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L^p−regularity results for parabolic equations with robin type boundary conditions in non-rectangular domains

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Abstract. This paper is devoted to the analysis of the following linear parabolic equation $\partial_t u - \partial_x^2 u = f$, subject to Robin type conditions $\partial_x u + \beta u = 0$, on the lateral boundary, where coefficient β satisfies suitable non-degeneracy assumptions and possibly depends on the time variable. The right-hand side f of the equation is taken in $L^p, 1 < p < \infty$. The problem is set in a domain of the form $\Omega = \{(t,x) \in \mathbb{R}^2 : 0 < t < 1, 0 < x < t^\alpha\}$, $\alpha > 1/2$. We use Labbas-Terreni results [23] on the operator's sum method in the non-commutative case. This work is an extension of the Hilbertian case studied in [15].

1. Introduction

This work is devoted to the study of the following parabolic problem

$$\begin{cases} \partial_t u(t,x) - \partial_x^2 u(t,x) = f(t,x) \\ \partial_x u + \beta_0(t) u\big|_{x=0} = 0 \\ \partial_x u + \beta_1(t) u\big|_{x=t^\alpha} = 0, \end{cases} \tag{1}$$

set in the non-cylindrical domain

$$\Omega = \left\{(t,x) \in \mathbb{R}^2 : 0 < t < 1, 0 < x < t^\alpha\right\}.$$

Here, $f \in L^p(\Omega)$, $1 + \alpha < p < \infty$ with $\alpha > 1/2$ and the coefficients $\beta_i,\ i = 0,1$, are real-valued functions on $[0,1]$ such that

$$\beta_0(t) < 0 \text{ and } \beta_1(t) > 0 \text{ for all } t \in [0,1], \tag{2}$$

$$\inf_{t\in[0,1]} (\beta_0(t) + \beta_1(t)) > 0. \tag{3}$$

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We also assume that

$$\beta_i \in C^{\theta_i}([0,1]), \theta_i \in]0,1[, i = 0,1. \tag{4}$$

Note that, for $i = 0,1$, the Robin type condition

$$\partial_x u + \beta_i(t) u = 0,$$

is a perturbation by β_i of the Neumann type one $\partial_x u = 0$ and it is well known that Dirichlet and Neumann type boundary conditions correspond to two extreme cases, namely $\beta_i = \infty$ and $\beta_i = 0$, respectively.

Since the mid of the last century, classical results on the resolution of Problem can be found in [27], [28] and [29] in the case of cylindrical domains. Some regularity results in the frameworks of anisotropic Sobolev-Slobodetskii or Hölder or Morrey spaces are given in [10], [11], [19], [30] and [31] and in the references therein.

The L^2-solvability of Problem (1) has been investigated in [15] by the *a priori* estimates technique. We can find another study of Problem (1) in the case where Ω is a disc in [17]. Hofmann and Lewis [13] have considered the heat equation with Neumann boundary condition in non-cylindrical domains under some conditions of Lipschitz's type on the geometrical behavior of the boundary which cannot include our domain. They showed that the optimal L^p regularity holds for $p = 2$ and the situation gets progressively worse as p approaches 1. The case of Robin-Neumann type boundary conditions is considered in [5]. We can find in Savaré [36] an abstract study for parabolic problems with mixed (Dirichlet-Neumann) lateral boundary conditions.

There are many works on the analysis of parabolic problems with Cauchy-Dirichlet boundary conditions in non-cylindrical domains both in Hilbertian and non-Hilbertian spaces. We quote, for instance, a series of papers by Sadallah and kheloufi *et al.* [33], [18], [35], [34], [16], [6], [7] where such problems were studied by the domain decomposition method, Kozlov [21], [22], for domains with conical points, Aref'ev and Bagirov [2], [3], Ivanova and Ushakov [14]. In the non-Hilbertian spaces, as L^p-spaces, with $p \in (1,\infty)$, $p \neq 2$ or Hölder spaces, we can mention that optimal regularity results are obtained for Problem (1) with Cauchy-Dirichlet boundary conditions, see [25] and [4] respectively. Further references are: [1], [20] and [26].

The difficulty with the space L^p, $p \neq 2$, is that this space is not a Hilbert space. So, the Hilbertian techniques used in the most of above mentioned papers cannot be generalized in this sense. An idea for this extension (to the case L^p, $p \in (1,\infty)$) can be found in [25], in which Problem (1) with Cauchy-Dirichlet boundary conditions is studied by another approach making use of the operators sum method.

In this work, we will prove that Problem (1) has a solution with optimal regularity, that is a solution u belonging to the natural anisotropic Sobolev space

$$H_p^{1,2}(\Omega) = \left\{u \in L^p(\Omega) : \partial_t u, \partial_x^j u \in L^p(\Omega),\ j = 1,2\right\}.$$

The organization of this paper is as follows. In Section 3, by using a change of variables conserving the spaces L^p and $H_p^{1,2}$, we transform Problem (1) into a degenerated parabolic problem in a cylindrical domain. Section 4 is concerned with the resolution of the transformed problem. Our approach is different from that used in [9]. It is based on the direct use of operators sums in a weighted L^p-Sobolev space. For more details and recent results concerning this method, see [8], [32] and the references therein. In Section 5 we return to our initial problem by using an inverse change of variable.

Note that this approach may be extended at least in the following directions:

1. The function f on the right-hand side of the equation of Problem (1), may be taken in Hölder or little Hölder spaces.

2. The unidimensional case in x, can be naturally extended to an upper dimension in x, such as, for example, the following problem

$$\partial_t u(t,x_1,x_2) - \partial_{x_1}^2 u(t,x_1,x_2) - \partial_{x_2}^2 u(t,x_1,x_2) = f(t,x_1,x_2)$$

in the domain

$$\left\{(t,x_1,x_2)\in\mathbb{R}^3 : 0<t<1, x_1,x_2>0 \text{ and } (x_1/t^{\alpha},x_2/t^{\alpha})\in G\right\},$$

where G is some given cylindrical domain in $\mathbb{R}^2_+$. These questions will be developed in forthcoming works. In the following section we recall the essential of the sum theory we will have to apply.

2. On the sum of linear operators

Let Λ be a closed linear operator in a complex Banach space E. Then, Λ is said to be sectorial if
(i) $D(\Lambda)$ and $Im(\Lambda)$ are dense in E,
(ii) $\ker(\Lambda)=\{0\}$,
(iii) $]-\infty,0[\subset\rho(\Lambda)$ ($\rho(\Lambda)$ is the resolvent set of Λ) and there exists a constant $K\geq 1$ such that

$$\forall t>0, \left\|t(\Lambda+tI)^{-1}\right\|_{L(E)}\leq K.$$

If Λ is sectorial it follows easily that $\rho(-\Lambda)$ contains an open sector

$$\Sigma_{\varphi}=\left\{z\in\mathbb{C}: z\neq 0, \left|\arg z\right|<\varphi\right\},$$

with $\varphi\in]0,\pi[$.

Consider two closed linear operators A and B with dense domains $D(A)$ and $D(B)$ respectively in E. Assume that both operators satisfy the following assumptions of Da Prato-Grisvard type [8].

