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Fixed point results in incomplete extended b-metric space with t-property

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Abstract. The main objective of this paper is to prove some fixed point theorems in partially ordered, extended b-metric spaces which are not complete. The idea behind the proof is based on the so-called t-property. Contractions of various kinds, including Boyd-Wong type contraction mappings defined on incomplete, extended b-metric spaces are studied. Several examples are provided in order to demonstrate the novel concepts and findings and also, to verify the fact that the existence theorems proved in this paper can be used whenever the well-known theorems in the literature are not applicable.

1. Introduction

The development of fixed point theory has been significantly influenced by the Banach Contraction Mapping Principle (BCMP). In 1922, Banach [15] proved his famous theorem on the existence and uniqueness of a fixed point of continuous contraction map on a complete metric space. Later, researchers widely extended the BCMP, either by generalizing the space or, the contraction map itself. It is very difficult to list all of these generalizations.

One very important generalization of the metric space has been proposed by Bakhtin [14] in 1989, and is known as b-metric space. The BCMP has numerous generalizations on the b-metric spaces [7, 18, 27, 30, 34]. The concepts of b-metric and b-metric spaces were further extended by Kamran in 2017 [20] by introducing the extended b-metric spaces. Some fixed point results on this new type space followed immediately [1, 3–5, 11, 12, 23, 35].

In other studies, the BCMP has been extended by broadening the definition of the contraction mapping. These generalizations include the works os Edelstein [19], Boyd-Wong [16], Meir-Keeler [28], Ćirić [17], Khan et al. [24], Kirk [26] and Kannan [21].

Yet another generalization of the BCMP on partially ordered metric spaces was proposed by Ran and Reuring in 2004 [32]. The primary distinction between the BCMP and the mappings proposed by Ran and

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Reurings is that the contraction condition is required to hold only for comparable elements of the space which is endowed with a partial ordering [6, 9, 25, 29, 31, 33, 37].

The idea of t-property and t-contraction on an ordered metric space has been introduced by Rashid et al.[33] in 2019. This new idea helps to study fixed points on metric spaces which are not complete. For a list of fixed point results obtained with this configuration, see [10, 34, 38].

This paper is organized as follows. The next section contains the definitions of some preliminary concepts. The main results are presented in Section 3, where the fixed point theorems are discussed in 3 subsections. Conclusion is given in Section 4.

2. Preliminaries

We start by presenting the definitions of the concepts to be used in the proof of our main results. First, we recall the definition of b-metric space.

Definition 2.1. *(**b-metric space**)[14]. Suppose $d_b : X \times X \to \mathbb{R}^+$ be a map on a nonvoid set X, satisfying*
($*_1$) $d_b(\vartheta, \eta) = 0$ *if and only if* $\vartheta = \eta$,
($*_2$) $d_b(\vartheta, \eta) = d_b(\eta, \vartheta)$,
($*_3$) $d_b(\vartheta, \eta) \leq s[d_b(\vartheta, z) + d_b(z, \eta)]$, *for some constant* $s \geq 1$
for all $\vartheta, \eta \in X$. Then d_b is called a b-metric on X and (X, d_b) is referred to as a b-metric space with a constant s.

Remark 2.2. *It should be noted that for $s = 1$, b-metric space reduces to a metric space. Thus, every metric space is a b-metric space with $s = 1$ but, in general, a b-metric space is not a metric space.*

This metric has been further generalized by replacing the constant s by a function. The concept of extended b-metric space is given as follows.

Definition 2.3. *(**Extended b-metric space**)[20]. Suppose that X is a nonvoid set and $b_\phi : X \times X \to \mathbb{R}^+$, where $\phi : X \times X \to [1, \infty)$, is a function satisfying*
($**_1$) $b_\phi(\vartheta, \eta) = 0$ *iff* $\vartheta = \eta$,
($**_2$) $b_\phi(\vartheta, \eta) = b_\phi(\eta, \vartheta)$,
($**_3$) $b_\phi(\vartheta, \eta) \leq \phi(\vartheta, \eta)[b_\phi(\vartheta, z) + b_\phi(z, \eta)]$,
for all $\vartheta, \eta \in X$. Then, b_ϕ is referred to as extended b-metric on X and (X, b_ϕ) is called an extended b-metric space.

