \leq& N_2(|\phi(0)| + N_\hbar\Psi_\hbar(\|y\|_C)) + N_1\left|y_1\right| \end{aligned}$$

$$
\begin{aligned}
&+ \int_0^t N_1 e^{-\lambda(t-s)} \Phi_{\mathcal{N}}(s) \Psi_{\mathcal{N}} \left(\sup |y_s| + \xi^* \sup |y_\tau|) d\tau \right) ds. \\
&\leq N_2(|\phi(0)| + N_\hbar \Psi_\hbar(\|y\|_C)) + N_1 \left| y_1 \right| \\
&\quad + \int_0^t N_1 e^{-\lambda(t-s)} \Phi_{\mathcal{N}}(s) \Psi_{\mathcal{N}} \left((1+\xi^*) \|y\|_\Omega \right) ds. \\
&\leq N_2(|\phi(0)| + N_\hbar \Psi_\hbar(\|y\|_C)) + N_1 \left| y_1 \right| \\
&\quad + \int_0^t N_1 e^{-\lambda(t-s)} \Phi_{\mathcal{N}}(s) \Psi_{\mathcal{N}} \left((1+\xi^*) \|y\|_\Omega \right) ds. \\
&\leq N_2(|\phi(0)| + N_\hbar \Psi_\hbar(\|y\|_C)) + N_1 \left| y_1 \right| \\
&\quad + N_1 \left(\int_0^t e^{-\frac{\lambda(t-s)}{1-p}} ds \right)^{1-p} \left(\int_0^t (\Phi_{\mathcal{N}}(s))^{\frac{1}{p}} ds \right)^p \Psi_{\mathcal{N}}((1+\xi^*)\|y\|_\Omega).
\end{aligned}
$$

Then,

$$
\begin{aligned}
|y)(t)| &\leq N_1 \left| y_1 \right| + N_2 |\phi(0)| + N_2 N_\hbar \Psi_\hbar(\|y\|_C) \\
&\quad + \frac{N_1 (1-p)^{1-p} \|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{p}}}}{\lambda^{1-p}} \Psi_{\mathcal{N}}((1+\xi^*)\|y\|_C).
\end{aligned}
$$

Dividing both sides by r and letting $r \to +\infty$, we get

$$
1 \leq N_2 N_\hbar \lim_{r \to +\infty} \frac{\Psi_\hbar((r)}{r} + \frac{N_1 (1-p)^{1-p} \|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{p}}}}{\lambda^{1-p}} \lim_{r \to +\infty} \frac{\Psi_{\mathcal{N}}((1+\xi^*)r)}{r}.
$$

Then,

$$
1 \leq N_2 N_\hbar \Lambda_\hbar + \frac{(1-p)^{1-p} \|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{p}}}}{\lambda^{1-p}} \Lambda_{\mathcal{N}},
$$

which is a contradiction to (3). Thus, there exists $r > 0$ such that $\mathbb{Q}(\mathfrak{A}_r) \subset \mathfrak{A}_r$.

Step 2. $\mathbb{Q}$ is continuous.
Let $(y_n)_{n \in N}$ be a sequence in $\mathfrak{C}(\mathfrak{I}, E)$ such that $y_n \to y$ in $\mathfrak{C}(\mathfrak{I}, E)$. Define

$$
\mathcal{N}_n(s) = \mathcal{N}(s, y_s^n, \int_0^s \mathcal{M}(s, \tau, y_\tau^n) d\tau):
$$

$$
\mathcal{N}(s) = \mathcal{N}(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau) d\tau).
$$

Using the assumptions $(\mathbb{H}_2)$ and $(\mathbb{H}_4)$, for $t \in \mathfrak{I}$, we obtain that

- $\mathcal{N}_n(t) \to \mathcal{N}(t)$ as $n \to +\infty$;
- $\hbar(y_n)(t) \to \hbar(y)(t)$ as $n \to +\infty$.

On the other hand, By (4), for every $t \in \mathfrak{I}$ and $y_n, y \in \mathfrak{A}_r$, we have

$$
\left| (\mathbb{Q}y_n)(t) - (\mathbb{Q}y)(t) \right| \leq N_2 \|\hbar(y_n)(t) \to \hbar(y)(t)\| + N_1 \int_0^t |\mathcal{N}_n(s) - \mathcal{N}(s)| ds.
$$

By the help of Lebesgue dominated convergence theorem, we have that

$$
\mathbb{Q}y_n \to \mathbb{Q}y \qquad \text{as} \quad n \to +\infty.
$$

From $(\mathbb{H}_2)(i)$, we find $\mathbb{Q}y_n \to \mathbb{Q}y \to 0$, as $n \to +\infty$ pointwise on $[-\eta, 0]$. Thus, we conclude that $\mathbb{Q}$ is continuous.

Step 3. $\mathbb{Q}$ is equicontinuous.
Take $-\eta \le t_1 \le t_2 \le 0$. For each $y \in \mathfrak{A}_r$ we have

$$|(\mathbb{Q}y)(t_2) - (\mathbb{Q}y)(t_1)| \le |\phi(t_2) - \phi(t_1)| + |\hbar(y)(t_2)) - \hbar(y)(t_1))|.$$

Since $\phi \in \Omega$, and from $(\mathbb{H}_2)(ii)$, when $t_2 \to t_1$ the right-hand side of the above inequality tends to zero.
Take $-\eta \le t_1 \le 0 \le t_2 \le d$ and for each $y \in \mathfrak{A}_r$, we get

$$\begin{aligned}
|(\mathbb{Q}y)(t_2) - (\mathbb{Q}y)(t_1)| &= \Big|\varphi(t_2) + \hbar(y)(t_2) - \frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\left(\varphi(0) + \hbar(y)(0)\right) + \mathcal{G}(t_1, 0)y_1 \\
&\quad + \int_0^{t_1} \mathcal{G}(t, s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau)d\tau\right)ds\Big| \\
&\le \left|\phi(t_2) - \phi(0)\right| + \left|\frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\phi(0) - \phi(0)\right| + \left|\hbar(y)(t_2) - \hbar(y)(0)\right| \\
&\quad + \left|\frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\hbar(y)(0) - \hbar(y)(0)\right| + \left|\mathcal{G}(t_2, 0)y_1 - \mathcal{G}(t_1, 0)y_1\right| \\
&\quad + N_1 \int_0^{t_1} e^{-\lambda(t-s)}\Phi_{\mathcal{N}}(s)\Psi_{\mathcal{N}}\left(|y_s| + \int_0^s \xi(s, \tau)|y_\tau|d\tau\right)ds.
\end{aligned}$$

It follows from the Hölder's inequality that

$$\begin{aligned}
&|(\mathbb{Q}y)(t_2) - (\mathbb{Q}y)(t_1)| \\
&\le \left|\phi(t_2) - \phi(0)\right| + \left|\frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\phi(0) - \phi(0)\right| + \left|\hbar(y)(t_2) - \hbar(y)(0)\right| \\
&\quad + \left|\frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\hbar(y)(0) - \hbar(y)(0)\right| + \left|\mathcal{G}(t_2, 0)y_1 - \mathcal{G}(t_1, 0)y_1\right| \\
&\quad + \frac{N_1\|\Phi_{\mathcal{N}}\|_{L^p}\Psi_{\mathcal{N}}((1 + \xi^*)r)(1 - p)^{1-p}}{\lambda^{1-p}}\left(e^{-\frac{\lambda}{1-p}(t-t_1)} - e^{-\frac{\lambda}{1-p}t}\right)^{1-p}.
\end{aligned}$$

