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A study of a class of p-type equations

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Abstract. In this paper, we give known embedding theorems in Sobolev spaces and Sobolev-Morrey spaces with dominant mixed derivatives. And as an application of the embedding theorems we study the problem of existence, uniqueness and smoothness of solutions of p-type equation.

1. Introduction and preliminaries

In this paper we study a p - type equation in the form

$$\sum{}' D^{1^e}\left(\left|D^{1^e}u\right|^{p-2} D^{1^e}u\right) = \sum{}' D^{1^e} f, \tag{1}$$

$$u|_{\partial G} = \varphi|_{\partial G}, \tag{2}$$

where $\sum{}' = \sum\limits_{\emptyset\neq e\subseteq e_n}$, $e_n = \{1, 2, ..., n\}$, $\emptyset \neq e$ any subset of the set e_n, $1^e = \{\omega_1^e, \omega_2^e, \dots \omega_n^e\}$, $\omega_j = 1$ $(j \in e)$, $\omega_j^e = 0$ $(j \in e_n \setminus e = e')$, $\left|D^{1^e}u\right| = \left(\sum{}'\left(D^{1^e}u\right)^2\right)^{\frac{1}{2}}$, $1 \leq p < \infty$, $u \in S_p^1W(G)$ the Sobolev spaces with dominant mixed derivatives is defined and studied in (see,[4, 22]), $f \in L_{p'}(G)$, $\frac{1}{p} + \frac{1}{p'} = 1$, $G \subset R^n$ is a bounded domain, with nonsmooth boundary. Denote by $S_p^1W(G)$ the Sobolev spaces with dominant mixed derivatives of locally summable functions $u(x)$ on G having the weak derivatives $D^{1^e}u$ $(e \subseteq e_n)$ with the finite norm

$$\|u\|_{S_p^1W(G)} = \sum_{e\subseteq e_n} \|D^{1^e}u\|_{L_p(G)}.$$

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More precisely in this paper using the variational method an existence and uniqueness of solution to Dirichlet problem for p type equation (1)-(2) in space $S_p^1 W(G)$. Also, using known embedding theorems in the space $S_{p,a,\varkappa}^l W(G)$ by the Riesz functional method we prove theorems that the solution of p -type equation belong to the Hölder class inside the domain, and has a zero boundary to Dirichlet problem condition up to bounds.

It should be noted that in [13] it was proved that the "smoothness exponent" in the case of parameters greater than in the non-parameters cases. Note that in this paper the smootness of solution of problem (1)-(2) is also studied in the parametrized space $S_{p,a,\varkappa}^l W(G)$. (see, Theorem 2.4 and Theorem 2.5).

The equation (1) in the case $p = 2$ takes the following form

$$\sum{}' D^{2^e} u = \sum{}' D^{1^e} f,$$

and in the case $p = n = 2$ haw the form

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^4 u}{\partial x^2 \partial y^2} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} + \frac{\partial^2 f}{\partial x \partial y}.$$

The existence and uniqueness of Dirichlet problem for the p-harmonic equation in the form

$$div\left(|\nabla u|^{p-2}\, \nabla u\right) = div f,$$

were studied in [1, 3, 5, 10, 11, 12, 19, 20, 27, 28]. Also, a similar and various problems of partial differential equations were studied in [6, 7, 8, 9, 10, 13-18, 21, 23-26, 29, 30] and soon.

Definition 1.1. A weak solution of the Dirichlet problem (1)-(2) on G is a function $u(x) \in S_p^1 W(G)$, if $u - \varphi \in \overset{\circ}{S}{}_p^1 W(G)$ such that

$$\sum{}' \int\limits_G \left|D^{1^e} u\right|^{p-2} D^{1^e} u D^{1^e} \psi dx = \sum{}' \int\limits_G f D^{1^e} \psi dx \tag{3}$$

for every $\psi \in \overset{\circ}{S}{}_p^1 W(G)$.

2. Main results

In this section we give main results of the paper.

Theorem 2.1. Let $G \subset R^n$ be a bounded domain, $1 \le p < \infty$, $g, h \in S_p^1 W(G)$, $\varphi \in S_p^1 W(G)$ and $f \in L_{p'}(G)$. Then the Dirichlet problem for p type equation (1) has a unique weak solution in $S_p^1 W(G)$.