There exist positive numbers r, M_A, M_B, θ_A,θ_B such that

$$\theta_A+\theta_B<\pi, \tag{5}$$

$$\begin{aligned}&\rho(-A)\supset\Sigma_{\pi-\theta_A}=\left\{z\in\mathbb{C}:|z|\geq r,\left|\arg z\right|<\pi-\theta_A\right\} \text{ and}\\ &\forall\lambda\in\Sigma_{\pi-\theta_A},\ \left\|(A+\lambda I)^{-1}\right\|_{L(E)}\leq\frac{M_A}{|\lambda|},\end{aligned} \tag{6}$$

$$\begin{aligned}&\rho(-B)\supset\Sigma_{\pi-\theta_B}=\left\{z\in\mathbb{C}:|z|\geq r,\left|\arg z\right|<\pi-\theta_B\right\} \text{ and}\\ &\forall\mu\in\Sigma_{\pi-\theta_B},\ \left\|(B+\mu I)^{-1}\right\|_{L(E)}\leq\frac{M_B}{|\lambda|}.\end{aligned} \tag{7}$$

We also assume that there are constants $m\in\mathbb{N}^*$, $C>0$, $\lambda_0>0$, (with $\lambda_0\in\rho(-A)$), $\tau_i>0$ and $\rho_i>0$, $i=1,...,m$ such that

$$\begin{cases}(i)\ \left\|(A+\lambda_0 I)(A+\lambda I)^{-1}\left[(A+\lambda_0 I)^{-1};(B+\mu I)^{-1}\right]\right\|_{L(E)}\\ \leq C\sum\limits_{i=1}^{m}\dfrac{1}{|\lambda|^{1-\tau_i}.\left|\mu\right|^{1+\rho_i}}\ \ \forall\lambda\in\rho(-A), \forall\mu\in\rho(-B),\\ (ii)\ 0\leq\tau_i<\rho_i\leq 1,\ \forall i=1,...,m.\end{cases} \tag{8}$$

For more details concerning this last Labbas-Terreni commutator assumption see [23], [24].

For any $\sigma\in]0,1[$ and $1\leq p\leq+\infty$, let us introduce the real Banach interpolation spaces $D_A(\sigma,p)$ between $D(A)$ and E (or $D_B(\sigma,p)$ between $D(B)$ and E) which are characterized by

$$D_A(\sigma,p)=\left\{\xi\in E: t\longmapsto\left\|t^{\sigma}A(A-tI)^{-1}\xi\right\|_E\in L^p_*\right\},$$

where L^p_* denotes the space of p-integrable functions on $(0,+\infty)$ with the measure dt/t. For $p=+\infty$,

$$D_A(\sigma,+\infty)=\left\{\xi\in E:\sup_{t>0}\left\|t^{\sigma}A(A-tI)^{-1}\xi\right\|_E<\infty\right\}.$$

For these spaces, see [12]. Then the main result proved in Labbas-Terreni [23] is the following :

Theorem 2.1. *Under the assumptions (5), (6), (7) and (8), there exists λ^* such that for any $\lambda\geq\lambda^*$ and for any $h\in D_A(\sigma,p)$, equation $Aw+Bw+\lambda w=h$, has a unique solution $w\in D(A)\cap D(B)$ with the regularities Aw, $Bw\in D_A(\theta,p)$ and $Aw\in D_B(\theta,p)$ for any θ verifying $\theta\leq\min\limits_i(\sigma,(\rho_i-\tau_i))$.*

3. Change of variables

The change of variables

$$(t,x) \quad \mapsto \quad (t,y) = (t, x/t^{\alpha})$$

transforms Ω into the square $Q =]0,1[\times]0,1[$. Putting $u(t,x) = v(t,y)$ and $f(t,x) = g(t,y)$, then Problem (1) is transformed, in Q, into the degenerate evolution problem

$$\begin{cases} t^{2\alpha}\partial_t v(t,y) - \partial_y^2 v(t,y) - \alpha t^{2\alpha-1} y \partial_y v(t,y) = t^{2\alpha} g(t,y) = h(t,y) \\ v|_{t=0} = 0, \\ \partial_y v + t^{\alpha}\beta_0(t) v\big|_{y=0} = 0, \\ \partial_y v + t^{\alpha}\beta_1(t) v\big|_{y=1} = 0. \end{cases} \tag{9}$$

It is easy to see that $f \in L^p(\Omega)$ if and only if $t^{\alpha/p} g \in L^p(Q)$. Indeed

$$\begin{aligned} f \in L^p(\Omega) \quad &\Leftrightarrow \quad \int_0^1 \int_0^{t^{\alpha}} \left|f(t,x)\right|^p dtdx < +\infty \\ &\Leftrightarrow \quad \int_0^1 \int_0^1 \left|g(t,y)\right|^p t^{\alpha} dtdy < +\infty \\ &\Leftrightarrow \quad \int_0^1 \int_0^1 \left|t^{\alpha/p} g(t,y)\right|^p dtdy < +\infty \\ &\Leftrightarrow \quad t^{\alpha/p} g \in L^p(Q). \end{aligned}$$

Consequently $f \in L^p(\Omega)$ if and only if $t^{-2\alpha+(\alpha/p)} h \in L^p(Q)$ which implies that $h \in L^p(Q)$, since

$$h = \left(t^{-2\alpha+(\alpha/p)} h\right) t^{2\alpha-(\alpha/p)}$$

and $2\alpha - (\alpha/p) > 0$. Then the function $h = t^{2\alpha} g$ lies in the closed subspace of $L^p(Q)$ defined by

$$E = \left\{ h \in L^p(0,1; L^p(0,1)) : t^{-2\alpha+(\alpha/p)} h \in L^p(0,1; L^p(0,1)) \right\}.$$

This space is equipped with the norm

$$\|h\|_E = \left\| t^{-2\alpha+(\alpha/p)} h \right\|_{L^p(0,1;L^p(0,1))}.$$

4. Resolution of Problem (9)

4.1. Writing Problem (9) in an operational form

Let $\alpha > 1/2$ and assume

$$p > 1 + \alpha. \tag{10}$$

Set $X = L^p(0,1)$ and $v(t) = v(t,.)$, then Problem (9) is equivalent to the following operational degenerate Cauchy problem in X

$$\begin{cases} t^{2\alpha} v'(t) + L(t) v(t) = h(t), \quad t \in (0,1), \\ v(0) = 0, \end{cases} \tag{11}$$

where the family $(L(t))_{t\in[0,1]}$ is defined by

$$\begin{array}{lll} D(L(t)) & = & \left\{\psi \in W^{2,p}(0,1) : \psi'(j) + t^{\alpha}\beta_j(t)\psi(j) = 0,\ j = 0,1\right\}, \\ (L(t)\psi)(y) & = & -\psi''(y) - \alpha t^{2\alpha-1} y\psi'(y) \ \text{ for a.e. } t \in (0,1). \end{array}$$