Remark 2.4. *It should be noted that for $\phi(\vartheta, \eta) = s$, for all $\vartheta, \eta \in X$ and some constant $s \geq 1$, the extended b-metric space reduces to a b-metric space. Each b-metric space is therefore an extended b-metric space, but not the other way around.*

Definition 2.5. *[36]. A sequence $\{\vartheta_n\}$ in an ordered set $(X, \preceq)$ is said to be increasing or ascending if, for all $m, n \in \mathbb{N}_0$ such that $m < n$, we have $\vartheta_m \preceq \vartheta_n$. It is said to be strictly increasing if $\vartheta_m \preceq \vartheta_n$ and $\vartheta_m \neq \vartheta_n$. We denote this as $\vartheta_m \prec \vartheta_n$.*

Definition 2.6. *[33]. Let $(X, d, \preceq)$ be any ordered metric space. X has the t-property if every strictly increasing Cauchy sequence $\{\vartheta_n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $\vartheta_n \preceq u$, for all $n \in \mathbb{N}_0$.*

Next, we state the definition of the t-property on extended b-metric spaces. We refer the readers to recall the convergence and Cauchy sequence on extended b-metric spaces (Definition 4 in [20]).

Definition 2.7. *An ordered extended b-metric space $(X, b_\phi, \preceq)$ is said to have the t-property, if every strictly increasing Cauchy sequence $\{\vartheta_n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $\vartheta_n \prec u$.*

Example 2.8. *Let $X = \mathbb{R}$ or $X = \mathbb{Q}$ or $X = (a, b], a, b \in \mathbb{R}$ be equipped with the natural ordering $\leq$ and the usual metric. Then X has the t-property.*

We complete this section by presenting some examples to demonstrate the concepts defined above.

Example 2.9. *Let $X = \mathbb{Q}$ be endowed with the metric $b_\phi : X \times X \to [0, \infty)$ given by $b_\phi(\vartheta, y) = |\vartheta - y|^2$, where $\phi : X \times X \to [1, \infty)$ is defined by $\phi(\vartheta, y) = \vartheta^2 + y^2 + 2$. If we take the sequence $\{\vartheta_n\}$ as an increasing Cauchy sequence in $\mathbb{Q}$ such that $\vartheta_n^2 < 2$, for all $n \in \mathbb{N}$, then $\{\vartheta_n\}$ is a Cauchy sequence and converges to $\sqrt{2}$. This shows that $(X, b_\phi, \preceq)$ is not complete but has t-property as every rational number greater than $\sqrt{2}$ is an upper bound of $\{\vartheta_n\}$.*

Example 2.10. *Suppose $X = [0, 2] \cap \mathbb{Q}$. We define an extended b-metric $b_\phi : X \times X \to [0, \infty)$ by*

$$\begin{aligned} b_\phi(\vartheta, \eta) &= 0, \quad \text{if} \quad \vartheta = \eta \\ &= \vartheta + \eta, \quad \text{if} \quad \vartheta \neq \eta. \end{aligned}$$

Let us define $\phi : X \times X \to [1, \infty)$ defined by $\phi(\vartheta, \eta) = 8 + \eta - \vartheta$.
Then clearly, (X, b_ϕ) is not a complete extended b-metric but has the t-property.

3. Main Results

3.1. Fixed points of ordered extended b-metric space having the t-property

Prior to establishing the main result, it is imperative to prove the following lemma.

Lemma 3.1. *Let $(X, b_\phi, \preceq)$ be an ordered extended b-metric space. Then for any sequence $\{\vartheta_n\}$ in X, we have*

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) \leq\ & \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m) b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots \\ & + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m) \cdots \phi(\vartheta_{m-1}, \vartheta_m) b_\phi(\vartheta_{m-1}, \vartheta_m), \end{aligned}$$

for all $n, m \in \mathbb{N}$ with $n < m$.