Proof. Let $g, h \in S_p^1 W(G)$. Then we consider the bilinear functional as the form

$$E(g,h) = \sum{}' \int\limits_G \left|D^{1^e} g\right|^{p-2} D^{1^e} g D^{1^e} h dx - \sum{}' \int\limits_G f D^{1^e} h dx =$$

$$= A(g,h) - \sum{}' \int\limits_G f D^{1^e} h dx = A(g,h) - (f,h). \tag{4}$$

Therefore, we have

$$\left|A(g,g)\right| = \left|A(g)\right| \le \left|\sum{}' \int\limits_G \left|D^{1^e} g\right|^{p-2} D^{1^e} g D^{1^e} g dx\right| \le$$

$$\leq \sum{}' \int\limits_G \left|D^{1^e} g\right|^{p-2} \left|D^{1^e} g\right|^2 dx = \sum{}' \int\limits_G \left|D^{1^e} g\right|^p dx < \infty,$$

$$\left|A(g)\right| \leq \|g\|^p_{S^1_p W(G)}.$$

The variational problem is stated as follows. Find a function $g \in S^1_p W(G)$ such that which gives the minimum value to the functional let $E(g,g) = E(g)$ and is unique. The Euler-Lagrange equation for the variational problem (4) is the equation (1), and we have

$$\left|E(g)\right| = \left|A(g) - \sum{}' \int\limits_G f D^{1^e} g dx\right| \geq \left|A(g)\right| - \left|\sum{}' \int\limits_G f D^{1^e} g dx\right| \geq$$

$$\geq \left|A(g)\right| - \sum{}' \left|\int\limits_G f D^{1^e} g dx\right| \geq \left|A(g)\right| - \sum{}' \int\limits_G \left|f D^{1^e} g\right| dx \geq$$

$$\geq C\|g\|_{S^1_p W(G)} - \sum{}' \left\{\left(\int\limits_G |f|^{p'} dx\right)^{\frac{1}{p'}} \left(\int\limits_G \left|D^{1^e} g\right|^p dx\right)^{\frac{1}{p}}\right\} = C\|g\|_{S^1_p W(G)} -$$

$$- \sum{}' \left(\|f\|_{L_{p'}(G)} + \|D^{1^e} g\|_{L_p}(G)\right) \geq C\|g\|_{S^1_p W(G)} - M_0 - \|g\|_{S^1_p W(G)} = -M^*,$$

$$\left|E(g)\right| \geq -M^*, \quad M^* = const.$$

This means that $E(g)$ is lower bounded on $S^1_p W(G)$. So there exists $g_0 \in S^1_p W(G)$ such that $E(g_0) = minE(g)$. Fix some sequence $\{g_m\} \in S^1_p W(G)$ $(m = 1, 2, ...)$ such that $\lim\limits_{m\to\infty} E(g_m) = m_0$. Let $\delta > 0$ choose m_δ so $m \geq m_\delta$ and $s = 1, 2, ...$ it holds $E(g_{m+s}) < m_0 + \delta$. Then noting that $\frac{1}{2}(g_{m+s} + g_m) \in S^1_p W(G)$ we have $E\left(\frac{g_{m+s}+g_m}{2}\right) \geq m_0$, and by direct calculations we show that $A\left(\frac{g_{m+s}-g_m}{2}\right) < 4\delta$, then we have $\|g_{m+s} + g_m\|_{S^1_p W(G)} \leq C_1$. This means that the sequence $\{g_m\}$ is fundamental in the spaces $S^1_p W(G)$. Thus there exist a function $g_0 \in S^1_p W(G)$ such that $\lim\limits_{m\to\infty} \|g_m - g_0\|_{S^1_p W(G)} = 0$. By theorem on trace in $S^1_p W(G)$ (see[4]), we get

$$|E(g_m) - E(g_0)| \leq C_2\|g_m - g_0\|_{S^1_p W(G)},$$

and hence it follows that $m_0 = \lim\limits_{m\to\infty} E(g_m) = E(g_0)$. Show that the function delivering minimum to the functional $E(g)$ is unique and satisfies equation (1) in the space $S^1_p W(G)$. Then $g \in S^1_p W(G)$ and $E(g_0) = m_0$. We have

$$0 \leq A\left(\frac{g - g_0}{2}\right) = \frac{1}{2}E(g) + \frac{1}{2}E(g_0) - E\left(\frac{g + g_0}{2}\right) \leq \frac{m_0}{2} + \frac{m_0}{2} - m_0 = 0,$$

$$A\left(g - g_0\right) = 0.$$

By $\|g_m - g_0\|_{S^1_p W(G)} \to 0$ $(m \to \infty)$, it follows that the function g coincides with g_0 as an element of the space $S^1_p W(G)$. And with the help of the theorem on trace in $S^1_p W(G)$ in [4], we have

$$\|\left(g_m - g_0\right)|_{\partial G}\|_{L_p(\partial G)} \leq C\|g_m - g_0\|_{S^1_p W(G)} \to 0 \quad (m \to \infty),$$

and

$$\|g_m|_{\partial G} - \varphi|_{\partial G}\|_{L_p(\partial G)} \to 0 \quad (m \to \infty),$$

then

$$\|g_0|_{\partial G} - \varphi|_{\partial G}\|_{L_p(\partial G)} \to 0 \quad (m \to \infty).$$