Observe that $\overline{D(L(t))} = X$. Set

$$\begin{cases} w(t) = e^{-\lambda t^{1-2\alpha}/(1-2\alpha)} v(t) \\ k(t) = e^{-\lambda t^{1-2\alpha}/(1-2\alpha)} h(t), \end{cases}$$

where λ is some positive number. Then w verifies

$$\begin{cases} t^{2\alpha} w'(t) + L(t) w(t) + \lambda w(t) = k(t), \quad t \in (0,1), \\ w(0) = 0, \end{cases} \tag{12}$$

where k belongs to the space E. We obtain then the new operational form of the previous problem, mainly

$$Aw + Bw + \lambda w = k,$$

where

$$\begin{array}{ll} D(A) & = \{w \in E : w \in D(L(t)),\ \text{a.e. } t \in (0,1)\} \\ (Aw)(t) & = L(t) w(t), t \in [0,1], \end{array}$$

and

$$\begin{array}{ll} D(B) & = \left\{w \in E : t^{2\alpha} w' \in E \text{ and } w(0) = 0\right\} \\ (Bw)(t) & = t^{2\alpha} w'(t), t \in [0,1]. \end{array}$$

Note that the trace $w(0)$ is well defined in $D(B)$. In fact, we have

$$t^{\alpha/p} w \in L^p(0,1;X), t^{\alpha/p} w' \in L^p(0,1;X),$$

and in virtue of (10) $\alpha/p + 1/p < 1$. Then w is continuous on $[0,1]$, (see [38, Lemma, p. 42]).

4.2. Application of the sums

Now we are in position to apply the result of the sums of operators. For this purpose we must verify the assumptions of Theorem 2.1. The spectral properties of A and B are as follows.

Proposition 4.1. *A and B are linear closed operators and their domains are dense in E. Moreover, they satisfy assumptions (5), (6) and (7).*

Proof. 1. First, concerning operator B, the proof can be found in [26].

2. Now, we are concerned with the operator A which has the same properties as its realization $L(t)$. The study uses the following perturbation result due to Lunardi ([29, Proposition 2.4.3, p. 65]).

Proposition 4.2. *Let L_0 be a linear operator of domain $D(L_0)$ dense in E. Assume that L_0 is sectorial and P a linear continuous operator on $D(L_0)$ which is compact. Then operator $L_0 + P : D(L_0) \to X$ is sectorial.*

For each $t \in [0,1]$ we write

$$(L(t)\psi) = L_0(t)\psi + P(t)\psi,$$

with

$$\begin{cases} L_0(t)\psi = -\psi'' \\ D(L_0(t)) = \left\{\psi \in W^{2,p}(0,1) : \psi'(j) + t^{\alpha}\beta_j(t)\psi(j) = 0,\ j = 0,1\right\} \end{cases}$$

and

$$\begin{cases} P(t)\psi &= -\alpha t^{2\alpha-1} y\psi' \\ D(P(t)) &= W^{1,p}(0,1). \end{cases}$$

Note that the domains $D(L_0(t))$ are variable and depend effectively on t.

It is well known that $\overline{D(L_0(t))} = L^p(0,1)$. Let us prove that $L_0(t)$ is sectorial. The spectral problem

$$\begin{cases} w''(y) - \lambda w(y) = f(y), y \in [0,1], \\ w'(0) + t^\alpha \beta_0(t) w(0) = 0, \\ w'(1) + t^\alpha \beta_1(t) w(1) = 0, \end{cases}$$

has the unique solution

$$w(t,y) = \int_0^1 K_t(\tau,y) f(\tau)\, d\tau,$$

where (assuming $Re\sqrt{\lambda} > 0$)

$K_t(\tau,y) =$

$$\begin{cases} \dfrac{1}{\sqrt{\lambda}} \cdot \dfrac{\left[\sqrt{\lambda}\cosh\sqrt{\lambda}\tau - t^\alpha\beta_0(t)\sinh\sqrt{\lambda}\tau\right]\left[\sqrt{\lambda}\cosh\sqrt{\lambda}(1-y) - t^\alpha\beta_1(t)\sinh\sqrt{\lambda}(1-y)\right]}{\left[t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda} - (\lambda + t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right]}, \\ \text{if } 0 \le \tau \le y, \\ \dfrac{1}{\sqrt{\lambda}} \cdot \dfrac{\left[\sqrt{\lambda}\cosh\sqrt{\lambda}y - t^\alpha\beta_0(t)\sinh\sqrt{\lambda}y\right]\left[\sqrt{\lambda}\cosh\sqrt{\lambda}(1-\tau) - t^\alpha\beta_1(t)\sinh\sqrt{\lambda}(1-\tau)\right]}{\left[t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda} - (\lambda + t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right]}, \\ \text{if } y \le \tau \le 1. \end{cases}$$

Since $\left|K_t(\tau,y)\right| = \left|K_t(y,\tau)\right|$, then in virtue of the Schur's Lemma

$$\left\|(L_0(t) - \lambda I)^{-1}\right\|_{L(E)} \le \sup_{y\in[0,1]} \int_0^1 \left|K_t(\tau,y)\right| d\tau. \tag{13}$$

Setting $\rho = Re\sqrt{\lambda}$, $\sigma = Im\sqrt{\lambda}$, we have $\rho > 0$ and

$$\left|K_t(\tau,y)\right| \le \begin{cases} \dfrac{1}{\left|\sqrt{\lambda}\right|} \cdot \dfrac{\left|\left[\left|\sqrt{\lambda}\right| e^{i\frac{\theta}{2}} + t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right| e^{i\frac{\theta}{2}} + t^\alpha\beta_1(t)\right]\right| \cosh\rho\tau\cosh\rho(1-y)}{\left|t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda} - (\lambda + t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right|}, \\ \text{if } 0 \le \tau \le y, \\ \dfrac{1}{\left|\sqrt{\lambda}\right|} \cdot \dfrac{\left|\left[\left|\sqrt{\lambda}\right| e^{i\frac{\theta}{2}} + t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right| e^{i\frac{\theta}{2}} + t^\alpha\beta_1(t)\right]\right| \cosh\rho y\cosh\rho(1-\tau)}{\left|t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda} - (\lambda + t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right|}, \\ \text{if } y \le \tau \le 1; \end{cases}$$

hence

$$\int_0^1 \left|K_t(\tau,y)\right| d\tau \le \left(\cosh\rho(1-y)\int_0^y \cosh\rho\tau d\tau + \cosh\rho y\int_y^1 \cosh\rho(1-\tau)d\tau\right)$$
$$\times\frac{\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1(t)\right]\right|}{\left|\sqrt{\lambda}\right|\left|t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda}-(\lambda+t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right|}$$
$$\le[\sinh\rho y\cosh\rho(1-y)+\cosh\rho y\sinh\rho(1-y)]$$
$$\times\frac{\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1(t)\right]\right|}{\rho\left|\sqrt{\lambda}\right|\left|t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda}-(\lambda+t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right|}$$
$$=\frac{\sinh\rho}{\rho\left|\sqrt{\lambda}\right|}\cdot\frac{\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1(t)\right]\right|}{\left|t^\alpha\sqrt{\lambda}(\beta_0(t)+\beta_1(t))\cosh\sqrt{\lambda}-(\lambda+t^{2\alpha}\beta_0(t)\beta_1(t))\sinh\sqrt{\lambda}\right|}.$$