Since $\phi \in \Omega$ and from $(\mathbb{H}_1)) - (\mathbb{H}_4))$, the right-hand side of the above inequality tends to zero when $t_1 \to 0$ and $t_2 \to 0$.
For any $y \in \mathfrak{A}_r$ and $0 \le t_1 \le t_2 \le d$, we get

$$\begin{aligned}
|(\mathbb{Q}y)(t_2) - (\mathbb{Q}y)(t_1)| &= \Big| - \frac{\partial}{\partial s}\mathcal{G}(t_2, 0)\left(\varphi(0) + \hbar(y)(0)\right) + \mathcal{G}(t_2, 0)y_1 \\
&\quad + \int_0^{t_2} \mathcal{G}(t, s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau)d\tau\right)ds \\
&\quad - \frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\left(\varphi(0) + \hbar(y)(0)\right) + \mathcal{G}(t_1, 0)y_1 \\
&\quad + \int_0^{t_2} \mathcal{G}(t, s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s, \tau, y_\tau)d\tau\right)ds\Big| \\
&\le \left|\frac{\partial}{\partial s}\mathcal{G}(t_2, 0)\phi(0) - \frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\phi(0)\right| \\
&\quad + \left|\frac{\partial}{\partial s}\mathcal{G}(t_2, 0)\hbar(y(0)) - \frac{\partial}{\partial s}\mathcal{G}(t_1, 0)\hbar(y)(0)\right|
\end{aligned}$$

$$
\begin{aligned}
&+\left|\mathcal{G}(t_2,0)y_1-\mathcal{G}(t_1,0)y_1\right| \\
&+\left|\int_0^{t_1}(\mathcal{G}(t_2,s)-\mathcal{G}(t_1,s))\mathcal{N}\left(s,y_s,\int_0^s\mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\right. \\
&\left.+\int_{t_1}^{t_2}\mathcal{G}(t_2,\tau)\mathcal{N}\left(s,y_s),\int_0^s\mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\right| \\
\leq&\left|\frac{\partial}{\partial s}\mathcal{G}(t_2,0)\phi(0)-\frac{\partial}{\partial s}\mathcal{G}(t_1,0)\phi(0)\right| \\
&+\left|\frac{\partial}{\partial s}\mathcal{G}(t_2,0)\hbar(y(0))-\frac{\partial}{\partial s}\mathcal{G}(t_1,0)\hbar(y)(0)\right| \\
&+\left|\mathcal{G}(t_2,0)y_1-\mathcal{G}(t_1,0)y_1\right| \\
&+\int_0^{t_1}\Phi_{\mathbb{F}}(s)\Psi_{\mathbb{F}}\left(|y_s|+\int_0^s\xi(s,\tau)|y_\tau|d\tau\right)ds \\
&\times\|\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)\|_{B(E)} \\
&+N_1\int_{t_1}^{t_2}e^{-\lambda(t-s)}\Phi_{\mathcal{N}}(s)\Psi_{\mathcal{N}}\left(|y_s|+\int_0^s\xi(s,\tau)|y_\tau|d\tau\right)ds.
\end{aligned}
$$

It follows from the Hölder's inequality that

$$
\begin{aligned}
|(\mathbb{Q}y)(t_2)-(\mathbb{Q}y)(t_1)|\leq&\|\frac{\partial}{\partial s}\mathcal{G}(t_2,0)-\frac{\partial}{\partial s}\mathcal{G}(t_1,0)\|_{B(E)}(\|\phi(0)\|_\Omega+\Psi_\hbar(\|y\|_C)) \\
&+\|\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)\|_{B(E)}\left|y_1\right| \\
&+\Psi_{\mathbb{F}}((1+\xi^*)r)\int_0^{t_1}\|\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)\|_{B(E)}\,\Phi_{\mathcal{N}}(s)ds \\
&+N_1\|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{p}}}\Psi_{\mathcal{N}}((1+\xi^*)r)\left(\int_{t_1}^{t_2}e^{-\frac{\lambda}{1-p}(t-s)}ds\right)^{1-p}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
|(\mathbb{Q}y)(t_2)-(\mathbb{Q}y)(t_1)|\leq&\left\|\frac{\partial}{\partial s}\mathcal{G}(t_2,0)-\frac{\partial}{\partial s}\mathcal{G}(t_1,0)\right\|_{B(E)}(\|\phi(0)\|_\Omega+\Psi_\hbar(\|y\|_C)) \\
&+\|\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)\|_{B(E)}\left|y_1\right|+\Psi_{\mathcal{N}}((1+\xi^*)r) \\
&\times\int_0^{t_1}\|\mathcal{G}(t_2,\tau)-\mathcal{G}(t_1,\tau)\|_{B(E)}\,\Phi_{\mathcal{N}}(s)ds \\
&+\frac{N_1\|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{p}}}\Psi_{\mathcal{N}}((1+\xi^*)r)(1-p)^{1-p}}{\lambda^{1-p}} \\
&\times\left(e^{-\frac{\lambda}{1-p}(t-t_2)}-e^{-\frac{\lambda}{1-p}(t-t_1)}\right)^{1-p}.
\end{aligned}
$$

When $t_2\to t_1$, the right-hand side of the above inequality tends to zero. Therefore, $\mathbb{Q}(\mathfrak{A}_r)$ is equicontinuous.

Step 4. Now, we show that $\mathbb{Q}:\mathfrak{A}_r\to\mathfrak{A}_r$ is a convex-power condensing operator.
Take $y_0\in\mathfrak{A}_r$, we shall prove that there exists a positive integral n_0 such that

$$
\beta_{\mathfrak{C}}(\mathbb{Q}^{(m_0,y_0)}(\mathfrak{S}))\leq\beta_{\mathfrak{C}}(\mathbb{Q}((\mathfrak{S})),
$$

for every nonprecompact bounded subset $\mathfrak{S}\subset\mathfrak{A}_r$. By the definition of operator $\mathbb{Q}^{(m,y_0)}$ and the equicontinuity of $\mathbb{Q}$, we get that $\mathbb{Q}^{(m,y_0)}(\mathfrak{S})\subset\mathfrak{A}_r$ is also equicontinuous. Therefore, we know from Lemma 2.6

that

$$\beta_{\mathfrak{C}}(\mathbb{Q}^{(m_0,y_0)}(\mathfrak{S})) = \sup_{t\in J} \beta_E(\mathbb{Q}^{(m_0,y_0)}(\mathfrak{S})(t)).$$