Taking into account the condition $\frac{d}{d\lambda}\left(E(g_0 + \lambda\omega)\right)_{\lambda=0} = 0$, show that the function $g_0 \in S_p^1 W(G)$, minimizing the integral $E(g)$, satisfies the equation

$$A\left(g_0, \omega\right) - (f, \omega) = 0, \tag{5}$$

where

$$\omega(x) = \gamma\left(\frac{r}{l_1}\right) - \gamma\left(\frac{r}{l_2}\right), \quad 0 < l_1 < l_2 < \delta, \quad r = \rho(x, x_0)$$

is a infinitely differentiable finite function with a support lying on a annular domain $\frac{l_1}{2} < r < l_2$, and therefore $\gamma, \omega \in C_0^\infty(G)$, and $D^{(s)}\omega|_{\partial G} = 0$ for all $s = 1, 2, ...$

Now prove that the function $g_0 \in S_p^1 W(G)$ minimizing the integral $E(g)$ is the weak solution of the Drichlet problem (1)-(2).

For the function $g_0(x)$ we can constucted Sobolev's [31] averaging g_{0,l_i}, $i = 1, 2$ on the ball l_i $(i = 1, 2)$ with centered at the point x as

$$g_{0,l_i}(x) = \frac{1}{\tau_n l_i^n} \int\limits_{R^n} K\left(\frac{|z - x|}{l_i}\right) g_0(z) dz, \quad i = 1, 2.$$

Note that the function

$$K\left(\frac{r}{l_i}\right) = \sum{}' D^{1^e}\left(\left|D^{1^e}\gamma\left(\frac{r}{l_i}\right)\right|^{p-2} D^{1^e}\gamma\left(\frac{r}{l_i}\right)\right) - \sum{}' D^{1^e} f \quad i = 1, 2.$$

Satisfy the following properties:

1. is an infinitelly differentiable function with support in the ball $r \le l_i$;
2. all its derivatives on sphere $R = h$ are zero;
3. $$\frac{1}{\tau_n l_i^n} \int\limits_G K\left(\frac{r}{l_i}\right) dx = 1$$
$$\left(\tau_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma\left(\frac{n}{2}\right)} \int\limits_0^1 \xi^{n-1} K(\xi) d\xi\right).$$

Then by (5), we can rewrite equality

$$\int\limits_G K\left(\frac{r}{l_1}\right) g(x) dx = \int\limits_G K\left(\frac{r}{l_2}\right) g(x) dx$$

in the form $g_{0,l_1}(x) = g_{0,l_2}(x)$. Thus for $l_1 < l_2 < \delta$

$$g_{0,l_1}(x) = g_{0,l_2}(x).$$

Since the average functions $g_{0,l_i}(x)$, $i = 1, 2$ are continuous and have continuous derivatives for any order, then $g_0(x)$ is a kernel. Integrating by parts in the equality $A(g_0, \omega) - (f, \omega) = 0$, whence is the limit case

$$\sum{}' \int_G \omega(x) D^{1^e} \left(\left| D^{1^e} g_0 \right|^{p-2} D^{1^e} g_0 \right) dx = \sum{}' \int_G \omega(x) D^{1^e} f dx.$$

Hence by the arbitrariness of the functions $\omega(x)$ it follows that

$$\sum{}' D^{1^e} \left(\left| D^{1^e} g_0 \right|^{p-2} D^{1^e} g_0 \right) = \sum{}' D^{1^e} f.$$

Thus, solution of the variational problem (4) in the space $S_p^1 W(G)$ is also solution of Dirichlet problem (1)-(2) and this solution is unique.

Thus completed the proof of Theorem 2.1 .

Now let's prove the next two theorems in order to study the problem of smoothness of the solution of the equation (1) with the help of the known embedding theorems in $S_{p,a,\varkappa}^l W(G)$ in (see, [13], $l = l_1, \ldots, l_n$, $l_j \in N$, $\varkappa = (\varkappa_1, \ldots, \varkappa_n)$, $\varkappa_j \in (0, \infty)$, $j \in e_n$, $a \in [0, 1]$).