On the other hand a direct calculation shows that (in the following estimation, we write $\beta_i, i=0,1$ instead of $\beta_i(t), i=0,1$)

$$\left|t^\alpha\sqrt{\lambda}(\beta_0+\beta_1)\cosh\sqrt{\lambda}-\left(\lambda+t^{2\alpha}\beta_0\beta_1\right)\sinh\sqrt{\lambda}\right|$$
$$=\left|t^\alpha\sqrt{\lambda}(\beta_0+\beta_1)\left(\frac{e^{\sqrt{\lambda}}+e^{-\sqrt{\lambda}}}{2}\right)-\left(\lambda+t^{2\alpha}\beta_0\beta_1\right)\left(\frac{e^{\sqrt{\lambda}}-e^{-\sqrt{\lambda}}}{2}\right)\right|$$
$$=\left|\frac{e^{\sqrt{\lambda}}}{2}\left[t^\alpha|\sqrt{\lambda}|(\beta_0+\beta_1)e^{i\frac{\theta}{2}}-|\lambda|e^{i\theta}-t^{2\alpha}\beta_0\beta_1\right]\right.$$
$$\left.+\frac{e^{-\sqrt{\lambda}}}{2}\left[t^\alpha|\sqrt{\lambda}|(\beta_0+\beta_1)e^{i\frac{\theta}{2}}+|\lambda|e^{i\theta}+t^{2\alpha}\beta_0\beta_1\right]\right|$$
$$\ge\frac{e^\rho}{2}\left|t^\alpha|\sqrt{\lambda}|(\beta_0+\beta_1)e^{i\frac{\theta}{2}+\Pi}+|\lambda|e^{i\theta}+t^{2\alpha}\beta_0\beta_1\right|-$$
$$\frac{e^{-\rho}}{2}\left|t^\alpha|\sqrt{\lambda}|(\beta_0+\beta_1)e^{i\frac{\theta}{2}}+|\lambda|e^{i\theta}+t^{2\alpha}\beta_0\beta_1\right|$$
$$=\frac{e^\rho}{2}\left|\left[\left|\sqrt{\lambda}\right|e^{i(\frac{\theta}{2}+\Pi)}+t^\alpha\beta_0\right]\left[\left|\sqrt{\lambda}\right|e^{i(\frac{\theta}{2}+\Pi)}+t^\alpha\beta_1\right]\right|-$$
$$\frac{e^{-\rho}}{2}\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_0\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1\right]\right|$$
$$\ge\frac{e^\rho}{2}\left|\left[\left|\sqrt{\lambda}\right|e^{i(\frac{\theta}{2}+\Pi)}+t^\alpha\beta_0\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1\right]\right|-$$
$$\frac{e^{-\rho}}{2}\left|\left[\left|\sqrt{\lambda}\right|e^{i(\frac{\theta}{2}+\Pi)}+t^\alpha\beta_0\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1\right]\right|$$
$$=\frac{e^\rho-e^{-\rho}}{2}\left|\left[\left|\sqrt{\lambda}\right|e^{i(\frac{\theta}{2}+\Pi)}+t^\alpha\beta_0\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1\right]\right|$$
$$=\sinh\rho\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1\right]\right|$$

which implies

$$\left|t^\alpha\sqrt{\lambda}(\beta_0+\beta_1)\cosh\sqrt{\lambda}-\left(\lambda+t^{2\alpha}\beta_0\beta_1\right)\sinh\sqrt{\lambda}\right|$$
$$\ge\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1\right]\right|\sinh\rho,$$

and consequently

$$\sup_{y\in[0,1]}\int_0^1 K_t(\tau,y)d\tau\le\frac{\sinh\rho}{\rho\left|\sqrt{\lambda}\right|}\cdot\frac{\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1(t)\right]\right|}{\left|\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha|\beta_0(t)|\right]\left[\left|\sqrt{\lambda}\right|e^{i\frac{\theta}{2}}+t^\alpha\beta_1(t)\right]\right|\sinh\rho},$$

which gives

$$\sup_{y\in[0,1]}\int_0^1 K_t(\tau,y)\,d\tau \le \frac{1}{\rho\left|\sqrt{\lambda}\right|} = \frac{1}{|\lambda|\cos\theta/2} \tag{14}$$

with $\left|\arg\lambda\right| = \theta$. Hence

$$\left\|(L_0(t)-\lambda I)^{-1}\right\|_{L(E)} \le \frac{1}{|\lambda|\cos\theta/2}.$$

Moreover, thanks to Hölder inequality, for $\psi \in D(L_0(t)) \subset D(P(t))$ we have

$$\begin{aligned}
&\left\|P(t)\psi\right\|_{L^p(0,1)}\\
&= \left(\int_0^1 \left|-\alpha t^{2\alpha-1} y\psi'(y)\right|^p dy\right)^{\frac{1}{p}}\\
&= \left(\int_0^1 \left|-\alpha t^{2\alpha-1} y\left[\int_0^y s\psi''(s)\,ds - \int_y^1 (1-s)\psi''(s)\,ds\right]\right|^p dy\right)^{\frac{1}{p}}\\
&\le \alpha t^{2\alpha-1}\left(\int_0^1 \left|y\int_0^y s\psi''(s)\,ds\right|^p dy\right)^{\frac{1}{p}} + \alpha t^{2\alpha-1}\left(\int_0^1 \left|y\int_y^1 (1-s)\psi''(s)\,ds\right|^p dy\right)^{\frac{1}{p}}\\
&\le \alpha t^{2\alpha-1}\left[C_1(p)\left\|\psi''\right\|_{L^p(0,1)} + C_2(p)\left\|\psi''\right\|_{L^p(0,1)}\right]\\
&\le C_3(p,\alpha)\left\|\psi''\right\|_{D(L_0(t))}.
\end{aligned}$$

On the other hand, let us set

$$\begin{array}{llll}
m(t): & L^p(0,1) & \to & L^p(0,1)\\
 & \psi & \mapsto & (m(t)\psi)(y) = -\alpha t^{2\alpha-1} y\psi(y),\\
 & & & \\
i: & W^{1,p}(0,1) & \to & L^p(0,1)\\
 & \psi & \mapsto & \psi,\\
 & & & \\
d: & W^{2,p}(0,1) & \to & W^{1,p}(0,1)\\
 & \psi & \mapsto & d(\psi) = \psi'.
\end{array}$$

Then one has $P(t) = m(t)\circ i \circ d$. Thus, $P(t)$ is compact from $D(L_0(t))$ into E since i is compact and d, $m(t)$ are continuous. So for any $t \in [0,1]$, operator $L(t)$ is sectorial and consequently there exist some $r_0 > 0$ and $\theta_1 \in \left]0, \frac{\pi}{2}\right[$ such that

$$\rho(-L(t)) \supset \Sigma_{\pi-\theta_1} = \left\{z : |z| \ge r_0, \left|\arg z\right| < \pi - \theta_1\right\}.$$

Now, for $k \in E$ and $z \in \Sigma_{\pi-\theta_1}$ the spectral equation

$$Aw + zw = k,$$

is equivalent to

$$L(t)w(t) + zw(t) = k(t), t \in [0,1],$$

which admits a unique solution,

$$w(t) = (L(t)+z)^{-1}k(t).$$

Hence

$$\|w(t)\|_{L^p(0,1)} \leq \frac{K}{|z|} \|k(t)\|_{L^p(0,1)},$$

which implies

$$\|w\|_E = \left(\int_0^1 \left\| t^{-2\alpha + (\alpha/p)} w(t) \right\|_X^p d\tau \right)^{1/p} \leq \frac{K}{|z|} \|k\|_E .$$

This ends the proof of Proposition 4.1. □

From the Agmon's result, see Tanabe ([37, p. 83, Lemma 3.8.1]), we can deduce the following.