Proof. By using the triangle inequality, we have

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) &\leq \phi(\vartheta_n, \vartheta_m)[b_\phi(\vartheta_n, \vartheta_{n+1}) + b_\phi(\vartheta_{n+1}, \vartheta_m)] \\ &= \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_{n+1}, \vartheta_m). \end{aligned}$$

Again using the triangle inequality, we have

$$b_\phi(\vartheta_n, \vartheta_m) \leq \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + \phi(\vartheta_n, \vartheta_m)\Big[\phi(\vartheta_{n+1}, \vartheta_m)[b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) + b_\phi(\vartheta_{n+2}, \vartheta_m)]\Big].$$

Continuing in this way, we get

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) \leq\ & \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m) b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots \\ & + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m) \cdots \phi(\vartheta_{m-2}, \vartheta_m) b_\phi(\vartheta_{m-2}, \vartheta_m) \\ & + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m) \cdots \phi(\vartheta_{m-2}, \vartheta_m) b_\phi(\vartheta_{m-1}, \vartheta_m). \end{aligned}$$

Since $\phi \geq 1$, we conclude that

$$\begin{aligned} & \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m) \cdots \phi(\vartheta_{m-2}, \vartheta_m) b_\phi(\vartheta_{m-1}, \vartheta_m) \\ & \leq \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m) \cdots \phi(\vartheta_{m-2}, \vartheta_m)\phi(\vartheta_{m-1}, \vartheta_m) b_\phi(\vartheta_{m-1}, \vartheta_m), \end{aligned}$$

and hence,

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) \leq\ & \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m) b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots \\ & + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m) \cdots \phi(\vartheta_{m-1}, \vartheta_m) b_\phi(\vartheta_{m-1}, \vartheta_m, \end{aligned}$$

which completes the proof. □

Now, we state and prove our first main theorem.

Theorem 3.2. *Let $(X, b_\phi, \preceq)$ be an ordered extended b-metric space having the t-property. Let $f : X \to X$ be a monotone non-decreasing self map such that,*
(C_1) there exists $\vartheta_0 \in X$ with $\vartheta_0 \preceq f\vartheta_0$,
(C_2) for all $\vartheta, y \in X$ with $\vartheta \prec y$,

$$d(y, fy) \leq \lambda d(\vartheta, f\vartheta), \text{ where } \lambda \in (0,1). \tag{1}$$

Further, suppose that the mapping $\phi : X \times X \to [1, \infty)$ satisfies

$$\phi(\vartheta, z) \geq \phi(y, z), \tag{2}$$

for all $\vartheta, y \in X$ with $\vartheta \prec y$ and anyz $\in X$. Suppose also that $\lim\limits_{m,n\to\infty} \phi(\vartheta_m, \vartheta_n) < \frac{1}{\lambda}$, where $\vartheta_n = f^n\vartheta_0$, $n \in \mathbb{N}$. Then f has a fixed point in X.

Proof. By the assumption (C_1), there exists $\vartheta_0 \in X$ with $\vartheta_0 \preceq f\vartheta_0$. Starting with this element ϑ_0, define the sequence $\{\vartheta_n\}, n \in \mathbb{N}_0$ as $\vartheta_{n+1} = f\vartheta_n$. If $\vartheta_{n+1} = \vartheta_n$ for some $n \geq 0$, the proof is done. Assume that $\vartheta_{n+1} \neq \vartheta_n$ for all $n \geq 0$. From the assumption (C_1) we have $\vartheta_0 \prec \vartheta_1$, and using the fact f is non-decreasing, we deduce $\vartheta_2 = f\vartheta_1 \prec f\vartheta_0 = \vartheta_1$. By continuing the process, we conclude that the sequence $\{\vartheta_n\}$ is strictly increasing.