Theorem 2.2 [13] Let $G \subset R^n$ be a domain satisfies the condition of flexible horn [2, 13]; $1 \le p \le q \le \infty$; $\overline{\varkappa} = c\varkappa$, $\frac{1}{c} = \max_{j \in e_n} l_j \varkappa_j$; $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ are integers, $j \in e_n$; $f \in S_{p,a,\varkappa}^l W(G)$ and let

$$\varepsilon_j = l_j - \nu_j - \left(1 - \varkappa_j a\right) \left(\frac{1}{p} - \frac{1}{q} \right) > 0, \quad j \in e_n.$$

Then

$$\|D^\nu f\|_{q,G} \le c_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j}} \|D^{l^e} f\|_{L p,a,\chi(G)}$$

$$\|D^\nu f\|_{q,b,\varkappa;U} \le c_2 \|f\|_{S_{p,a,\varkappa}^l W(G)}, \quad p \le q < \infty.$$

In patricular, if

$$\varepsilon_{j,0} = l_j - \nu_j - \left(1 - \varkappa_j a\right) \frac{1}{p} > 0, \quad j \in e_n$$

then $D^\nu f$ is continuous on G and

$$\sup_{x \in G} \left| D^\nu f(x) \right| \le c_1 \sum_{e \subseteq e_n} \prod_{j \in e_n} T_j^{s_{e,j,0}} \|D^{l^e} f\|_{L_{p,a,\varkappa(G)}},$$

where

$$\|f\|_{S_{p,a,\varkappa}^l W(G)} = \sum_{e \subseteq e_n} \|D^{l^e} f\|_{L_{p,a,\chi(G)}},$$

$$\|f\|_{L_{p,a,\varkappa(G)}} = \sup_{\substack{x \in G, \\ t_j > 0, \\ j \in e_n}} \left(\frac{1}{\prod_{j \in e_n} [t_j]_1^{\frac{\varkappa_j a}{p}}} \|f\|_{p, G_{t^\varkappa}(x)} \right),$$

$$G_{t^\varkappa}(x) = G \cap I_{t^\varkappa}(x) = G \cap \left\{ y : \left| y_j - x_j \right| < \frac{1}{2} t_j^{\varkappa_j}, \ \ j \in e_n \right\},$$

$$s_{e,j} = \begin{cases} \varepsilon_j, & j \in e \\ -\nu_j - (1 - \varkappa_j a) \left(\frac{1}{p} - \frac{1}{q} \right), & j \in e' \end{cases}$$

$$\left(s_{e,j,0} = \begin{cases} \varepsilon_{j,0}, & j \in e \\ -\nu_j - (1-\varkappa_j a)\dfrac{1}{p}, & j \in e' \end{cases} \right),$$

$[t_j]_1 = min\{1, t_j\}$, $T_j \in (0, min(1, t_{0,j}))$, $j \in e_n$; t_0 is a fixed positive vector, c_1, c_2 are constants, independent of f and c_1 is also independent of T.

Theorem 2.3 [13] Let all conditions of Theorem 2.2 be satisfied. If $\varepsilon_j > 0$ $(j \in e_n)$, then $D^\nu f$ satisfies the Hölder condition in $L_q(G)$ with exponent δ_j. Moreover

$$\|\triangle(\xi, G)D^\nu f\|_{q,G} \leq C\|f\|_{S^l_{p,a,\varkappa}W(G)} \prod_{j\in e_n} |\xi_j|^{\delta_j},$$

where

$$\begin{array}{l} 0 \leq \delta_j \leq 1,\ if\ \ \varepsilon_j > 1,\ \ j \in e, \\ 0 \leq \delta_j < 1,\ if\ \ \varepsilon_j = 1,\ \ j \in e;\ \ 0 \leq \delta_j \leq 1, j \in e, \\ 0 \leq \delta_j < \varepsilon_j,\ if\ \ \varepsilon_j < 1,\ \ j \in e. \end{array}$$

If $\varepsilon_{j,0} > 0$ $(j \in e_n)$, then

$$\sup_{x\in G} \left|\triangle(\xi, G)D^\nu f(x)\right| \leq C\|f\|_{S^l_{p,a,\varkappa}W(G)} \prod_{j\in e_n} |\xi_j|^{\delta_{j,0}},$$

where $\delta_{j,0}$, $j \in e_n$ satisfy the similar conditions as δ_j, with $\varepsilon_{j,0}$ instead of ε_j.

Theorem 2.4. If $p > n$, then every weak solution to (1) in $S^1_p W(G)$ belongs to the space $C_{\delta_0}(G_b), \overline{G}_b \subset G$.