Case 1. The operator $\hbar : \mathfrak{C}(\mathfrak{I}, E) \to \Omega$ is compact ($K_\hbar = 0$).
For every bounded subset $\mathfrak{S} \subset \mathfrak{C}(\mathfrak{I}, E)$, by Lemma 2.4, there exists a countable set $\mathfrak{S}_0 = \{z^n\}_{n=1}^{\infty} \subset \mathfrak{S}$, such that for each $t \in \mathfrak{I}$, and from $(\mathbb{H}_1)$, $(\mathbb{H}_3)(iii)$, $(\mathbb{H}_4)(ii)$, $(\mathbb{H}_5)(ii)$, Lemma 2.4, Lemma 2.5 and properties of the measure β_E, we obtain

$$\begin{aligned}
\beta_E((\mathbb{Q}^{(1,y_0)}\mathfrak{S})(t) &= \beta_E((\mathbb{Q}(\mathfrak{S})(t)\\
&\le 2\beta_E((\mathbb{Q}(\mathfrak{S}_0)(t)\\
&\le 2\beta_E\Bigg(-\frac{\partial}{\partial s}\mathcal{G}(t,0)\,(\varphi(0)+\hbar(y)(0)) + \mathcal{G}(t,0)y_1\\
&\quad + \int_0^t \mathcal{G}(t,s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\Bigg)\\
&\le 2\beta_E\left(\int_0^t \mathcal{G}(t,s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\right)\\
&\le 4\int_0^t \mathcal{G}(t,s)\beta_E\left(\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)\right)ds\\
&\le 2\beta_E\left(\int_0^t \mathcal{G}(t,s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\right)\\
&\le 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{t\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi))\\
&\quad + \beta_E\left(\int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)\Bigg)ds\\
&\le 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{t\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi))\\
&\quad + 2\int_0^s \beta_E\left(\mathcal{M}(s,\tau,y_\tau)\right)d\tau\Bigg)ds\\
&\le 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{t\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi))\\
&\quad + 2\int_0^s \beta_E\left(\mathcal{M}(s,\tau,y_\tau)\right)d\tau\Bigg)ds\\
&\quad + 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{\varpi\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi))\\
&\quad + 2\int_0^s K^* \sup_{\varpi\in[-\eta,0]} \beta_E(\mathfrak{S}_0(\tau+\varpi))d\tau\Bigg)ds\\
&\quad + 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{s\in[0,t]} \beta_E(\mathfrak{S}_0(s))\\
&\quad + 2\int_0^s K^* \sup_{\tau\in[0,s]} \beta_E(\mathfrak{S}_0(\tau))d\tau\Bigg)ds
\end{aligned}$$

$$
\begin{aligned}
&+ 4\int_0^t \mathcal{G}(t,s)\rho(s)(1+2K^*s)\sup_{s\in[0,d]}\beta_E(\mathfrak{S}_0(s))ds\\
&\leq \left(2N_1\|\rho\|_{L^\infty}\frac{1}{2\times 2k}(1+2kt)^2\right)\beta_{\mathfrak{C}}((\mathfrak{S}))\\
&\leq \frac{\left((1+2kd)\sqrt{2N_1\|\rho\|_{L^\infty}}\right)^2}{2\times(2k)}\beta_{\mathfrak{C}}(\mathfrak{S}).
\end{aligned}
$$

Again by Lemma 2.6, there exists a countable set $D_2=\left\{y_n^2\right\}\subset\overline{co}\mathbb{Q}^{(2,y_0)}(D),y_0)$, such that

$$
\begin{aligned}
\beta_E(\mathbb{Q}^{(2,y_0)}(\mathfrak{S}))(t)) &= \beta_E(\mathbb{Q}^{(2,y_0)}(\overline{co}\left\{\mathbb{Q}^{(2,y_0)}(\mathfrak{S}),y_0\right\}))(t))\\
&\leq 2\beta_E(\mathbb{Q}(\mathfrak{S}_2))(t))\\
&\leq 2\beta_E\left(\frac{\partial}{\partial s}\mathcal{G}(d,0)\hbar(\left\{y_n^2)(0)\right\}_{n=0}^{\infty}\right))\\
&\quad + 4M\int_0^t \beta_E\left(f\left(s,\left\{y_n^2(s)\right\}_{n=0}^{\infty},\int_0^s \mathcal{H}\left(s,\tau,\left\{y_n^2(\tau)\right\}_{n=0}^{\infty}\right)d\tau\right)\right)\\
&\quad \times \mathcal{G}(t,s)\rho(t)ds\\
&\leq 4M\|\rho\|_{L^\infty}\int_0^t (1+2K^*s)\sup_{\zeta\in[0,t]}\beta_E(\overline{co}\mathbb{Q}^{(1,y_0)}(\mathfrak{S}),y_0))(s)))\\
&\leq \int_0^t (1+2K^*s)\left(2N_1\|\rho\|_{L^\infty}\frac{1}{2\times 2k}(1+2ks)^2\right)\beta_{\mathfrak{C}}((\mathfrak{S})ds\\
&\leq \int_0^t (2N_1\|\rho\|_{L^\infty})^2\frac{1}{2\times 2k}(1+2ks)^3\beta_{\mathfrak{C}}((\mathfrak{S})ds\\
&\leq \left((2N_1\|\rho\|_{L^\infty})^2\frac{1}{2\times 4\times(2k)^2}(1+2kt)^4\right)\beta_{\mathfrak{C}}(\mathfrak{S})\\
&\leq \frac{\left((1+2kd)\sqrt{2N_1\|\rho\|_{L^\infty}}\right)^4}{2\times 4\times(2k)^2}\beta_{\mathfrak{C}}(\mathfrak{S}).
\end{aligned}
$$

If for any $t\in J$, we assume

$$
\beta_E(\mathbb{Q}^{(m,y_0)}(\mathfrak{S}))(t))\leq \frac{\left((1+2kd)\sqrt{2N_1\|\rho\|_{L^\infty}}\right)^{2m}}{2\times 4\times\cdots 2m\times(2k)^m}\beta_{\mathfrak{C}}(\mathfrak{S})
$$

Then, by Lemma 2.4, there exists a countable set set

$$
\mathfrak{S}_{m+1}=\left\{y_n^{m+1}\right\}\subset\overline{co}\mathbb{Q}^{(m,y_0)}(\mathfrak{S}),y_0),
$$

such that

$$
\begin{aligned}
\beta_E(\mathbb{Q}^{(m+1,y_0)}(\mathfrak{S}))(t) &= \beta_E(\mathbb{Q}^{(m+1,y_0)}(\overline{co}\left\{\mathbb{Q}^{(2,y_0)}(\mathfrak{S}),y_0\right\}))(t)\\
&\leq 2\beta_E(\mathbb{Q}(\mathfrak{S}_{m+1}))(t)\\
&\leq 2\beta_E\left(\frac{\partial}{\partial s}\mathcal{G}(d,0)\hbar(\left\{y_n^{m+1})(0)\right\}_{n=0}^{\infty}\right)\\
&\quad + 4M\int_0^t \mathcal{G}(t,s)\rho(t)\\
&\quad \times\beta_E\left(f(s,\left\{y_n^{m+1}(s)\right\}_{n=0}^{\infty}),\int_0^s \mathcal{H}\left(s,\tau,\left\{y_n^{m+1}(\tau)\right\}_{n=0}^{\infty}\right)d\tau\right)ds
\end{aligned}
$$

$$\leq 4M\|\rho\|_{L^{\inf(ty}}\int_0^t(1+2K^*s)\sup_{\zeta\in[0,t]}\beta_E(\overline{co}\mathbb{Q}^{(m,y_0)}(\mathfrak{S}),y_0))(s)))$$
$$\leq \frac{(2N_1\|\rho\|_{L^\infty})^m(1+2kd)^{2m}}{2\times 4\times\cdots 2m\times(2k)^m}\beta_{\mathfrak{C}}(\mathfrak{S}).$$