Now, since $\vartheta_0, \vartheta_1 \in X$ with $\vartheta_0 \prec \vartheta_1$, then by (1), we have

$$b_\phi(\vartheta_1, f\vartheta_1) \leq \lambda b_\phi(\vartheta_0, f\vartheta_0). \tag{3}$$

Again, since $\vartheta_1, \vartheta_2 \in X$ with $\vartheta_1 \prec \vartheta_2$, then by (1), we have

$$b_\phi(\vartheta_2, f\vartheta_2) \leq \lambda b_\phi(\vartheta_1, f\vartheta_1). \tag{4}$$

Using (3) in (4) , we get

$$b_\phi(\vartheta_2, f\vartheta_2) \leq \lambda^2 b_\phi(\vartheta_0, f\vartheta_0).$$

Continuing in this way, we get

$$b_\phi(\vartheta_n, f\vartheta_n) \leq \lambda^n b_\phi(\vartheta_0, f\vartheta_0), \quad n \in \mathbb{N}. \tag{5}$$

Now we will show that $\{\vartheta_n\}$ is a Cauchy sequence.
Let $n, m \in \mathbb{N}$ with $n < m$. Then by Lemma 3.1, we have

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) \leq\ & \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m).b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) + \cdots \\ & + \phi(\vartheta_n, \vartheta_m)\phi(\vartheta_{n+1}, \vartheta_m)\phi(\vartheta_{n+2}, \vartheta_m)\cdots\phi(\vartheta_{m-1}, \vartheta_m) b_\phi(\vartheta_{m-1}, \vartheta_m). \end{aligned} \tag{6}$$

Since $\{\vartheta_n\}$ is strictly increasing sequence, then by using the property (2) of ϕ, we have

$$\phi(\vartheta_{m-1}, \vartheta_m) \leq \phi(\vartheta_{m-2}, \vartheta_m) \leq ... \leq \phi(\vartheta_{n+1}, \vartheta_m) \leq \phi(\vartheta_n, \vartheta_m), \tag{7}$$

for all $n, m \in \mathbb{N}$ with $n < m$. Now by using (7) in (6), we get

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) \leq\ & \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + [\phi(\vartheta_n, \vartheta_m)]^2 b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) \\ & + [\phi(\vartheta_n, \vartheta_m)]^3 b_\phi(\vartheta_{n+2}, \vartheta_{n+3}) + \cdots + [\phi(\vartheta_n, \vartheta_m)]^{m-n-1} b_\phi(\vartheta_{m-1}, \vartheta_m) \end{aligned} \tag{8}$$

By using (5) in (8), we have

$$\begin{aligned}
b_\phi(\vartheta_n,\vartheta_m) &\leq [\phi(\vartheta_n,\vartheta_m)]b_\phi(\vartheta_n,\vartheta_{n+1}) + [\phi(\vartheta_n,\vartheta_m)]^2 b_\phi(\vartheta_{n+1},\vartheta_{n+2}) + \cdots \\
&\quad + [\phi(\vartheta_n,\vartheta_m)]^{m-n-1} b_\phi(\vartheta_{m-1},\vartheta_m), \\
&\leq [\phi(\vartheta_n,\vartheta_m)]\lambda^n b_\phi(\vartheta_0, f\vartheta_0) + [\phi(\vartheta_n,\vartheta_m)]^2 \lambda^{n+1} b_\phi(\vartheta_0, f\vartheta_0) + \cdots \\
&\quad + [\phi(\vartheta_n,\vartheta_m)]^{m-n-1}\lambda^{m-1} b_\phi(\vartheta_0, f\vartheta_0), \\
&\leq \Big[[\phi(\vartheta_n,\vartheta_m)\lambda]^n + [\phi(\vartheta_n,\vartheta_m)\lambda]^{n+1} + \cdots + [\phi(\vartheta_n,\vartheta_m)\lambda]^{m-1}\Big] b_\phi(\vartheta_0, f\vartheta_0) \\
&= [t^n + t^{n+1} + \cdots + t^{m-1}] b_\phi(\vartheta_0, f\vartheta_0) \\
&= t^n \frac{(1-t^{m-n-1})}{1-t} b_\phi(\vartheta_0, f\vartheta_0),
\end{aligned}$$

where $t = \phi(\vartheta_n,\vartheta_m)\lambda$. Using the fact that $\lim\limits_{n,m\to\infty} \phi(\vartheta_n,\vartheta_m) < \frac{1}{\lambda}$, we have $\lim\limits_{n,m\to\infty} t < 1$. Hence, passing to limit as $n, m \to \infty$ in the inequality above, we conclude that,

$$\lim_{n,m\to\infty} b_\phi(\vartheta_n,\vartheta_m) \leq \lim_{n,m\to\infty} t^n \frac{(1-t^{m-n-1})}{1-t} b_\phi(\vartheta_0, f\vartheta_0) = 0. \tag{9}$$

