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Boundary value problems for hybrid Caputo's exponential fractional diffrential equations

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Abstract. In this paper, we investigate the existence and uniqueness for a new class of impulsive fractional boundary value problems with nonlocal and boundary hybrid conditions. Our main theorem regarding existence and uniqueness is established by applying fixed-point theorems, specifically the Banach fixed-point theorem and the Leray-Schauder alternative fixed-point theorem. Additionally, Two examples are included to show the applicability of our results.

1. Introduction

Fractional calculus involves the integration or differentiation of any order, and it has a history that dates back to the origins of calculus. Despite its significance, it was not given much attention for a long time. However, in recent decades, the study of fractional differential equations has gained momentum as it has proven to be a useful tool in various fields such as technical sciences, economics, and physics (see for example [1, 4–6, 9, 11, 13]).

Impulsive effects are a common occurrence resulting from short-term disturbances that are considerably shorter in duration than the original process [10]. These disturbances can be reasonably approximated as sudden changes in state or impulses. The equations that govern such phenomena can be modeled as impulsive differential equations. Recently, the exploration of impulsive differential equations has provided a practical framework for the mathematical modeling of a wide range of real-life phenomena across various domains such as physics, control theory, chemistry, biotechnology, population dynamics, economics, and medicine.

Dhage and Lakshmikantham [3], examined a first-order hybrid differential equation, which can be expressed as follows:

$$\begin{cases} \frac{d}{dt}\left(\frac{v(t)}{\varphi(t,v(t))}\right) = \psi(t, v(t)), \quad t \in J = [0, T], \\ v(t_0) = v_0 \in \mathbb{R}, \end{cases}$$

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where $\varphi \in C(J\times\mathbb{R},\mathbb{R}\backslash 0)$ and $\psi \in C(J\times\mathbb{R},\mathbb{R})$. They contributed to the study of hybrid differential equations by demonstrating the existence and uniqueness of solutions, as well as establishing fundamental differential inequalities for these systems. Additionally, they demonstrated the existence of extremal solutions and comparison results by using the theory of inequalities.

Benchohra et al.[2] investigated boundary value problems associated with fractional-order differential equations.

$$\begin{cases} {}^cD^{\delta}v(t) = \psi(t,v(t)), \quad t\in J=[0,T], \ \ 0<\delta<1, \\ av(0)+bv(T)=c, \end{cases}$$

where ${}^cD^{\delta}$ denote the Caputo fractional derivative, $\psi : [0,T]\times\mathbb{R}\longrightarrow\mathbb{R}$ is a continuous function and a ,b, c are real constants with $a+b\neq 0$.

Zhao, Sun, Han, and Li [16] investigated fractional hybrid differential equations involving the Riemann-Liouville differential operator.

$$\begin{cases} D^{\delta}\Big(\frac{v(t)}{\varphi(t,v(t))}\Big) = \psi(t,v(t)), \quad t\in J=[0,T], \\ v(0)=0\in\mathbb{R}, \end{cases}$$

where $\varphi \in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$ and $\psi\in C(J\times\mathbb{R},\mathbb{R})$. The study presented an existence theorem for fractional hybrid differential equations and established several fundamental differential inequalities. Additionally, the existence of extremal solutions was demonstrated.

In [8], K. Hilal and A. Kajouni investigated the boundary value problems for hybrid differential equations with the Caputo differential derivative of order $0<\delta<1$.

$$\begin{cases} D^{\delta}\Big(\frac{v(t)}{\varphi(t,v(t))}\Big) = \psi(t,v(t)), \quad t\in J=[0,T], \\ a\frac{v(0)}{\varphi(0,v(0))}+b\frac{v(T)}{\varphi(T,v(T))}=c, \end{cases}$$

where $\psi\in C(J\times\mathbb{R},\mathbb{R})$ and $\varphi\in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$, and real constants a, b, c are with $a+b\neq 0$.

Motivated by recent research on hybrid fractional differential equations, we investigate the following boundary value problem for nonlinear equations of hybrid fractional differential equations.

$$\begin{cases} {}^e_cD^{\delta}_0\Big(\frac{v(t)}{\varphi(t,v(t))}\Big) = \psi(t,v(t)) \,, \quad t\in J=[0,1] \,, \ t\neq t_k, \ k=1,2,....,n, \ 0<\delta<1, \\ v(t_k^+)=v(t_k^-)+I_k(v(t_k^-)), \\ \frac{v(0)}{\varphi(0,v(0))}=\Phi(v), \end{cases} \tag{1}$$

where ${}^e_cD^{\delta}_{\eta}$ is the exponential fractional derivatives of Caputo type of order δ , $\varphi\in C(J\times\mathbb{R},\mathbb{R}\backslash\{0\})$, $\psi\in C(J\times\mathbb{R},\mathbb{R})$ and $\Phi(v) : C(J,\mathbb{R})\longrightarrow\mathbb{R}$ are continuous functions such that $\Phi(v)=\sum_{i=1}^{n}\lambda_i v(\zeta_i)$, where $\zeta_i\in(0,1)$ for $i=1,2,...,n$, and $I_k:\mathbb{R}\longrightarrow\mathbb{R}$ with $v(t_k^-)=\lim_{\varepsilon\to 0^-}v(t_k+\varepsilon)$ and $v(t_k^+)=\lim_{\varepsilon\to 0^+}v(t_k+\varepsilon)$ represent the left and right limits of $v(t)$ at $t=t_k$, $k=i$.

In the following part of this study, we assume that $\sum_{i=1}^{n}\lambda_i v(\zeta_i)^{\delta-1}<1$.