Therefore, we obtain the following inequality

$$\beta_{\mathfrak{C}}(\mathbb{Q}^{(m,y_0)}(\mathfrak{S}))(t)) \quad\leq\quad \frac{\left((1+2kd)\sqrt{2N_1\|\rho\|_{L^\infty}}\right)^{2m}}{2\times 4\times\cdots 2m\times(2k)^m}\beta_{\mathfrak{C}}(\mathfrak{S}).$$

By the fact that

$$\frac{\left((1+2kd)\sqrt{2N_1\|\rho\|_{L^\infty}}\right)^{2m}}{2\times 4\times\cdots 2m\times(2k)^m}\to 0 \text{ as } n\to+\infty.$$

Therefore, there must exist a positive integer m_0, which is large enough, such that

$$\frac{\left((1+2kd)\sqrt{2N_1\|\rho\|_{L^\infty}}\right)^{2m_0}}{2\times 4\times\cdots 2n_0\times(2k)^{m_0}}\leq 1.$$

By Lemma 2.4and Lemma 2.5. From $(\mathbb{H}_5)(ii)$, we have, for any $t\in[-\eta,0]$,

$$\beta_E((\mathbb{Q}(\mathfrak{S})(t)=\beta_E\left(\varphi(t)+\hbar(y)(t)\right)=0.$$

Therefore, $\mathbb{Q}:\mathfrak{S}\to\mathfrak{S}$ is a convex-power condensing operator.

Case 2. The operator $\hbar:\ \mathfrak{C}(\mathfrak{I},E)\to\Omega$ is noncompact ($K_\hbar\neq 0$).
Take $y_0\in\mathfrak{A}_r$, we shall prove that there exists a positive integral m_0 such that

$$\beta_{\mathfrak{C}}(\mathbb{Q}^{(m_0,y_0)}(\mathfrak{S}))\leq\beta_{\mathfrak{C}}(\mathbb{Q}((\mathfrak{S}))$$

for every nonprecompact bounded subset $\mathfrak{S}\subset\mathfrak{A}_r$. By the definition of operator $\mathbb{Q}^{(m,y_0)}$ and the equicontinuity of $\mathbb{Q}$, we get that $\mathbb{Q}^{(m,y_0)}(\mathfrak{S})\subset\mathfrak{A}_r$ is also equicontinuous. Therefore, we know that

$$\beta_{\mathfrak{C}}(\mathbb{Q}^{(m_0,y_0)}(\mathfrak{S}))=\sup_{t\in J}\beta_E(\mathbb{Q}^{(m_0,y_0)}(\mathfrak{S})(t)).$$

For every bounded subset $\mathfrak{S}\subset\mathfrak{C}(\mathfrak{I},E)$, by Lemma 2.4, there exists a countable set $\mathfrak{S}_0=\{z^n\}_{n=1}^\infty\subset\mathfrak{S}$, such that for each $t\in\mathfrak{I}$, and from $(\mathbb{H}_1)$, $(\mathbb{H}_3)(iii)$, $(\mathbb{H}_4)(ii)$, $(\mathbb{H}_5)(ii)$ and by using Lemma 2.4, Lemma 2.5 and properties of the measure β_E, we obtain

$$\begin{aligned}\beta_E((\mathbb{Q}^{(1,y_0)}\mathfrak{S})(t)&=\beta_E((\mathbb{Q}(\mathfrak{S})(t)\\&\leq 2\beta_E((\mathbb{Q}(\mathfrak{S}_0)(t)\\&\leq 2\beta_E\Bigg(-\frac{\partial}{\partial s}\mathcal{G}(t,0)\left(\varphi(0)+\hbar(y)(0)\right)+\mathcal{G}(t,0)y_1\\&\quad+\int_0^t\mathcal{G}(t,s)\mathcal{N}\left(s,y_s,\int_0^s\mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\Bigg)\\&\leq 2\beta_E\left(\frac{\partial}{\partial s}\mathcal{G}(t,0)\hbar(y)(0)\right)\\&\leq 2\beta_E\left(\int_0^t\mathcal{G}(t,s)\mathcal{N}\left(s,y_s,\int_0^s\mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\right)\\&\leq N_2K_\hbar\beta_E(\mathfrak{S}_0(t))\end{aligned}$$

$$
\begin{aligned}
&\leq 4\int_0^t \mathcal{G}(t,s)\beta_E\left(\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)\right)ds\\
&\leq 2\beta_E\left(\int_0^t \mathcal{G}(t,s)\mathcal{N}\left(s, y_s, \int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)ds\right)\\
&\leq N_2K_\hbar \sup_{t\in[0,d]} \beta_E(\mathfrak{S}_0(t))\\
&\leq 4\int_0^t \left(\sup_{t\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi) + \beta_E\left(\int_0^s \mathcal{M}(s,\tau,y_\tau)d\tau\right)\right)\\
&\quad \times \mathcal{G}(t,s)\rho(t)ds\\
&\leq N_2K_\hbar \sup_{t\in[0,d]} \beta_E(\mathfrak{S}_0(t))\\
&\leq 4\int_0^t \left(\sup_{t\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi)) + 2\int_0^s \beta_E\left(\mathcal{M}(s,\tau,y_\tau)\right)d\tau\right)\\
&\quad \times \mathcal{G}(t,s)\rho(t)ds\\
&\leq N_2K_\hbar \sup_{t\in[0,d]} \beta_E(\mathfrak{S}_0(t))\\
&\leq 4\int_0^t \left(\sup_{t\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi)) + 2\int_0^s \beta_E\left(\mathcal{M}(s,\tau,y_\tau)\right)d\tau\right)\\
&\quad \times \mathcal{G}(t,s)\rho(t)ds\\
&\leq N_2K_\hbar \sup_{t\in[0,d]} \beta_E(\mathfrak{S}_0(t))\\
&\quad + 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{\varpi\in[-\eta,0]} \beta_E(\mathfrak{S}_0(s+\varpi))\\
&\quad + 2\int_0^s K^* \sup_{\varpi\in[-\eta,0]} \beta_E(\mathfrak{S}_0(\tau+\varpi))d\tau\Bigg)ds\\
&\leq N_2K_\hbar \sup_{t\in[0,d]} \beta_E(\mathfrak{S}_0(t))\\
&\quad + 4\int_0^t \mathcal{G}(t,s)\rho(t)\Bigg(\sup_{s\in[0,t]} \beta_E(\mathfrak{S}_0(s))\\
&\quad + 2\int_0^s K^* \sup_{\tau\in[0,s]} \beta_E(\mathfrak{S}_0(\tau))d\tau\Bigg)ds\\
&\leq N_2K_\hbar \sup_{t\in[0,d]} \beta_E(\mathfrak{S}_0(t))\\
&\quad + 4\int_0^t \mathcal{G}(t,s)\rho(s)(1+2K^*s) \sup_{s\in[0,d]} \beta_E(\mathfrak{S}_0(s))ds\\
&\leq \left(N_2K_\hbar + 2N_1\|\rho\|_{L^\infty}\frac{1}{2\times 2k}(1+2kt)^2\right)\beta_{\mathfrak{C}}((\mathfrak{S}))\\
&\leq L\left(1+\frac{1}{2\times 2k}(1+2kt)^2\right)\beta_{\mathfrak{C}}((\mathfrak{S})),
\end{aligned}
$$

where

$$L = \max(N_2K_\hbar, 2N_1\|\rho\|_{L^\infty}).$$

Again by Lemma 2.4, there exists a countable set $D_2 = \left\{y_n^2\right\} \subset \overline{co}\mathbb{Q}^{(2,y_0)}(D), y_0)$, such that