This proves that $\{\vartheta_n\}$ is a strictly increasing Cauchy sequence and since $(X, b_\phi, \preceq)$ has the t-property, there is a $w \in X$, such that $\vartheta_n \prec w$, $\forall n \in \mathbb{N}$. Thus, by using (1) and (5), we have

$$\begin{aligned}
b_\phi(w, fw) &\leq \lambda b_\phi(\vartheta_n, f\vartheta_n), \\
&\leq \lambda^{n+1} b_\phi(\vartheta_0, f\vartheta_0), \\
&\to 0, \quad \text{as} \quad n \to \infty,
\end{aligned}$$

implying that $b_\phi(w, fw) = 0$ and hence, f has a fixed point in X which completes the proof. □

We provide some illustrations to support our theoretical result.

Example 3.3. *Let* $X = \{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{8}, ...\} \cup \{0\}$ *and "* $\preceq$ *" is defined as natural ordering "* $\leq$ *". We define the metric* $b_\phi : X \times X \to [0,\infty)$ *by*

$$\begin{aligned}
b_\phi(\vartheta,\eta) &= 0, \quad \textit{iff} \quad \vartheta = \eta \\
&= 3 + \vartheta + \eta, \quad \textit{iff} \quad \vartheta \neq \eta.
\end{aligned}$$

Further, if we specify $\phi : X \times X \to [1,\infty)$ *by* $\phi(\vartheta,\eta) = 5 + \eta - \vartheta$*, then we can easily verify that* $(X, b_\phi, \preceq)$ *is an ordered extended b-metric space.*
Now, we consider $f : X \to X$ *by* $f\vartheta = \frac{\vartheta}{2}$*. and* $\lambda = \frac{1}{7}$*.*
Here, for any $\vartheta_0 \in X$*, we can show that* $\vartheta_k = f^k(\vartheta_0) = -\frac{1}{2^k}$ *for some* $k \in \mathbb{N} \cup \{0\}$*. Thus,*

$$\phi(\vartheta_m,\vartheta_n) = 5 - \frac{1}{2^n} + \frac{1}{2^m}$$

and hence $\lim\limits_{m,n\to\infty} \phi(\vartheta_m,\vartheta_n) = 5 < 7 = \frac{1}{\lambda}$*.*
Now, it remains to prove (1). Let $\vartheta, \eta \in X$ *with* $\vartheta \prec \eta$*, we have*

$$\begin{aligned}
\lambda b_\phi(\vartheta, f\vartheta) - b_\phi(\eta, f\eta) &= \frac{1}{7}[b_\phi(\vartheta, \frac{\vartheta}{2})] - b_\phi(\eta, \frac{\eta}{2}) \\
&= \frac{1}{7}[3 + \vartheta + \frac{\vartheta}{2}] - [3 + \eta + \frac{\eta}{2}] \\
&\geq 0.
\end{aligned}$$

Thus, all of the requirements of Theorem 3.2 have been met. Hence, f *has a fixed point in* X *which is* 0*.*

3.2. Fixed points in $\overline{O}$-complete ordered extended b-metric spaces

We will first review some definitions before moving on to the major finding.