First of all, let's note that we will prove the theorem by the Riesz method, and accordingly, we set the right side of (1) equal to zero, i.e. $f \equiv 0$. Suppose that $I_d(x_0) = \left\{x :\ |x_j - x_{j,0}| < d_j^{\varkappa_j},\ \ j \in e_n\right\}$ and let $x_0 \in G_d$, if $I_d(x_0) \subset G$. By the principle of variation

$$\sum{}' \int\limits_{I_d(x_0)} \left|D^{1^e}\left(\theta(x)(u(x) - v(x))\right)\right|^{p-2} D^{1^e}\left(\theta(x)(u(x) - v(x))\right) D^{1^e}\left(\theta(x)(u(x) - v(x))\right) dx \geq$$

$$\geq \sum{}' \int\limits_{I_a(x_0)} \left|D^{1^e}\left(u(x) - v(x)\right)\right|^{p-2} D^{1^e}\left(u(x) - v(x)\right) D^{1^e}\left((u(x) - v(x)\right) dx \geq$$

$$\geq \sum{}' \int\limits_{I_a(x_0)} \left|D^{1^e}\left(u(x) - v(x)\right)\right|^{p-2} \left|D^{1^e}\left(u(x) - v(x)\right)\right|^2 dx =$$

$$= \sum{}' \int\limits_{I_a(x_0)} \left|D^{1^e}\left(u(x) - v(x)\right)\right|^p dx = A\left(u(x) - v(x), I_a(x_0)\right)$$

we can write, for every $0 < a_j \leq d_j < 1$, $j \in e_n$,

$$\theta(x) = 1 - \prod_{j\in e_n} r_j\left(\frac{x_j - x_{j,0}}{a_j}\right)$$

such that $\theta(x) \equiv 1$ in a neighborhood of $I_a(x_0)$ and for every polynomial $v(x) = \sum\limits_{e\subseteq e_n} c_e x^e$. Let $r_j(t_j) = 1$ for $|t_j| < \frac{1}{2}$; $r_j(t_j) = 0$, $t_j \geq \frac{1}{2}$, $j \in e_n$, and $0 \leq r_j(t) \leq 1$, $j \in e_n$.

It is clear that $\Theta(x) \equiv 0$ in $I_{\frac{a}{2}}(x_0)$, we choose the coefficients of $v(x)$ so that.

Now let's prove the next two theorems in order to study the problem of smoothness of the solution of the equation (1) with the help of the embedding theorems in $S^l_{p,a,\chi}W(G)$ (see, [13]).

$$\int\limits_{I_a(x_0)\setminus I_{\frac{a}{2}}(x_0)} (u(x)-v(x))\, x^e dx = 0.$$

Therefore

$$A(u(x)-v(x) \le qA\left(u(x)-v(x), I_a(x_0)\setminus I_{\frac{a}{2}}(x_0)\right),$$

and since $A(u(x)-v(x), G) = A(u(x), G)$,

$$A\left(u(x), I_{\frac{a}{2}}(x_0)\right) \le A\left(u(x), I_a(x_0)\right) - A\left(u(x), I_a(x_0)\setminus I_{\frac{a}{2}}(x_0)\right) \le$$

$$\le A\left(u(x), I_a(x_0)\right) - \frac{1}{q}A\left(u(x), I_a(x_0)\right) = \left(1-\frac{1}{q}\right)A\left(u(x), I_a(x_0)\right),$$

and hence, by inducation, we get

$$A\left(u(x), I_{\frac{a}{2^k}}(x_0)\right) \le \left(1-\frac{1}{q}\right)^k A\left(u(x), I_a(x_0)\right).$$

Let $0<\sigma_j<\frac{a_j}{2^k}$, $j\in e_n$, $0<\prod\limits_{j\in e_n}\sigma_j<\frac{\prod\limits_{j\in e_n} a_j}{2^k}$, then $I_\sigma(x_0)\subset I_{\frac{a}{2^k}}(x_0)$, and $k\; ln\,2 < ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}$. Let

$$k = \left[\frac{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}{ln 2}\right]$$

and let $s = 1-\frac{1}{q}$. Then for every $x_0\in G_d$

$$A\left(u(x), I_\sigma(x_0)\right) \le s^k A\left(u(x), G\right) < s^{\frac{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}{ln2}-1} A\left(u(x), G\right) =$$

$$= e^{\frac{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}{ln2}ln\, s - ln\, s} A\left(u(x), G\right) = \left(e^{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}\right)^{\left(\frac{ln\, s}{ln\, 2}-\frac{ln\, s}{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}\right)} A\left(u(x), G\right) \le$$

$$\le \left(e^{ln\prod\limits_{j\in e_n}\frac{1}{\sigma_j}}\right)^{\frac{ln\, s}{ln\, 2}-\frac{ln\, s}{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}} A\left(u(x), G\right) =$$

$$= \left(\prod_{j\in e_n}\frac{1}{\sigma_j}\right)^{\left(\frac{ln\, s}{ln\, 2}-\frac{ln\, s}{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}\right)} A\left(u(x), G\right) =$$

$$= \left(\prod_{j\in e_n}\sigma_j\right)^{\left|\frac{ln\, s}{ln\, 2}-\frac{ln\, s}{ln\prod\limits_{j\in e_n}\frac{a_j}{\sigma_j}}\right|} A\left(u(x), G\right),$$

$$A\left(u(x), I_\sigma(x_0)\right) \le \prod_{j\in e_n}\sigma_j^{\xi_j} A\left(u(x), G\right),$$

$$\prod_{j\in e_n}\left(\frac{1}{\sigma_j}\right)^{\xi_j}\int\limits_{I_\sigma(x_0)}|D'^e u(x)|^p\,dx \le CA(u(x),G).$$