$$\begin{aligned}
\beta_E(\mathbb{Q}^{(2,y_0)}(\mathfrak{S}))(t)) &= \beta_E(\mathbb{Q}^{(2,y_0)}(\overline{co}\left\{\mathbb{Q}^{(2,y_0)}(\mathfrak{S}), y_0\right\}))(t)) \\
&\leq 2\beta_E(\mathbb{Q}(\mathfrak{S}_2))(t) \\
&\leq 2\beta_E\left(\frac{\partial}{\partial s}\mathcal{G}(d,0)\hbar\left(\left\{y_n^2\right\}_{n=0}^{\infty}\right)(0)\right) \\
&\quad + 4M\int_0^t \beta_E\left(f(s,\left\{y_n^2(s)\right\}_{n=0}^{\infty}), \int_0^s \mathcal{H}\left(s,\tau,\left\{y_n^2(\tau)\right\}_{n=0}^{\infty}\right)d\tau\right) \\
&\quad \times \mathcal{G}(t,s)\rho(t)ds \\
&\leq 4N_2\beta_E\left(\hbar\left(\overline{co}\mathbb{Q}^{(1,y_0)}(\mathfrak{S}), y_0)\right)(\zeta))\right)\Big) \\
&\quad + 4M\|\rho\|_{L^\infty}\int_0^t (1+2K^*s)\sup_{\zeta\in[0,t]}\beta_E(\overline{co}\mathbb{Q}^{(1,y_0)}(\mathfrak{S}), y_0))(s))) \\
&\leq L^2\left(1+\frac{1}{2\times 2k}(1+2kt)^2\right) \\
&\quad + L^2\int_0^t (1+2K^*s)\left(1+\frac{1}{2\times 2k}(1+2ks)^2\right)\beta_{\mathfrak{C}}((\mathfrak{S})ds \\
&\leq L^2\left(1+\frac{2}{2\times 2k}(1+2kt)^2+\frac{1}{2\times 4\times(2k)^2}(1+2kt)^4\right)\beta_{\mathfrak{C}}(\mathfrak{S}) \\
&\leq L^2\left(1+\sum_{j=1}^{j=2}\left(\frac{(2-j+1)(1+2kd)^{2j}}{2\times 4\times\cdots\times(2j)\times(2kd)^j}\right)\right)\beta_{\mathfrak{C}}((\mathfrak{S}).
\end{aligned}$$

If for any $t \in J$, we assume

$$\beta_{\mathfrak{C}}(\mathbb{Q}^{(m,y_0)}(\mathfrak{S}))(t)) \leq L^m\left(1+\sum_{j=1}^{j=2}\left(\frac{(2-j+1)(1+2kd)^{2j}}{2\times 4\times\cdots\times(2j)\times(2kd)^j}\right)\right)\beta_{\mathfrak{C}}((\mathfrak{S}).$$

Then by Lemma 2.4, there exists a countable set set $\mathfrak{S}_{m+1} = \left\{y_n^{m+1}\right\} \subset \overline{co}\mathbb{Q}^{(m,y_0)}(\mathfrak{S}), y_0)$, such that

$$\begin{aligned}
&\beta_E(\mathbb{Q}^{(m+1,y_0)}(\mathfrak{S}))(t) \\
&= \beta_E(\mathbb{Q}^{(m+1,y_0)}(\overline{co}\left\{\mathbb{Q}^{(2,y_0)}(\mathfrak{S}), y_0\right\}))(t)) \\
&\leq 2\beta_E(\mathbb{Q}(\mathfrak{S}_{m+1}))(t)) \\
&\leq 2\beta_E\left(\frac{\partial}{\partial s}\mathcal{G}(d,0)\hbar(\left\{y_n^{m+1})(0)\right\}_{n=0}^{\infty}\right)) \\
&\quad + 4M\int_0^t \mathcal{G}(t,s)\rho(t)\beta_E\left(f(s,\left\{y_n^{m+1}(s)\right\}_{n=0}^{\infty}), \int_0^s \mathcal{H}\left(s,\tau,\left\{y_n^{m+1}(\tau)\right\}_{n=0}^{\infty}\right)d\tau)ds)\right)ds \\
&\leq 4N_2\beta_E\left(\hbar\left(\overline{co}\mathbb{Q}^{(m,y_0)}(\mathfrak{S}), y_0)\right)(\zeta))\right)\Big) \\
&\quad + 4M\|\rho\|_{L^\infty}\int_0^t (1+2K^*s)\sup_{\zeta\in[0,t]}\beta_E(\overline{co}\mathbb{Q}^{(m,y_0)}(\mathfrak{S}), y_0))(s))) \\
&\leq L^{m+1}\Bigg(1+\frac{m}{2\times 2k}(1+2kt)^2+\frac{m-1}{2\times 4\times(2k)^2}(1+2kt)^4+\cdots \\
&\quad + \frac{1}{2\times 4\times\cdots 2m\times(2k)^m}(1+2kd)^{2m}\Bigg)\beta_{\mathfrak{C}}((\mathfrak{S})
\end{aligned}$$

$$+ L^{m+1}\int_0^t (1+2K^*s)\Bigg(1+\frac{m}{2\times 2k}(1+2ks)^2+\frac{m-1}{2\times 4\times (2k)^2}(1+2ks)^4+\cdots+$$

$$+\frac{1}{2\times 4\times\cdots 2m\times (2k)^m}(1+2ks)^{2m}\Bigg)\beta_{\mathfrak{C}}((\mathfrak{S})ds$$

$$\leq L^{m+1}\Bigg(1+\frac{m+1}{2\times 2k}(1+2kt)^2+\frac{m}{2\times 4\times (2k)^2}(1+2kt)^4$$

$$+\cdots+\frac{1}{2\times 4\times\cdots 2(m+1)\times (2k)^{m+1}}(1+2kt)^{2(m+1)}\Bigg)\beta_{\mathfrak{C}}(\mathfrak{S}).$$

Therefore, we obtain the following inequality

$$\beta_{\mathfrak{C}}(\mathbb{Q}^{(m,y_0)}(\mathfrak{S}))(t))\leq L^m\left(1+\sum_{j=1}^{j=m}\left(\frac{(m-j+1)\,(1+2kd)^{2j}}{2\times 4\times\cdots\times(2j)\times(2kd)^j}\right)\right)\beta_{\mathfrak{C}}((\mathfrak{S}).$$

By the fact that

$$L^m\to 0 \text{ as } m\to+\infty,$$

and

$$\sum_{j=1}^{j=m}\left(\frac{(m-j+1)\,(1+2kd)^{2j}}{2\times 4\times\cdots\times(2j)\times(2k)^j}\right)\ \text{ is convergent,}$$

there must exist a positive integer m_0, which is large enough, such that

$$L^{m_0}\left(1+\sum_{j=1}^{j=m_0}\left(\frac{(m_0-j+1)\,(1+2kd)^{2j}}{2\times 4\times\cdots\times(2j)\times(2k)^j}\right)\right)\leq 1.$$