Proposition 4.3. *For every $t \in [0,1]$, the problem*

$$\begin{cases} -w'' - \alpha t^{2\alpha-1} y w' + \lambda w = f \in L^p(0,1), \\ w'(0) + t^\alpha \beta_0(t) w(0) = g_0, \\ w'(1) + t^\alpha \beta_1(t) w(1) = g_1, \end{cases}$$

admits a unique solution $w \in W^{2,p}(0,1)$, provided λ belongs to a suitable sector (depending on p)

$$\Sigma_{\varphi_0,\lambda_0} = \left\{ \lambda \in \mathbb{C} : |\lambda| \geq \lambda_0, \left|\arg \lambda\right| \leq \varphi_0 \right\},$$

with $\lambda_0 > 0$ and $\varphi_0 \in]\pi/2, \pi[$. Moreover, the following estimate holds

$$\begin{aligned} &|\lambda| \|w\|_{L^p(0,1)} + |\lambda|^{1/2} \|w'\|_{L^p(0,1)} + \|w''\|_{L^p(0,1)} \\ &\quad \leq C(p) \left\{ \left\|f\right\|_{L^p(0,1)} + |\lambda|^{1/2} \left(\|G_0\|_{L^p(0,1)} + \|G_1\|_{L^p(0,1)} \right) \right. \\ &\quad \left. + \|G_0\|_{W^{1,p}(0,1)} + \|G_1\|_{W^{1,p}(0,1)} \right\}, \end{aligned} \tag{15}$$

where $G_0 = g_0$ on $y = 0$ and $G_1 = g_1$ on $y = 1$.

Let us prove now:

Proposition 4.4. *A and B satisfy the Labbas-Terreni condition (8).*

Proof. Fix σ and t such that

$$0 \leq \sigma \leq t \leq 1.$$

Note that

$$\begin{aligned} &L(t)(L(t)+\lambda)^{-1}\left(L(t)^{-1} - L(\sigma)^{-1}\right) \\ &= (L(t)+\lambda)^{-1} - L(t)(L(t)+\lambda)^{-1} L(\sigma)^{-1} \\ &= -L(\sigma)^{-1} + (L(t)+\lambda)^{-1} + L(\sigma)^{-1} - L(t)(L(t)+\lambda)^{-1} L(\sigma)^{-1} \\ &= -L(\sigma)^{-1} + (L(t)+\lambda)^{-1} + (\lambda + L(t) - L(t))(L(t)+\lambda)^{-1} L(\sigma)^{-1} \\ &= -L(\sigma)^{-1} + (L(t)+\lambda)^{-1} + \lambda (L(t)+\lambda)^{-1} L(\sigma)^{-1} \\ &= -L(\sigma)^{-1} + (L(t)+\lambda)^{-1} (L(\sigma)+\lambda) L(\sigma)^{-1}. \end{aligned}$$

Let $g \in L^p(0,1)$ and set $v = L(\sigma)^{-1} g$, $u = (L(t)+\lambda)^{-1}(L(\sigma)+\lambda) v$. We will estimate $\|u - v\|_{L^p(0,1)}$. We have

$$\begin{cases} -v''(y) - \alpha\sigma^{2\alpha-1} y v'(y) = g(y), \ y \in (0,1), \\ v'(0) + \sigma^\alpha \beta_0(\sigma) v(0) = 0, \\ v'(1) + \sigma^\alpha \beta_1(\sigma) v(1) = 0, \end{cases}$$

and

$$\begin{cases} -u''(y) - \alpha t^{2\alpha-1} y u'(y) + \lambda u(y) = -v''(y) - \alpha \sigma^{2\alpha-1} y v'(y) + \lambda v(y), \ y \in (0,1), \\ u'(0) + t^{\alpha}\beta_0(t) u(0) = 0, \\ u'(1) + t^{\alpha}\beta_1(t) u(1) = 0; \end{cases}$$

therefore $u - v$ is solution of the following problem

$$\begin{cases} -(u-v)'' - \alpha t^{2\alpha-1} y (u-v)' + \lambda(u-v) = \alpha\left[t^{2\alpha-1} - \sigma^{2\alpha-1}\right] y v', \ y \in (0,1), \\ (u-v)'(0) + t^{\alpha}\beta_0(t)(u-v)(0) = \left[t^{\alpha}\beta_0(t) - \sigma^{\alpha}\beta_0(\sigma)\right] v(0) = g_0, \\ (u-v)'(1) + t^{\alpha}\beta_1(t)(u-v)(1) = \left[t^{\alpha}\beta_1(t) - \sigma^{\alpha}\beta_1(\sigma)\right] v(1) = g_1. \end{cases}$$

Consider now two cut-functions $\Phi_0, \Phi_1 \in D(0,1)$ such that

$$\begin{cases} \Phi_0(y) = 1 \text{ if } y \leq 1/2, \\ \Phi_0(y) = 0 \text{ if } 1/2 < y \end{cases}$$

and

$$\begin{cases} \Phi_1(y) = 0 \text{ if } y \leq 1/2, \\ \Phi_1(y) = 1 \text{ if } 1/2 < y. \end{cases}$$

Applying estimate (15) in the previous proposition by taking

$$G_0 = \Phi_0 g_0, \ G_1 = \Phi_1 g_1,$$

(which verify $G_0 = g_0$ on $y = 0$ and $G_1 = g_1$ on $y = 1$, by construction), we then obtain

$$\begin{aligned} |\lambda| \|u - v\|_{L^p(0,1)} \leq \ & C(p)\Big(\left\|\alpha\left[t^{2\alpha-1} - \sigma^{2\alpha-1}\right] y v'(y)\right\|_{L^p(0,1)} \\ & + |\lambda|^{1/2}\left(\left\|\Phi_0 g_0\right\|_{L^p(0,1)} + \left\|\Phi_1 g_1\right\|_{L^p(0,1)}\right) \\ & + \left\|\Phi_0 g_0\right\|_{W^{1,p}(0,1)} + \left\|\Phi_1 g_1\right\|_{W^{1,p}(0,1)}\Big). \end{aligned}$$