It follows that $\xi_j = \varkappa_j a,\ \ j\in e_n,\ D^{1^e}\in L_{p,a,\varkappa}(G_d),\ e\subseteq e_n$. Since $p>n$, then $\varepsilon_j>0$ and $\varepsilon_{j,0}>0$ $(j\in e_n)$. So the conditions of Theorem 2.2 and Theorem 2.3 are satisfied. Thus by Theorem 2.2 a weak solution $u(x)$ is continuous on G_d and by Theorem 2.3 is satisfied the Hölder condition.

We consider a non-homogeneous p- type equations corresponding to (1), and let u_{a,x_0} be a solution to equation (1) in $S_p^1W\left(I_a(x_0)\right)$. Putting $\psi\equiv u_{a,x_0}$ in (3), we have

$$\int\limits_{I_a(x_0)}\sum{}'\left|D^{1^e}u_{a,x_0}\right|^p dx \le \int\limits_{I_a(x_0)}\sum{}'\left|f\right|\left|D^{1^e}u_{a,x_0}\right|dx \le$$

$$\le C_1\left(mesI_a(x_0)\right)^{\frac{1}{p'}} \le C_2\prod_{j\in e_n}a_j^{\zeta_j},$$

$$\int\limits_{I_a(x_0)}\sum{}'\left|fD^{1^e}u_{a,x_0}\right|^p dx \le \left(\int\limits_{I_a(x_0)}1^{p'}\right)^{\frac{1}{p'}}\sum{}'\left(\int\limits_{I_a(x_0)}\left|fD^{1^e}u_{a,x_0}\right|^p dx\right)^{\frac{1}{p}} \le$$

$$\le C_1\left(mesI_a(x_0)\right)^{\frac{1}{p'}} \le C_2\prod_{j\in e_n}a_j^{\zeta_j},$$

where $\zeta_j \le \frac{\varkappa_j}{p'},\ \ j\in e_n$, therefore,

$$A\left(u(x),I_a(x_0)\right)\le C_2\prod_{j\in e_n}a_j^{\zeta_j},$$

C_2 is a constant independent of u and x_0. The function $\overline{u}(x)=u(x)-u_{a,x_0}$ is a solution to the equation (1) in $I_a(x_0)$ an so $\overline{u}(x)$ satisfies the inequality

$$A\left(\overline{u}(x),I_\sigma(x_0)\right)\le C_2\prod_{j\in e_n}\left(\frac{\sigma_j}{a_j}\right)^{\zeta_j}A\left(\overline{u}(x),G\right),$$

and for every $\sigma_j\le a_j$ $(j\in e_n)$, $x_0\in G_d$, we have

$$A\left(u(x),I_\sigma(x_0)\right)\le C_3A\left(\overline{u}(x),I_\sigma(x_0)\right)+C_4A\left(u_{\sigma,x_0},I_\sigma(x_0)\right)\le\le C_5\prod_{j\in e_n}\left(\frac{\sigma_j}{a_j}\right)^{\zeta_j}A\left(\overline{u}(x),G\right),$$

therefore,

$$\sum{}'\prod_{j\in e_n}\frac{1}{\sigma_j}\int\limits_{I_\sigma(x_0)}\left|D^{1^e}u\right|^p dx\le C^1A\left(u(x),G\right),$$

for all $x_0\in G_d$. Applying Theorem 2.2 and Theorem 2.3 we find that $u(x)$ is continuous and satisfied the Hölder condition.

This completed the proof of Theorem 2.4.

Theorem 2.5. Let $G\subset R^n$ be a domain such that there exists $m>0$ for any $x_0\in\partial G$ and the number $k_0<1$ there exists a parallelepiped

$$\prod{}_{mk_0}(x^1)\subset\prod{}_{k_0}(x_0)\cap\left(R^n\setminus G\right)$$

and $u(x)$ is a weak solution to (1) from the space $\overset{\circ}{S}{}_p^1W(G)$. If $p > n$, then $u(x)$ belongs to the space $C_{\delta_0}(\overline{G})$, ($C_{\delta_0}(G)$ is Hölder's space).