By Lemma 2.4 and Lemma 2.5 and from $(\mathbb{H}_5)(ii)$, we have for any $t\in[-\eta,0]$,

$$\begin{aligned}\beta_E((\mathbb{Q}^{(1,y_0)}\mathfrak{S})(t) &= \beta_E((\mathbb{Q}(\mathfrak{S})(t)\\ &\leq 2\beta_E((\mathbb{Q}(\mathfrak{S}_0)(t)\\ &\leq 2\beta_E\,(\varphi(t)+\hbar(y)(t))\\ &\leq 2K_\hbar\beta_E(\mathfrak{S}_0(t))\\ &\leq 2K_\hbar\sup_{t\in[0,d]}\beta_E(\mathfrak{S}(t)).\end{aligned}$$

Again by Lemma 2.6, there exists a countable set $\mathfrak{S}_2=\left\{y_n^2\right\}\subset\overline{co}\mathbb{Q}^{(2,y_0)}(D),y_0)$, such that

$$\begin{aligned}\beta_E(\mathbb{Q}^{(2,y_0)}(\mathfrak{S}))(t) &= \beta_E(\mathbb{Q}^{(2,y_0)}(\overline{co}\left\{\mathbb{Q}^{(2,y_0)}(\mathfrak{S}),y_0\right\}))(t))\\ &\leq 2\beta_E(\mathbb{Q}(\mathfrak{S}_2))(t))\\ &\leq 2\beta_E\,(\varphi(t)+\hbar(\mathfrak{S}_2)(t))\\ &\leq 2K_\hbar\beta_E(\mathfrak{S}_2(t))\\ &\leq (2K_\hbar)^2\sup_{t\in[0,d]}\beta_E(\mathfrak{S}(t)).\end{aligned}$$

Continuing this iterative procedure and for $t\in\mathfrak{I}$, we get the m^{th} iterative therm,

$$\beta_E(\mathbb{Q}^{(m,y_0)}(\mathfrak{S}))(t))\leq(2K_\hbar)^m\beta_{\mathfrak{C}}(\mathfrak{S}).$$

Therefore, by the fact that

$$(2K_\hbar)^m\to 0\quad\text{as}\quad m\to+\infty,$$

we know that there exists a large enough positive integer m_0 such that

$$(2K_{\hbar})^{m_0} < 1.$$

Thus, $\mathbb{Q} : \mathfrak{S} \to \mathfrak{S}$ is a convex-power condensing operator. Thus, $\mathbb{Q}$ has at least one fixed point, which is a mild solution of the nonlocal impulsive problem (1). $\square$

4. An example

Let $X = L^2([0,\pi],\mathbb{R})$. Consider the following finite delay partial differential equation:

$$\begin{cases} \dfrac{\partial^2}{\partial \tau^2}x(\tau,\xi) = \dfrac{\partial^2}{\partial \xi^2}x(\tau,\xi) + \gamma(\tau)\dfrac{\partial}{\partial \xi}x(\tau,\xi) + \pi^{-\tau}\left|\displaystyle\int_{-\eta}^{0}\ln\left(1+\pi^{-\varpi}\left|\phi(\varpi)(\xi)\right|\right)d\varpi\right|^{\frac{1}{2}} \\ \quad + \displaystyle\int_0^{\pi}\left|\int_{-\eta}^{0}\pi^{-st-|x_t(\varpi,\xi)|}d\varpi\right|\left(1+\left|\int_{-\eta}^{0}\pi^{-st-|x_t(\varpi,\xi)|}d\varpi\right|\right)^{-1}d\xi,\ \tau \in J,\ \ \xi \in [0,\pi], \\ x(\tau,0) = x(\tau,\pi) = 0,\ \tau \in [0,\pi], \\ x(\tau,\xi) = \psi(\varpi,\xi) + \displaystyle\int_0^t \frac{\pi^{-s-\varpi}\left|x_t(\varpi,\xi)\right|}{1+|x_t(\varpi,\xi)|}ds,\ \tau \in [-\eta,0],\ \tau \in [0,\pi], \\ \dfrac{\partial x(0,\xi)}{\partial \tau} = x_1,\ \xi \in [0,\pi]. \end{cases} \tag{5}$$

Assume that $\eta \in \left[0, \frac{2\ln 2+\ln\ln\pi}{\ln\pi}\right]$ and $d \geq 1 + \frac{\eta}{2}$. Define the operator $\mathbb{A} :\ \ D(\mathbb{A}) \to E$ by

$$\mathbb{A}v(\tau) = v''(\tau),$$

with domain $D(A_1) = \mathbb{H}^2(\mathbb{R},\mathbb{C})$. It is well known that $\mathbb{A}$ is the infinitesimal generator of a C_0-semigroup and of a strongly continuous cosine function on E, which will be denoted by $(C(t))$. From [40], for all $x \in \mathbb{H}^2([0,\pi],\mathbb{R}), t \in \mathbb{R}, \|C(t)\|_{B(E)} \leq 1$. Define also the operator $\mathbb{B} : \mathbb{H}^1([0,\pi],\mathbb{R}) \to E$ by

$$\mathbb{B}(t)v(s) = \gamma(t)v'(s),$$

where $\gamma : [0,1] \to \mathbb{R}$ is a Hölder continuous function.

Consider the closed linear operator $\mathcal{A}(t) = \mathbb{B}(t) + \mathbb{A}$. It has been proved by Henríquez in [25] that the family $\{\mathcal{A}(t) : t \in J\}$ generates an evolution operator $\{S(t,s)\}_{t,s\in\Delta}$. Moreover, $S(\cdot,\cdot)$ is well defined and satisfies the conditions $(\mathbb{H}_1)$ with $N_1 = N_2 = \lambda = 1$.

Define $x(\tau)(\xi) = x(\tau,\xi)$, $x_\tau(\varpi)(\xi) = z(\tau+\varpi,\xi)$. Denote

$$\mathcal{N}(t,\varphi,y)(\xi) = \pi^{-\tau}\left|\int_{-\eta}^{0}\ln\left(1+\pi^{-\varpi}\left|\phi(\varpi)(\xi)\right|\right)d\varpi\right|^{\frac{1}{2}} + \frac{\pi^{-\tau+\frac{\eta}{2}}}{\sqrt{\ln\pi}}\int_0^{\pi}\frac{\left|y(\xi)\right|}{1+\left|y(\xi)\right|}d\xi,$$

$$\mathcal{M}(t,s,y(s))(\xi) = \int_{-\eta}^{0}\pi^{-st-\varpi-|x_t(\varpi,\xi)|}d\varpi,$$

$$(\hbar)(x)(\varpi)(\xi) = \frac{1}{(\ln\pi)^2}e^{-|x_t(s,\xi)|} + \int_0^t \frac{\pi^{-s-\varpi}\left|x_t(s,\xi)\right|}{\ln\pi + |x_t(s,\xi)|^2}ds,$$

and

$$\phi(\varpi)(\xi) = \psi(\varpi,\xi).$$

Thus, above nonlocal fractional partial differential equations with finite delay (1) can be written as the abstract form (5).

Next, we verify the assumptions $(\mathbb{H}_1)-(\mathbb{H}_4)$ for the above system (5) one by one.