Definition 3.4. *[2]. An ordered metric space $(X, b_\phi, \preceq)$ is said to be $\overline{O}$-complete if every increasing Cauchy sequence in X converges in X. In an ordered metric space, completeness implies $\overline{O}$-completeness, but the converse is not true in general.*

Example 3.5. *Let $X = (0, \infty)$ induced with the natural ordering and $b_\phi(\vartheta, y) = |\vartheta - y|$, then clearly $(X, b_\phi, \preceq)$ is $\overline{O}$-complete but not complete.*

Theorem 3.6. *Let $(X, b_\phi, \preceq)$ be an $\overline{O}$-complete ordered extended b-metric space. Let $f : X \to X$ be a self-mapping which is continuous, monotone non-decreasing and satisfies,*
(D_1) there exists $\vartheta_0 \in X$ such that $\vartheta_0 \preceq f\vartheta_0$.
(D_2) for all $\vartheta, y \in X$ with $\vartheta \prec y$, $\vartheta \neq f(\vartheta)$, the inequality

$$b_\phi(y, fy) \leq \lambda[b_\phi(\vartheta, y) + b_\phi(f\vartheta, fy], \tag{10}$$

holds for some $\lambda \in (0, \frac{1}{2})$. Further, suppose that the mapping $\phi : X \times X \to [1, \infty)$ is such that

$$\phi(\vartheta, z) \geq \phi(y, z), \tag{11}$$

holds for all $\vartheta, y \in X$ with $\vartheta \prec y$ and for any $z \in X$. Let also $\lim_{m,n\to\infty} \phi(\vartheta_m, \vartheta_n) < \dfrac{1-\lambda}{\lambda}$, where $\vartheta_n = f^n\vartheta_0, \quad n \in \mathbb{N}$. Then f has a fixed point in X.

Proof. As in Theorem 3.2, starting with $\vartheta_0 \in X$ in the condition (D_1), we construct a strictly increasing sequence $\{\vartheta_n\}$ in X as

$$\vartheta_{n+1} = f\vartheta_n, \quad n \in \mathbb{N}. \tag{12}$$

Since $\vartheta_0 \prec \vartheta_1$, we replace y, ϑ in (10) by ϑ_1, ϑ_0 respectively, and we get

$$\begin{aligned} b_\phi(\vartheta_1, f\vartheta_1) &\leq \lambda[b_\phi(\vartheta_0, \vartheta_1) + b_\phi(f\vartheta_0, f\vartheta_1] \\ &= \lambda b_\phi(\vartheta_0, f\vartheta_0) + \lambda b_\phi(\vartheta_1, f\vartheta_1). \end{aligned}$$

Then

$$b_\phi(\vartheta_1, f\vartheta_1) \leq \frac{\lambda}{1-\lambda} b_\phi(\vartheta_0, f\vartheta_0). \tag{13}$$

Again as $\vartheta_1 \prec \vartheta_2$, by using (10) with $y = \vartheta_2, \vartheta = \vartheta_1$ and (12), we obtain

$$\begin{aligned} b_\phi(\vartheta_2, f\vartheta_2) &\leq \lambda[b_\phi(\vartheta_1, \vartheta_2) + b_\phi(f\vartheta_1, f\vartheta_2], \\ &= \lambda b_\phi(\vartheta_1, f\vartheta_1) + \lambda b_\phi(\vartheta_2, f\vartheta_2). \end{aligned}$$

Then,

$$b_\phi(\vartheta_2, f\vartheta_2) \leq \frac{\lambda}{1-\lambda} b_\phi(\vartheta_1, f\vartheta_1),$$

which implies,

$$b_\phi(\vartheta_2, f\vartheta_2) \leq \left(\frac{\lambda}{1-\lambda}\right)^2 b_\phi(\vartheta_0, f\vartheta_0),$$

upon using (13). Continuing this process, we get

$$b_\phi(\vartheta_n, f\vartheta_n) \leq \left(\frac{\lambda}{1-\lambda}\right)^n b_\phi(\vartheta_0, f\vartheta_0), \tag{14}$$

for all $n \in \mathbb{N}$. Since $0 < \lambda < \frac{1}{2}$, then $0 < k = \frac{\lambda}{1-\lambda} < 1$. Now, the inequality (14) becomes

$$b_\phi(\vartheta_n, f(\vartheta_n)) \leq k^n b_\phi(\vartheta_0, f\vartheta_0). \tag{15}$$

As in the proof of Theorem 3.2, we can show that $\{\vartheta_n\}$ is an increasing Cauchy sequence in X. Since $(X, b_\phi, \preceq)$ is $\overline{O}$-complete, there exists $w \in X$ such that

$$\lim_{n\to\infty} \vartheta_n = w. \tag{16}$$

Since f is continuous, we conclude that

$$w = \lim_{n\to\infty} \vartheta_{n+1} = \lim_{n\to\infty} f(\vartheta_n) = f(w). \tag{17}$$