Proof: Let $x_0 \in \partial G$ and $f \equiv 0$ in $I_\sigma(x_0)$, $u(x) \equiv 0$ outside of G. From the variational principle it follows that

$$A\left(u(x), I_a(x_0)\right) \leq A\left(\Theta(x)u(x), I_a(x_0)\right).$$

Let $\sigma_j < a_j < 1$ $(j \in e_n)$, for all $x_0 \in \partial G$, $f \equiv 0$ in $I_a(x_0)$. Since $\Theta(x) \equiv 0$ in $I_{\frac{a}{2}(x_0)}$, then by Theorem 2.4, we have

$$A\left(u(x), I_\sigma(x_0)\right) \leq C_1 \prod_{j \in e_n} \sigma_j{}^{\zeta_j} A\left(u(x), G\right). \tag{6}$$

Let for all $0 < \sigma_j < a_j$, $x_0 \in G$ and we consider two cases:

1. $x_0 \in G_{\sqrt{\sigma}}$;
2. $x_0 \notin G_{\sqrt{\sigma}}$.

1) In this case for all $\sigma_j < a_j$ $(j \in e_n)$ assuming that $a_j = \sqrt{\sigma_j}$ $(j \in e_n)$, we have

$$A\left(u(x), I_\sigma(x_0)\right) \leq C_2 \prod_{j \in e_n} \sigma_j{}^{\zeta_j} A\left(u(x), G\right). \tag{7}$$

2) In this case there is $x^1 \in \partial G$ such that $I_{2\sqrt{\sigma}}(x^1) \supset I_{\sqrt{\sigma}}(x^0)$. Let $a_j > 2\sqrt{\sigma}$ $(j \in e_n)$, u_{a,x^1} is a solution of equation (1) in the space $\overset{\circ}{S}{}_p^1W\left(I_a(x^1) \cap G\right)$. Then the inequality

$$A\left(u_{a,x^1}, I_a(x_0)\right) \leq C_3 \prod_{j \in e_n} a_j{}^{\zeta_j} \tag{8}$$

is hold, if assuming that $u_{a,x^1} \equiv 0$ outside of $I_a(x^1) \cap G$.

The function $u(x) - u_{a,x^1}$ is a solution of equation (1) in $I_a(x^1)$ and $f = 0$. From the inequalities (6)-(8), we have

$$A\left(u(x), I_{2\sqrt{\sigma}}(x^1)\right) \leq C_4 A\left(u(x) - u_{a,x^1}, I_{2\sqrt{\sigma}}(x^1)\right) + +C_5 A\left(u_{a,x^1}, I_{2\sqrt{\sigma}}(x^1)\right) \leq C_6 \prod_{j \in e_n} \sigma_j{}^{\zeta_j} A\left(u(x), G\right),$$

$$A\left(u(x), I_\sigma(x_0)\right) \leq C_7 \prod_{j \in e_n} \sigma_j{}^{\zeta_j} A\left(u(x), G\right).$$

Therefore,

$$\sum{}' \frac{1}{\prod\limits_{j \in e_n} \sigma_j{}^{\zeta_j}} \int\limits_{I_\sigma(x_0)} \left|D^{1^e} u(x)\right|^p dx \leq CA\left(u(x), G\right),$$

for all $x_0 \in G$. It implies that $u \in S^1_{p,a,\chi}W(\overline{G})$ and by the conditions of Theorem 2.2 and Theorem 2.3 it follows that $u \in C_{\delta_0}(\overline{G})$.

This completed the proof of Theoerem 2.5.

References

[1] G.Afrouzi and A.Hadjian, *Non trivial solutions for p-harmonic type equations via a local minimum theorem for functionals*, Taiwanese Journal of Math., 19 (6), 1731-1742, 2015.

[2] O.V.Besov, V.P.Ilyin and S.M.Nikolskii, *Integral representations of functions and embedding theorems*, M.Nauka, Moscow, 1996.

[3] Y.Deng and H.Pi, *Multiple solutions for p- harmonic type equations*, Nonlinear Anal, 71, 4952-4959, 2009.

[4] A.D.Djabrailov, *Families of spaces of functions whose mixed derivatives satisfy the multiple Hölder integral condition*, Tr. Math. Inst. AN SSSR, 117, 139-158, 1972.

[5] A. Fiorenza and C.Sbordone, *Existence and uniqueness results for solutions of nonlinear equations with right hand side in* L^1,Stud. Math., 127 (3), 4959-4669, 1998.

[6] P.S.Filatov, *Local anisotropic Hölder estimates for solutions to a quasielliptic equation*, Sib. Math. J., 38 (6), 1397-1409, 1997 (Russian).