Verification of $(\mathbb{H}_2)$**:** for a.e. $t\in\Im$, the function $\mathcal{N}(t,\cdot,\cdot):\Im\times\Omega\times E\to E$ is continuous, and for each $(y_1,y_2)\in\Omega\times E$, the function $\mathcal{N}$ is strongly measurable; By a simple computation, we have

$$|\mathcal{N}(t,\varphi,y)(\xi)|\leq\frac{\pi^{-\tau+\frac{\eta}{2}}}{\sqrt{\ln\pi}}\left(\left|\varphi(\varpi)(\xi)\right|^{\frac{1}{2}}+|y(t,\xi)|\right).\tag{6}$$

Then for any bounded sets $W_1\subset\Omega$, $W_2\subset X$, we get

$$\beta_E(f(t,W_1,W_2))\leq\frac{\pi^{-\tau+\frac{\eta}{2}}}{\sqrt{\ln\pi}}\left(\sup_{\theta\in[-\eta,0]}\beta_E(V_1(\theta))+\beta_E(W_2)\right).$$

We shall show that condition $(H6)$ holds with

$$\Psi_{\mathbb{F}}(t)=\sqrt{t}+\pi t,\ \ \Phi_{\mathcal{N}}(t)=\frac{\pi^{-\tau+\frac{\eta}{2}}}{\sqrt{\ln\pi}},\ \ \rho(t)=\frac{\pi^{-\tau+\frac{\eta}{2}}}{\sqrt{\ln\pi}},$$

and

$$\lim_{r\to+\infty}\inf\frac{\Psi_{\mathcal{N}}(r)}{r}=\pi.\|\Phi_{\mathcal{N}}\|_{L^{\frac{1}{2}}}=\frac{\pi^{-d+\frac{\eta}{2}}}{2(\ln\pi)^{\frac{3}{2}}}.$$

The functions $\rho\in L^{\infty}(\Im,\mathbb{R}^+)$, and $\|\rho\|_{L^\infty}\in\left(0,\frac{1}{2}\right]$.

Verification of $(\mathbb{H}_3)$**:**

For each $t\in[0,d]$, $\varphi\in\Omega$, we obtain

$$|g(\tau,s,\varphi)(\tau)|\leq\pi^{-ts}\int_{-\eta}^{0}\pi^{-\varpi}|\varphi(\varpi)(\xi)|d\varpi\leq\frac{\pi^{-ts+\eta}}{\ln\pi}|\varphi|.\tag{7}$$

for any bounded and countable set $V\subset\Omega$

$$\beta_E(\mathcal{M}(t,s,V)\leq\pi^{\eta}\sup_{t\in[-\eta,0]}\beta_E(V(t))\text{ for a.e }t\in J.$$

From the above discussion, we obtain

$$\xi(t,s)=\frac{\pi^{-ts+\eta}}{\ln\pi},K^*=\pi^{\eta}$$

and

$$\xi^*=\sup_{t\in\Im}\int_0^t\xi(t,s)ds=\frac{\pi^{\eta}}{\ln\pi}<+\infty.$$

Verification of $(\mathbb{H}_5)$**:** For each $t\in[0,d]$, $\varphi\in\Omega$, we obtain

$$\begin{aligned}|(\hbar)(\varphi)(t)| &\leq \frac{1}{(\ln\pi)^2}\cos\left(\frac{1}{5}\left|\varphi\right|\right)+\pi^{-\varpi}\int_{-\eta}^{0}\pi^{-7s}|\varphi(\varpi)(\xi)|d\varpi\\ &\leq \frac{1}{(\ln\pi)^2}\cos\left(\frac{1}{5}\|\varphi\|\right)+\frac{\pi^{-\varpi-7\eta}}{7\ln\pi}\|\varphi\|\\ &\leq \frac{1}{(\ln\pi)^2}\left(\cos\left(\frac{1}{5}\|\varphi\|\right)+\|\varphi\|\right),\end{aligned}$$

and

$$
\begin{aligned}
|(\hbar\varphi_2)(t) - (\hbar\varphi_1)(t)| &\leq \frac{1}{5(\ln\pi)^2}\varphi_2|(\varpi)(\xi) - \varphi_1(\varpi)(\xi)| \\
&\quad + \pi^{-\varpi}\int_{-\eta}^{0} \pi^{-7s}|\varphi_2(\varpi)(\xi) - \varphi_1(\varpi)(\xi)|d\varpi. \\
&\leq \left(\frac{1}{5(\ln\pi)^2} + \frac{1}{7\ln\pi}\right)\|\varphi_2 - \varphi_1\|,
\end{aligned}
$$

which implies for each $t \in [-\eta, 0]$, the operator $\Gamma : \mathfrak{C}(\Im, E)$ defined by $\Gamma_t(y) = (\hbar y)(t)$ is continuous and the subset $\hbar(\Lambda) \subset \Omega$ is equicontinuous for each bounded set $\Lambda \subset \mathfrak{C}(\Im, E)$. Moreover, for any bounded and countable set $V \subset E$

$$\beta_E(\hbar(V) \leq \left(\frac{1}{5(\ln\pi)^2} + \frac{1}{7\ln\pi}\right)\sup_{\theta\in[-\eta,0]}\beta_E(V(\theta)) \text{ for a.e } t \in J.$$

Hence $(\mathbb{H}_4)$ is satisfied with

$$\Psi_\hbar = \cos\left(\frac{1}{5}t\right) + t, \quad N_\hbar = \frac{1}{(\ln\pi)^2} \text{ and } \quad K_\hbar = \left(\frac{1}{5(\ln\pi)^2} + \frac{1}{7\ln\pi}\right).$$

$\Psi_\hbar$ satisfies $\lim_{r\to+\infty}\inf\frac{\Psi_\hbar((r)}{r} = 1 \in [0, (\ln\pi)^2]$ and $K_\hbar \in \left[0, \frac{1}{2}\right]$.

With all the parameters discussed above, it is easy to check that conditions stated in Theorem 3.2.

$$N_2 N_\hbar \Lambda_\hbar + \frac{(1-p)^{1-p}\|\Phi_{\mathbb{F}}\|_{L^{\frac{1}{p}}}}{\lambda^{1-p}}\Lambda_{\mathbb{F}} = \frac{1}{(\ln\pi)^2} + \frac{\pi^{-d+1+\frac{\eta}{2}}}{2\sqrt{2}(\ln\pi)^{\frac{3}{2}}} < 1. \tag{8}$$

Now all the assumptions in Theorem 3.2 are satisfied, and so there is at least one solution of the problem (5) on $[-\eta, d]$.

Declarations

Ethical approval: This article does not contain any studies with human participants or animals performed by any of the authors.

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References

[1] S. Abbas and M. Benchohra, *Advanced Functional Evolution Equations and Inclusions*, Springer, Cham, 2015.

[2] H. Afshari and E. Karapinar, A solution of the fractional differential equations in the setting of *b*-metric space. *Carpathian Math. Publ.* **13** (3) (2021), 764-774.

[3] H. Afshari, V. Roomi and M. Nosrati, Existence and uniqueness for a fractional differential equation involving Atangana-Baleanu derivative by using a new contraction. *Lett. Nonl. Anal. Appl.* **1** (2) (2023), 52-56.

[4] P. Aviles, J. Sandefur, Nolinear second order equations wtih applications to partial differential equations, *J. Differential Equations* **58**(1985), 404-427.

[5] K. Balachandran, J.Y. Park, Nonlocal Cauchy problem for abstract fractional semilinear evolution equation. *Nonlinear Anal. Theory Methods Appl.* **71**(10) (2009), 4471–4475.

[6] A. Bellen, N. Guglielmi, A. E. Ruehli. Methods for linear systems of circuit delay differential equations of neutral type. *IEEE Trans Circuits Syst.* 1999;46(1):212-6.

[7] J. Banaś and K. Goebel, *Measures of Noncompactness in Banach Spaces,* Lecture Note in Pure App. Math. **60**, Dekker, New York, 1980.