Hence w is a fixed point of f in X which ends the proof. $\square$

3.3. *Fixed points of extended b-metric space in the sense of Boyd-Wong*

In this section, we state and prove our last main result which is a fixed point theorem for contractions of Boyd-Wong type on ordered, extended b-metric spaces with the t-property. The Boyd-Wong contractions [16] are known to be one important extension of the Banach contractions and studied by many authors [13, 22].

First, we recall the auxiliary functions involved in the definition of Boyd-Wong contractions.

Let Ψ be set of all functions $\psi : [0, \infty) \to [0, \infty)$ satisfying,

(i) ψ is non-decreasing,

(ii) $\psi(x) < x, \ \forall\, x > 0$,

(iii) $\lim_{r\to x^+} \psi(r) < x, \ \forall\, x > 0$.

We will use the following lemma, the proof of which can be found in [8].

Lemma 3.7. *[8]. Let $\psi \in \Psi$ and $\{u_n\}$ be a given sequence such that $u_n \to 0^+$ as $n \to \infty$. Then $\psi(u_n) \to 0^+$ as $n \to \infty$. Also $\psi(0) = 0$.*

Theorem 3.8. *Let $(X, b_\phi, \preceq)$ be an ordered extended b-metric space having the t-property and $f : X \to X$ be a monotone non-decreasing self-mapping. Assume that for all $\vartheta, y \in X$ with $\vartheta \prec y$, we have*

$$b_\phi(y, fy) \leq \psi(b_\phi(\vartheta, f\vartheta)), \tag{18}$$

where $\psi \in \Psi$. Suppose that the series $\sum_{n\geq 1} \psi^n(t)$ converges for all $t > 0$ and there exists $\vartheta_0 \in X$ such that $\vartheta_0 \preceq f(\vartheta_0)$. Suppose also that the mapping $\phi : X \times X \to [1, \infty)$ satisfies for all $\vartheta, y \in X$ with $\vartheta \prec y$

$$\phi(\vartheta, z) \geq \phi(y, z),$$

for all $\vartheta, y \in X$ with $\vartheta \prec y$ and any $z \in X$, and

$$\lim_{m,n\to\infty} \phi^m(\vartheta_m, \vartheta_n) = L,$$

where $L < \infty$, and $\vartheta_n = f^n\vartheta_0, \ n \in \mathbb{N}$. Then f has a fixed point in X. Moreover, every strict upper bound of fixed point of f is also a fixed point of f.

Proof. The proof starts as the proof of Theorem 3.2 by constructing a strictly increasing sequence $\{\vartheta_n\}$ in X defined by

$$\vartheta_{n+1} = f\vartheta_n. \tag{19}$$

Denote $T_n = b_\phi(\vartheta_n, f\vartheta_n)$, for all $n \in \mathbb{N}_0$. Since $\vartheta_n \neq f\vartheta_n$, we have $T_n > 0$ for all $n \in \mathbb{N}_0$. Also, using the fact that $\vartheta_n \prec \vartheta_{n+1}$ for all $n \in \mathbb{N}$, from (18), we have

$$T_{n+1} = b_\phi(\vartheta_{n+1}, f\vartheta_{n+1}) \leq \psi(b_\phi(\vartheta_n, f\vartheta_n)) = \psi(T_n) < T_n. \tag{20}$$

This shows that $\{T_n\}$ is a monotone decreasing sequence in $\mathbb{R}^+$ so, there exists $r \geq 0$ such that

$$\lim_{n\to\infty} T_n = r. \tag{21}$$

Letting $n \to \infty$ in (20), we get

$$r \leq \lim_{n\to\infty} \psi(T_n) < r,$$

which implies

$$\lim_{n\to\infty} \psi(T_n) = r. \tag{22}$$

Suppose that $r > 0$. By (22) and the property (iii) of the function ψ, we get

$$r = \lim_{n\to\infty} \psi(T_n) = \lim_{T_n\to r^+} \psi(T_n) < r,$$

which is a contradiction, so that, $r = 0$, and

$$\lim_{n\to\infty} T_n = 0. \tag{23}$$

Now, the condition (18) with $y = \vartheta_1$, $\vartheta = \vartheta_0$, we have

$$b_\phi(\vartheta_1, f\vartheta_1) \leq \psi(b_\phi(\vartheta_0, f\vartheta_0).$$