We have

$$\begin{aligned} \left\|\alpha\left[t^{2\alpha-1} - \sigma^{2\alpha-1}\right] y v'\right\|_{L^p(0,1)} & \leq \alpha\left|t^{2\alpha-1} - \sigma^{2\alpha-1}\right| \left\|y v'\right\|_{L^p(0,1)} \\ & \leq M_1 |t - \sigma|^{\min(1,2\alpha-1)} \|v\|_{W^{2,p}(0,1)} \\ & \leq M |t - \sigma|^{\min(1,2\alpha-1)} \left\|g\right\|_{L^p(0,1)}, \end{aligned}$$

$$\begin{aligned} & |\lambda|^{1/2} \left\|\Phi_0 g_0\right\|_{L^p(0,1)} \\ & \leq |\lambda|^{1/2} \left|\left[t^{\alpha}\beta_0(t) - \sigma^{\alpha}\beta_0(\sigma)\right] v(0)\right| \\ & \leq |\lambda|^{1/2} \left|t^{\alpha}\beta_0(t) - t^{\alpha}\beta_0(\sigma) + t^{\alpha}\beta_0(\sigma) - \sigma^{\alpha}\beta_0(\sigma)\right| |v(0)| \\ & \leq C|\lambda|^{1/2} \left[\left|\beta_0(t) - \beta_0(\sigma)\right| + |t^{\alpha} - \sigma^{\alpha}|\right] |v(0)| \\ & \leq C|\lambda|^{1/2} \left[|t - \sigma|^{\theta_0} + |t - \sigma|^{\min(1,\alpha)}\right] |v(0)| \\ & \leq C|\lambda|^{1/2} |t - \sigma|^{\min(1,\alpha,\theta_0)} |v(0)| \\ & \leq C|\lambda|^{1/2} |t - \sigma|^{\min(1,\alpha,\theta_0)} \|v\|_{L^{\infty}(0,1)} \\ & \leq C|\lambda|^{1/2} |t - \sigma|^{\min(1,\alpha,\theta_0)} \|v\|_{W^{1,p}(0,1)} \\ & \leq C|\lambda|^{1/2} |t - \sigma|^{\min(1,\alpha,\theta_0)} \|v\|_{W^{2,p}(0,1)} \\ & \leq C|\lambda|^{1/2} |t - \sigma|^{\min(1,\alpha,\theta_0)} \left\|g\right\|_{L^p(0,1)} \end{aligned}$$

and in the similar manner we get

$$|\lambda|^{1/2} \left\|\Phi_1 g_1\right\|_{L^p(0,1)} \leq C|\lambda|^{1/2} |t - \sigma|^{\min(1,\alpha,\theta_1)} \left\|g\right\|_{L^p(0,1)}.$$

Therefore

$$\begin{aligned}
&\|u-v\|_{L^p(0,1)} \\
&\leq \left\{\frac{C}{|\lambda|}|t-\sigma|^{\min(1,2\alpha-1)}+\frac{C}{|\lambda|^{1/2}}|t-\sigma|^{\min(1,\alpha,\theta_0,\theta_1)}\right\}\|g\|_{L^p(0,1)} \\
&\leq \left\{\frac{C}{|\lambda|}|t-\sigma|^{\alpha_1}+\frac{C}{|\lambda|^{1/2}}|t-\sigma|^{\alpha_2}\right\}\|g\|_{L^p(0,1)},
\end{aligned}$$

where $\alpha_1 = \min(1, 2\alpha - 1)$ and $\alpha_2 = \min(1, \alpha, \theta_0, \theta_1)$.
To prove (8), it is sufficient to estimate

$$\left\|A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right]\right\|_{L(E)}$$

where $\lambda \in \rho(-A)$ and $z \in \rho(-B)$. Let $k \in E$, then

$$\begin{aligned}
\Delta &= \left(t^{-2\alpha+(\alpha/p)}A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right]k\right)(t) \\
&= t^{-2\alpha+(\alpha/p)}(A(A+\lambda)^{-1}\left(A^{-1}(B+z)^{-1}-(B+z)^{-1}A^{-1}\right)k)(t) \\
&= t^{-2\alpha+(\alpha/p)}L(L+\lambda)^{-1}\left[L^{-1}\left((B+z)^{-1}k\right)(t)-\left((B+z)^{-1}L^{-1}k\right)(t)\right] \\
&= L(t)(L(t)+\lambda)^{-1}\int_0^1 \sigma^{-2\alpha+(\alpha/p)}K_z(t,\sigma)\left(L(t)^{-1}-L(\sigma)^{-1}\right)k(\sigma)\,d\sigma \\
&= \int_0^1 \sigma^{-2\alpha+(\alpha/p)}K_z(t,\sigma)L(t)(L(t)+\lambda)^{-1}\left(L(t)^{-1}-L(\sigma)^{-1}\right)k(\sigma)\,d\sigma
\end{aligned}$$

where

$$K_z(t,\sigma) = \begin{cases} \dfrac{1}{t^{2\alpha-(\alpha/p)}\sigma^{\alpha/p}}\exp\left\{\dfrac{z}{(2\alpha-1)}\left(t^{1-2\alpha}-\sigma^{1-2\alpha}\right)\right\} \text{ if } t>\sigma \\ 0 \text{ if } t<\sigma, \end{cases}$$

see [26]. Then

$$\begin{aligned}
\|\Delta\|_X \leq\ & \frac{K}{|\lambda|}\int_0^1 \left|K_\mu(t,\sigma)\right||t-\sigma|^{\alpha_1}\sigma^{-2\alpha+(\alpha/p)}\|k(\sigma)\|_X\,d\sigma \\
&+\frac{K}{|\lambda|^{1/2}}\int_0^1 \left|K_\mu(t,\sigma)\right||t-\sigma|^{\alpha_2}\sigma^{-2\alpha+(\alpha/p)}\|k(\sigma)\|_X\,d\sigma,
\end{aligned}$$

with $\alpha_1 = \min(1, 2\alpha - 1)$, $\alpha_2 = \min(\alpha, \theta_0, \theta_1)$. and $\mu = z/(2\alpha - 1)$. We have

$$\begin{aligned}
&\int_0^1 \left|K_\mu(t,\sigma)\right||t-\sigma|^{\alpha_1}\,d\sigma \\
&= \frac{1}{t^{2\alpha-(\alpha/p)}}\exp\left(Re\mu t^{1-2\alpha}\right)\int_0^t \sigma^{-\alpha/p}(t-\sigma)^{\alpha_1}\exp\left(-Re\mu\sigma^{1-2\alpha}\right)d\sigma.
\end{aligned}$$

Then by Hölder inequality, one has

$$\begin{aligned}
&\int_0^t \sigma^{-\alpha/p}(t-\sigma)^{\alpha_1}\exp\left(-Re\mu\sigma^{1-2\alpha}\right)d\sigma \\
&\leq \left(\int_0^t \sigma^{-\alpha/p}\exp\left(-Re\mu\sigma^{1-2\alpha}\right)d\sigma\right)^{1-\alpha_1}\times\left(\int_0^t \sigma^{-\alpha/p}(t-\sigma)\exp\left(-Re\mu\sigma^{1-2\alpha}\right)d\sigma\right)^{\alpha_1}
\end{aligned}$$

and

$$\begin{aligned}
J_1 &= \left(\int_0^t \sigma^{2\alpha-\alpha/p}\sigma^{-2\alpha}\exp\left(-Re\mu\sigma^{1-2\alpha}\right)d\sigma\right)^{1-\alpha_1} \\
&\leq \frac{\left(t^{2\alpha-\alpha/p}\right)^{1-\alpha_1}}{(2\alpha-1)^{1-\alpha_1}}\frac{1}{(Re\mu)^{1-\alpha_1}}\left(\exp\left(-Re\mu t^{1-2\alpha}\right)\right)^{1-\alpha_1}
\end{aligned}$$