[7] S.Gala and M.A.Ragusa, *Logarithmically improved regularity criterion for the Boussinesq equations in Besov spaces with negative indices*, Applicable Analysis, 95 (6), 1271-1279, 2016.

[8] E.Giusti, *Equazioni quasi-elliptic espazi* $S^{l}_{p,\theta}(\Omega,\delta)$, I. Ann. Mater. Pure. Appl. ser., 4 (75), 313-353, 1967.

[9] R.V.Guseynov, *On smoothness of solutions of a class of quasielliptic equations*, Vestnik Moscow Univ. Ser. I, Math. Mech., 6, 10-14, 1992 (Russian).

[10] L.Greco and A.Verde, *A regularity property of p- harmonic functions*, Annal. Academ. Scien. Fennicae Math., 25, 317-323, 2000.

[11] H.Luiro and M.Parvianen, *Gradient walkes and p-harmonic functions*, Proc. Amer. of the Math. Soc., 145, 4313-4324, 2017.

[12] J.Manfredi, *p-harmonic functions in the plane*, Proc. of the Amer. Math. Soc., 103 (2), 473-479, 1988.

[13] A.M.Najafov, *Embedding theorems in spaces of type Sobolev - Morrey* $S^{l}_{p,a,\chi}W(G)$ *with dominant mixed derivatives*, Seb. Math. J., 47 (3), 613-625, 2006.

[14] A.M.Najafov, *Problem on smoothness of solution of one class of hypoelliptic equations*,Proc. A. Razm. Math. Inst., 140, 131-139, 2006.

[15] A.M.Najafov, *Smooth solutions of a class of quasielliptic equations*, Sarajevo J. Math., 3 (16), 193-206, 2007.

[16] A.M.Najafov, *The differential properties of functions from Sobolev-Morrey type spaces of fractional arder*, Jour. Math. Res., 7(3), 1-10, 2015.

[17] A.M.Najafov and O.T.Orucova, *On the solution of a class of partial differential equations*, Electron. Jour. Qual. Theory Diff. Equ., 44, 1-9, 2017.

[18] A.M.Najafov and N.R.Rustamova and S.T.Alekberli, *On solvability of a quasielliptic partial differential equations*, Jour. of Ellip. and Parab.Equ., 5(1), 175-187, 2019.

[19] A.M.Najafov and S.T.Alekberli, *On solvability of p-harmonic type equations in grand Sobolev spaces*, Eur. J.Pure Appl. Math., 13 (3).

[20] A.M.Najafov and S.T.Alekberli, *On smoothness of solution of a class of p-harmonic type equations*, Jour. of Ellip. and Parab. Equ., https:doi.org/10.1007/s41808-022-00170-z.

[21] A.M.Najafov and S.T.Alekberli, *A solution of p-harmonic type equations of fractional order*, Lobachevski Jour. of Math., 43 (8), 2244-2249, 2022.

[22] S.M.Nikolski, *Functions with dominant mixed derivatives satisfying the multiple Hölder condition*, Seb. M. Jour., 4(6), 1342-1364, 1963.

[23] N.S.Papageorgiou and A.Scapellato, *Nonlinear Robin problems with general potential and crossing reaction, Rend. Lincei-Math. Appl., 30, 1-29, 2019.*

[24] N.S.Papageorgiou and A.Scapellato, *Concave-Convex problems for the Robin p-laplasion plus an indefinite potential*, Mathematics, 8(3, 421), 1-27, 2020.

[25] M.A.Ragusa, *Local Hölder regularity for solutions of elliptic systems*, Duke Math. J., 113 (2), 385-397, 2002.

[26] M.A.Ragusa, and A.Scopellato, *Mixed Morrey spaces and their applications to partial differential equations*, Nonlinear Anal. Theory Methods Appl., 151, 51-65, 2017.

[27] M.A.Ragusa and A.Tachikawa, *Boundary regularity of minimizers of* $p(x)$*-energy functionals*, Ann. Inst. H.Poincare C Anal. Non Lineaire, 33 (2), 451-476, 2016.

[28] M.A.Ragusa and A.Tachikawa, *Regularity of minimizers for functionals of double plase with variable exponents*, Advances in Nonlinear Analysis, 9 (1), 710-728, 2020.

[29] A.Scapellato, *Homogeneous Herz spaces with variable exponents and regularity results*, Electron. J. Qual. Theory Diff. Equ., 82, 1-11, 2018.

[30] A.Scapellato, *New perspectives in the theory of some function spaces and their applications*, AIP Conf. Proced., 1978 (1), 140002, 2018; https:// doi.org/10.1063/1.5043782.

[31] S.L.Sobolev, *Some applications of functional analysis in mathematical physics*, Novosibirsk, Russian, 1950