Repeating this process n times, we deduce

$$T_n = b_\phi(\vartheta_n, f\vartheta_n) \leq \psi^n(b_\phi(\vartheta_0, f\vartheta_0)), \text{ for all } n \geq 1.$$

Since $\sum_{n\geq 1} \psi^n(t)$ converges for all $t > 0$, we have that $\sum_{n\geq 1} T_n$ converges.
We shall show that $\{\vartheta_n\}$ is a Cauchy sequence in X. As $\{\vartheta_n\}$ is a strictly increasing sequence, for $n, m \in \mathbb{N}$ with $n < m$, by using the triangle inequality, (18), (19), (23) and the definition of ϕ, we obtain

$$\begin{aligned} b_\phi(\vartheta_n, \vartheta_m) &\leq \phi(\vartheta_n, \vartheta_m) b_\phi(\vartheta_n, \vartheta_{n+1}) + [\phi(\vartheta_n, \vartheta_m)]^2 b_\phi(\vartheta_{n+1}, \vartheta_{n+2}) \\ &\quad + [\phi(\vartheta_n, \vartheta_m)]^3 b_\phi(\vartheta_{n+2}, \vartheta_{n+3}) + \cdots + [\phi(\vartheta_n, \vartheta_m)]^{m-n-1} b_\phi(\vartheta_{m-1}, \vartheta_m) \\ &\leq [\phi(\vartheta_n, \vartheta_m)]^m [T_n + T_{n+1} + \cdots + T_{m-1}] \\ &\leq [\phi(\vartheta_n, \vartheta_m)]^m \sum_{k=n}^{\infty} T_k, \end{aligned} \tag{24}$$

Due to the fact that $\lim_{n,m\to\infty} \phi(\vartheta_n, \vartheta_m)]^m$ is finite and the series $\sum_{n\geq 1} T_n$ is convergent, its tail $\sum_{k_n}^{\infty} T_k = 0 \to 0$ as $n \to \infty$ and we have,

$$\lim_{n,m\to\infty} \phi(\vartheta_n, \vartheta_m)]^m \sum_{k_n}^{\infty} T_k = 0,$$

which implies that

$$\lim_{n,m\to\infty} b_\phi(\vartheta_n, \vartheta_m) = 0.$$

Hence, $\{\vartheta_n\}$ is a monotone increasing Cauchy sequence in X, which has the t-property, so there exists $w \in X$ such that $\vartheta_n \prec w$ for all n. By using (18) and (22), we have

$$b_\phi(w, fw) \leq \psi(b_\phi(\vartheta_n, f\vartheta_n)) = \psi(T_n) \to 0, \text{ as } n \to \infty.$$

This shows that w is a fixed point of f in X. Let $z \in X$ be any strict upper bound of w, *i.e.*, $w \prec z$. By using (18) and Lemma 3.7, we have

$$b_\phi(z, fz) \leq \psi(b_\phi(w, fw)) = \psi(0) = 0.$$

Hence z is also a fixed point of f in X. □

4. Conclusion

This article provides a thorough analysis of fixed point theorems in ordered extended b-metric spaces, with a focus on incomplete spaces. The primary goal was to demonstrate the existence of fixed points for different contractive mappings in such spaces, even though conventional theorems of existing literature are not applicable. The t-property notion was presented and investigated as a basic instrument to accomplish this goal.

More in-depth applications and generalizations of the existing findings may arise from future studies in this area. Overall, the study of fixed point theorems in incomplete ordered extended b-metric spaces extends the knowledge in this field and opens up fascinating new research directions.

5. Competing interests

The authors claim to have no interests that would be in conflict.

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