$$\begin{aligned} J_2 &= \left(\int_0^t \sigma^{2\alpha-\alpha/p} (t-\sigma) \sigma^{-2\alpha} \exp\left(-Re\mu\sigma^{1-2\alpha} \right) d\sigma \right)^{\alpha_1} \\ &\leq \frac{\left(t^{2\alpha-\alpha/p}\right)^{\alpha_1}}{(2\alpha-1)^{\alpha_1}} \frac{1}{(Re\mu)^{\alpha_1}} \left(\int_0^t (t-\sigma) \chi'(\sigma) \right)^{\alpha_1} \end{aligned}$$

where $\chi(\sigma) = \exp\left(-Re\mu\sigma^{1-2\alpha}\right)$. Using an integration by parts, we obtain

$$\begin{aligned} \int_0^t (t-\sigma)\chi'(\sigma)\,d\sigma &= \int_0^t (t-\sigma)\exp\left(-Re\mu\sigma^{1-2\alpha}\right) d\sigma \\ &= \int_0^t \sigma^{2\alpha}(t-\sigma)\sigma^{-2\alpha}\exp\left(-Re\mu\sigma^{1-2\alpha}\right) d\sigma \\ &\leq \frac{t^{2\alpha}}{(2\alpha-1)} \frac{1}{(Re\mu)^{\alpha_1}} \exp\left(-Re\mu t^{1-2\alpha}\right) \end{aligned}$$

from which we deduce that

$$J_2 \leq \frac{\left(t^{2\alpha-\alpha/p}\right)^{\alpha_1}}{(2\alpha-1)^{\alpha_1}} \frac{1}{(Re\mu)^{\alpha_1}} \frac{t^{2\alpha}}{(2\alpha-1)} \frac{1}{(Re\mu)^{\alpha_1}} \left(\exp\left(-Re\mu t^{1-2\alpha}\right)\right)^{\alpha_1}.$$

Finally we have

$$\begin{aligned} &\int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^{\alpha_1} d\sigma \\ \leq\; & \frac{\exp\left(Re\mu t^{1-2\alpha}\right)}{t^{2\alpha-(\alpha/p)}} \frac{\left(t^{2\alpha-\alpha/p}\right)^{1-\alpha_1}}{(2\alpha-1)^{1-\alpha_1}} \frac{\left(\exp\left(-Re\mu t^{1-2\alpha}\right)\right)^{1-\alpha_1}}{(Re\mu)^{1-\alpha_1}} \\ &\times \frac{\left(t^{2\alpha-\alpha/p}\right)^{\alpha_1}}{(2\alpha-1)^{\alpha_1}} \frac{1}{(Re\mu)^{\alpha_1}} \frac{t^{2\alpha}}{(2\alpha-1)} \frac{\left(\exp\left(-Re\mu t^{1-2\alpha}\right)\right)^{\alpha_1}}{(Re\mu)^{\alpha_1}} \\ \leq\; & \frac{\left(t^{2\alpha}\right)^{\alpha_1}}{(2\alpha-1)^{1+\alpha_1}} \frac{1}{(Re\mu)^{1+\alpha_1}}, \end{aligned}$$

and

$$\max_{t\in[0,1]} \int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^{\alpha_1} d\sigma \leq \frac{C}{(Re\mu)^{1+\alpha_1}}. \tag{16}$$

In a similar manner we obtain

$$\max_{\sigma\in[0,1]} \int_0^1 \left|K_\mu(t,\sigma)\right| |t-\sigma|^{\alpha_1} dt \leq \frac{C}{(Re\mu)^{1+\alpha_1}}. \tag{17}$$

Now, using Schur interpolation Lemma together with (16) and (17), we obtain

$$\begin{aligned} \left\| A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right] \right\|_{L(E)} &\leq \frac{C}{|\lambda|(Re\mu)^{1+\alpha_1}} + \frac{C}{|\lambda|^{1/2}(Re\mu)^{1+\alpha_2}} \\ &= \frac{C}{|\lambda|(Rez)^{1+\alpha_1}} + \frac{C}{|\lambda|^{1/2}(Rez)^{1+\alpha_2}} \end{aligned}$$

which implies

$$\left\| A(A+\lambda)^{-1}\left[A^{-1};(B+z)^{-1}\right] \right\|_{L(E)} \leq \frac{C}{|\lambda||z|^{1+\alpha_1}} + \frac{C}{|\lambda|^{1/2}|z|^{1+\alpha_2}}$$

for any $\lambda \in \rho(-A)$ and any z belonging to a suitable sectorial curve.
Then (8) is verified with $(\tau_1, \rho_1) = (0, \alpha_1)$ and $(\tau_2, \rho_2) = (1/2, \alpha_2)$. □

Using Theorem 2.1, we deduce the following result

Proposition 4.5. *There exists λ^* such that for all $\lambda \geq \lambda^*$ and for all $k \in D_A(\sigma, p)$, Problem (12) admits a unique solution $w \in D(A) \cap D(B)$ such that for all $\theta \leq \min(\sigma, \delta)$ with $\delta = \min\left(\alpha_1, \alpha_2 - \frac{1}{2}\right)$*

i) $L(.)\, w \in D_A(\theta, p)$,
ii) $t^{2\alpha} w' \in D_A(\theta, p)$,
iii) $L(.)\, w \in D_B(\theta, p)$.

Observe that we have a similar result when $k \in D_B(\sigma, p)$. To make precise the time and the space regularity of w we need to specify the space $D_A(\sigma, p)$. One has

$$D_A(\sigma, p) = \begin{cases} \left\{w \in E : \ t^{-2\alpha + (\alpha/p)} w \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right), w'(t,j) + t^{\alpha}\beta_j(t)\, w(t,j) = 0\right\} \\ j = 0, 1 \text{ if } 2\sigma > 1/p \\ \left\{w \in E : \ t^{-2\alpha + (\alpha/p)} w \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)\right\} \text{ if } 2\sigma < 1/p. \end{cases}$$

Indeed we know that

$$D_A(\sigma, p) = \left\{w \in E : \left\|\zeta^{1-\sigma} A e^{-\zeta A} w\right\|_E \in L_*^p\right\},$$

because $-A$ is a generator of the analytic semigroup $\left\{e^{-\zeta A}\right\}_{\zeta \geq 0}$. Now, $w \in D_A(\sigma, p)$ implies

$$\left\|\zeta^{1-\sigma} A e^{-\zeta A} w\right\|_E \in L_*^p.$$

Or $\left\|\zeta^{1-\sigma} A e^{-\zeta A} w\right\|_E \in L_*^p$ is equivalent to

$$\begin{aligned} \int_0^{\infty} \left\|\zeta^{1-\sigma} A e^{-\zeta A} w\right\|_E^p \frac{d\zeta}{\zeta} &= \int_0^{\infty} \left\|t^{-2\alpha + (\alpha/p)} \zeta^{1-\sigma} A e^{-\zeta A} w\right\|_{L^p(0,1;L^p(0,1))}^p \frac{d\zeta}{\zeta} \\ &= \int_0^{\infty} \left(\int_0^1 \left\|t^{-2\alpha + (\alpha/p)} \zeta^{1-\sigma} \left(A e^{-\zeta A} w\right)(t)\right\|_{L^p(0,1)}^p dt\right) \frac{d\zeta}{\zeta} \\ &< +\infty. \end{aligned}$$