[8] M. Benchohra, F. Bouazzaoui, E. Karapınar and A. Salim, Controllability of second order functional random differential equations with delay. *Mathematics* **10** (2022), 16pp. https://doi.org/10.3390/math10071120

[9] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Advanced Topics in Fractional Differential Equations: A Fixed Point Approach*, Springer, Cham, 2023.

[10] M. Benchohra, E. Karapınar, J. E. Lazreg and A. Salim, *Fractional Differential Equations: New Advancements for Generalized Fractional Derivatives*, Springer, Cham, 2023.

[11] M. Benchohra, N. Rezoug, Y. Zhou. Semilinear mixed type integro-differential evolution equations via Kuratowski measure of noncompactness. *Z. Anal. Anwend.* **38** (2019), no. 2, 143-156

[12] N. Benkhettou, K. Aissani, A. Salim, M. Benchohra, C. Tunc. Controllability of fractional integro-differential equations with infinite delay and non-instantaneous impulses. *Appl. Anal. Optim.* **6** (2022), no. 1, 79-94.

[13] N. Benkhettou, A. Salim, K. Aissani, M. Benchohra and E. Karapınar, Non-instantaneous impulsive fractional integro-differential equations with state-dependent delay, *Sahand Commun. Math. Anal.* **19** (2022), 93-109. https://doi.org/10.22130/scma.2022.542200.1014

[14] A. Bensalem, A. Salim, M. Benchohra and G. N'Guérékata, Functional integro-differential equations with state-dependent delay and non-instantaneous impulsions: existence and qualitative results, *Fractal Fract.* **6** (2022), 1-27. https://doi.org/10.3390/fractalfract6100615

[15] J. Bochenek, Existence of the fundamental solution of a second order evolution equation, *Ann. Pol. Math.***66** (1997), 15-35,

[16] N. Can, L. Long, H. Binh and N. Luc, Biharmonic heat equation with gradient non-linearity on L^p space. *Thermal Science* (2021), 6 pp.

[17] C. M. Chen, A. Abkar, S. Ghods and E. Karapınar, Fixed point theory for the α-admissible Meir-Keeler type set contractions having $KKM*$ property on almost convex sets. *Appl. Math. Inf. Sci.* **11** (1) (2017), 171-176.

[18] C. M. Chen, E. Karapınar and G. T. Chen, On the Meir-Keeler-Khan set contractions. *J. Nonlinear Sci. Appl.* **9** (8) (2016), 5271-5280.

[19] P. Chen, Y. Li, Monotone iterative technique for a class of semilinear evolution equations with nonlocal conditions, *Results Math.* **63** (2013), 731-744.

[20] H. O. Fattorini, *Second Order Linear Differential Equations in Banach Spaces*, North-Holland Mathematics Studies, Vol. **108**, North-Holland, Amsterdam, 1985.

[21] G. Gautam, J. Dabas. Mild solution for nonlocal fractional functional differential equation with not instantaneous impulse, *Int. J. Nonlinear Sci.* **21** (3)(2016), 151-160.

[22] H. Gou, and Y. Li, A study on non-autonomous second order evolution equations with nonlocal conditions. *Qual. Theory Dyn. Syst.* **22** (2023), no. 111.

[23] Q. Han, On robust stability of neutral systems with time-varying discrete delay and norm-bounded uncertainty. *Autom J. IFAC.* 2004; 40(6): 1087–92.

[24] H. P. Heinz, On the behaviour of measure of noncompactness with respect to differentiation and integration of rector-valued functions, *Nonlinear Anal.* **7**(1983), 1351-1371.

[25] H. Henríquez. Existence of solutions of non-autonomous second order functional differential equations with infinite delay, *Nonlinear Anal.* **74**(10)(2011), 3333-3352.

[26] H. Henríquez, V. Poblete, J. Pozo, Mild solutions of non-autonomous second order problems with nonlocal initial conditions, *J. Math. Anal. Appl.* **412**(2)(2014), 1064-1083.

[27] H. R. Henríquez and J. C. Pozo, Existence of solutions of abstract non-autonomous second order integro-differential equations, *Bound. Value Probl.* **168** (2016), 24 pp.

[28] A. Heris, A. Salim, M. Benchohra and E. Karapınar, Fractional partial random differential equations with infinite delay. *Results in Physics.* (2022). https://doi.org/10.1016/j.rinp.2022.105557

[29] J. Kamal, D. Bahuguna. Approximate controllability of nonlocal neutral fractional integro-differential equations with finite delay. *J. Dyn. Control Syst.* **22** (2016), no. 3, 485-504.

[30] E. Karapinar, H. D. Binh, N. H. Luc and N. H. Can, On continuity of the fractional derivative of the time-fractional semilinear pseudo-parabolic systems. *Adv. Difference Equ.* **2021** (2021), no. 70.

[31] M. Kozak, A fundamental solution of a second-order differential equation in a Banach space, *Univ. Iagel. Acta Math.* 32(1995), 275-289.

[32] F. Li, Nonlocal cauchy problem for delay fractional integrodifferential equations of neutral type, *Adv. Difference Equ.* **2012** (2012), 1–23.

[33] N. H. Luc, L. D. Long, H. L. T. Diem, D. Baleanu, N. H. Can, Identifying the initial condition for space-fractional sobolev equation. *J. Appl. Anal. Comput.* **11** (5) (2021), 2402-2422.

[34] M. Nosrati and H. Afshari, Triangular functions in solving weakly singular Volterra integral equations. *Adv. Theor. Nonl. Anal. Appl.* **7** (1) (2023), 195-204.

[35] L. Rabhi, H. Al, K. R. Mohammed. Existence results of mild solutions for nonlocal fractional delay integro-differential evolution equations via Caputo conformable fractional derivative. *AIMS Math.* **7** (2022), no. 7, 11614-11634.

[36] A. Salim, F. Mesri, M. Benchohra and C. Tunç, Controllability of second order semilinear random differential equations in Fréchet

spaces. *Mediterr. J. Math.* **20** (84) (2023), 1-12. https://doi.org/10.1007/s00009-023-02299-0
[37] H. Serizawa and M. Watanabe, Time-dependent perturbation for cosine families in Banach spaces, *Houston J. Math.* **12** (1986), 579-586.
[38] J. X. Sun and X. Y. Zhang, A fxed point theorem for convexpower condensing operators and its applications to abstract semilinear evolution equations, *Acta Mathe. Sinica*, **48**, no. 3, pp. 439-446, 2005
[39] X. Su, X. Fu. Approximate controllability of second-order semilinear evolution systems with finite delay. *Acta Math. Appl. Sin. Engl. Ser.* **37** (2021), no. 3, 573-589.
[40] C. C. Travis and G.F. Webb, Second order differential equations in Banach spaces, in: Nonlinear Equations in Abstract Spaces, *Proc. Internat. Sympos. (Univ. Texas, Arlington, TX, 1977), Academic Press, New-York*, 1978, 331–361.
[41] N. Valliammal, C. Ravichandran, J. H. Park, On the controllability of fractional neutral integrodifferential delay equations with nonlocal conditions, *Math. Methods Appl. Sci.* **40** (2017), 5044–5055.
[42] T. Winiarska, Evolution equations of second order with operator dependent on t, *Selected Problems of Math. Cracow Univ. Tech.* **6** (1995) 299-314.