On the other hand, thanks to the Dunford representation of the semigroup $\left\{e^{-\zeta A}\right\}_{\zeta \geq 0}$, we have

$$e^{-\zeta A} = \frac{1}{2i\pi} \int_{\gamma} e^{\zeta \lambda} (A + \lambda)^{-1} d\lambda,$$

where γ is a sectorial curve lying in $\rho(-A)$ such that $Re(-\lambda) < 0$ for a larger $\lambda \in \gamma$. Moreover

$$\left(A e^{-\zeta A} w\right)(t) = L(t)\, e^{\zeta L(t)} (w(t)).$$

Then, by Fubini's Theorem, we obtain

$$\begin{aligned} &\int_0^{\infty} \left\|\zeta^{1-\sigma} A e^{-\zeta A} w\right\|_E^p \frac{d\zeta}{\zeta} \\ &= \int_0^{\infty} \left[\int_0^1 \left\|t^{-2\alpha + (\alpha/p)} \zeta^{1-\sigma} L(t)\, e^{\zeta L(t)} (w(t))\right\|_{L^p(0,1)}^p dt\right] \frac{d\zeta}{\zeta} \\ &= \int_0^1 \left\|t^{-2\alpha + (\alpha/p)}\right\|^p \left[\int_0^{\infty} \left\|\zeta^{1-\sigma} L(t)\, e^{\zeta L(t)} w(t)\right\|_{L^p(0,1)}^p \frac{d\zeta}{\zeta}\right] dt < +\infty \end{aligned}$$

which means that, for almost every t, the function

$$y \quad \mapsto \quad t^{-2\alpha + (\alpha/p)} w(t)(y)$$

is in $D_{L(t)}(\sigma, p)$. It is well known that this last space is the following:

$$D_{L(t)}(\sigma, p) = \left(W_*^{2,p}(0,1); L^p(0,1)\right)_{1-\sigma,p}$$

with

$$W_*^{2,p}(0,1) = \left\{w \in W^{2,p}(0,1) : w'(j) + t^\alpha \beta_j(t) w(j) = 0, j = 0,1\right\}$$

and

$$\left(W_*^{2,p}(0,1); L^p(0,1)\right)_{1-\sigma,p}$$

$$= \begin{cases} \left\{w \in W^{2\sigma,p}(0,1) : w'(j) + t^\alpha \beta_j(t) w(j) = 0, j = 0,1\right\} \text{ if } 2\sigma > 1/p, \\ W^{2\sigma,p}(0,1) \text{ if } 2\sigma < 1/p. \end{cases}$$

Let σ be a fixed positive number satisfying $\sigma < 1/2p$ and $\sigma \leq \delta$. From the above proposition, we deduce the following result.

Proposition 4.6. *For all h with $t^{-2\alpha+(\alpha/p)}h \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$, Problem*(11) *admits a unique solution fulfilling the following regularity properties:*
(*i*) $w \in L^p(Q)$, $t^{-2\alpha+(\alpha/p)}w \in L^p(Q)$, $w(0) = 0$,
(*ii*) $t^{-2\alpha+(\alpha/p)}\partial_y^2 w \in L^p(Q)$,
(*iii*) $t^{\alpha/p}\partial_t w \in L^p(Q)$,
(*iv*) $t^{-2\alpha+(\alpha/p)}\partial_y^2 w \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$,
(*v*) $t^{\alpha/p}\partial_t w \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$.

Indeed, Problem (11) is equivalent to Problem (12).

5. Back to the initial problem (1)

We now return to our original problem. Let us recall that $h(t,y) = t^{2\alpha} g(t,y)$, $g(t,y) = f(t,x)$ and $v(t,y) = u(t,x)$ where $(t,y) = \left(t, \frac{x}{t^\alpha}\right)$.
The assumption $t^{-2\alpha+(\alpha/p)}h \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$ means that

$$\int_0^1 \left\| t^{-2\alpha+(\alpha/p)} h(t,.) \right\|_{W^{2\sigma}(0,1)}^p dt < \infty.$$

So,

$$\begin{aligned} \int_0^1 \left\| t^{-2\alpha+(\alpha/p)} h(t,.) \right\|_{W^{2\sigma}(0,1)}^p dt &= \int_0^1 t^{\alpha-2\alpha p} \int_0^1 \int_0^1 \frac{\left|h(t,y) - h(t,y')\right|^p}{\left|y-y'\right|^{2\sigma p+1}} dy dy' dt \\ &= \int_0^1 t^{2\sigma\alpha p} \int_0^{t^\alpha} \int_0^{t^\alpha} \frac{\left|f(t,x) - f(t,x')\right|^p}{|x-x'|^{2\sigma p+1}} dx dx' dt. \end{aligned}$$

Let us introduce the following subspace of $L^p(\Omega)$ (with a slight abuse):

$$L_{t^{2\sigma\alpha}}^p\left(0,1; W_{t^\alpha}^{2\sigma,p}\right) = \left\{ f \in L^p(\Omega) : \int_0^1 t^{2\sigma\alpha p} \int_0^{t^\alpha} \int_0^{t^\alpha} \frac{\left|f(t,x) - f(t,x')\right|^p}{|x-x'|^{2\sigma p+1}} dx dx' dt < \infty \right\}.$$

Then, we are in position to prove the main result of this work.

Theorem 5.1. *For given $\sigma \in]0,1[$ such that $0 < \sigma < \dfrac{1}{2p}$ and $\sigma \leq \delta$, and for any $f \in L^p_{t^{2\sigma\alpha}}\left(0,1; W^{2\sigma,p}_{t^\alpha}\right)$, Problem (1) has a unique solution $u \in H^{1,2}_p(\Omega)$ with the regularities: u, $\partial_t u$, $\partial_x u$ and $\partial_x^2 u$ belong to $L^p_{t^{2\sigma\alpha}}\left(0,1; W^{2\sigma,p}_{t^\alpha}\right)$.*

The proof of Theorem 5.1 can be easily deduced from the following equivalences.

Proposition 5.2. (i) $t^{-2\alpha+(\alpha/p)}h \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$ *if and only if* $f \in L^p_{t^{2\sigma\alpha}}\left(0,1; W^{2\sigma,p}_{t^\alpha}\right)$,

(ii) $t^{-2\alpha+(\alpha/p)}w \in L^p(0,1; L^p(0,1))$ *if and only if* $u \in L^p(\Omega)$,

(iii) $t^{-2\alpha+(\alpha/p)}\partial_y^2 w \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$ *if and only if* $\partial_x^2 u \in L^p_{t^{2\sigma\alpha}}\left(0,1; W^{2\sigma,p}_{t^\alpha}\right)$,

(iv) $t^{\alpha/p}\partial_t v \in L^p\left(0,1; W^{2\sigma,p}(0,1)\right)$ *if and only if* $\partial_t u \in L^p_{t^{2\sigma\alpha}}\left(0,1; W^{2\sigma,p}_{t^\alpha}\right)$.

Note that the equivalence (iv) is a consequence of the equation $\partial_t u(t,x) = \partial_x^2 u(t,x) + f(t,x)$ and the equivalences (i) and (iii).

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