The structure of this paper is as follows: Section 2 reviews the important tools and key results concerning fractional calculus. Section 3 presents the main results, while Section 4 provides examples of applications of the main results.

2. Preliminaries

In this section, we present introductory definitions, facts, and notations that will be used in the rest of this paper.
Let $J_0 = [0, t_1],\ J_1 = (t_1, t_2], ..., J_{n-1} = (t_{n-1}, t_n],\ J_n = (t_n, 1],\ n \in \mathbb{N},\ n > 1$.
For $t_k \in (0, 1)$ such that $t_1 < t_2 < \ < t_n$, we define the following spaces:
$J' = J \backslash \{t_1, t_2, ..., t_n\}$,
$Y = \{v \in C([0, 1], \mathbb{R}) : v \in C(J')$ and left $v(t_i^+)$ and right limit $v(t_k^-)$ exist and $v(t_k^-) = v(t_k), 1 \leq k \leq n\}$.
Then, $(Y, \| \, . \, \|)$ is a Banach space under the norm $\|v\| = \max_{t\in[0,1]}\{|v(t)|\}$.

Definition 2.1. *[12] The exponential type fractional integral of a function $h \in L^1(E)$ of order $\delta > 0$ is defined by*

$$^{e}J^{\delta}_{\eta^+}h(\theta) = \frac{1}{\Gamma(\delta)} \int_{\eta}^{\theta} \left(e^{\theta} - e^{\mu}\right)^{\delta-1} h(\mu)e^{\mu}d\mu \quad for\ each\ \theta \in E.$$

Let $\mathcal{AC}^n_e[k, m] = \{h : [k, m] \longrightarrow \mathbb{C} : {}^{e}D^{n-1}h(x) \in \mathcal{AC}[k, m],\ {}^{e}D = e^{-x}\frac{d}{dx}\}$. For $h \in \mathcal{AC}^n_e[k, m]$, where $-\infty \leq k < m \leq +\infty$, we define the exponential fractional derivatives of Caputo and Riemann-Liouville types as follow.

Definition 2.2. *[12] The exponential fractional derivatives of Caputo type of order $\delta \geq 0$ for a function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as*

$${}^{e}_{c}D^{\delta}_{\eta}h(\theta) = \frac{1}{\Gamma(n-\delta)} \int_{\eta}^{\theta} \left(e^{\theta} - e^{\mu}\right)^{n-\delta-1} \left(e^{-\mu}\frac{d}{d\mu}\right)^{n} h(\mu)\frac{d\mu}{e^{\mu}}.$$

Definition 2.3. *[12] The exponential fractional derivatives of Riemann-Liouville type of order $\delta \geq 0$ for a function $h : \mathbb{R} \longrightarrow \mathbb{R}$ is defined as*

$${}^{e}_{c}D^{\delta}_{\eta}h(\theta) = \frac{1}{\Gamma(n-\delta)}\left(e^{-\mu}\frac{d}{d\mu}\right)^{n} \int_{\eta}^{\theta} \left(e^{\theta} - e^{\mu}\right)^{n-\delta-1} h(\mu)\frac{d\mu}{e^{\mu}}.$$

Lemma 2.4. *[12] Let $h \in \mathcal{AC}^n_e[k, m]$ and $\delta > 0$. Then,*

$$^{e}J^{\delta}_{\eta^+}\left({}^{e}_{c}D^{\delta}_{\eta}h(\theta)\right) = h(\theta) - \sum_{k=0}^{n-1} \frac{(e^{\theta} - e^{\eta})^k}{k!},$$

where $n = \delta + 1$.

Lemma 2.5. *[12] Let $\delta > 0$. The solution of the differential equation:*

$${}^{e}_{c}D^{\delta}_{\eta}h(\theta) = 0$$

is given by $h(\theta) = \sigma_0 + \sigma_1(e^{\theta} - e^{\eta}) + \sigma_2(e^{\theta} - e^{\eta})^2 + ... + \sigma_{n-1}(e^{\theta} - e^{\eta})^{n-1}$.
Where $n = [\delta] + 1$ and $\sigma_i \in \mathbb{R}$, $i = 1, ..., n$.

Theorem 2.6. ***Leray-Schauder alternative*** *[7]*
Let a completely continuous operator $P : \Xi \longrightarrow \Xi$ (a restricted map is compact for any bounded set in Ξ).
Let $M(P) = \{v \in \Xi : v = \rho P v \ \ for\ some\ \ 0 < \rho < 1\}$. Then either the set $M(P)$ is unbounded or P has at least one fixed point.

3. Results

In this section, we prove the existence of a solution for Cauchy problem (1). To do so, we will need the following assumptions.

(H0) The function $v \longrightarrow \frac{v}{\varphi(t,v)}$ is increasing in $\mathbb{R}$ for every $t \in [0,1]$.

(H1) The function φ is continuous and bounded, there exists a number $l > 0$ such that

$$|\varphi(t,v)| \leq l, \text{ for all } (t,v) \in [0,1] \times \mathbb{R}.$$

(H2) There exists a constant $\lambda > 0$ such that

$$|\phi(v)| \leq \lambda, \text{ for all } v \in Y.$$

(H3) There exists a constant $d > 0$, such that

$$I_i(v) \leq d, \text{ for } i = 1,2,.....,n \text{ and } v \in \mathbb{R}.$$

(H4) There exists a $\beta_0, \beta_1 > 0$, such that

$$|\psi(t,v)| \leq \beta_0 + \beta_1 \|v\|, \text{ for all } v \in Y.$$

(H5) There exists a constants $K_\phi > 0$ and $K_I > 0$, such that

$$|\phi(v)| \leq K_\phi \|v\|, \quad |I_i(w)| \leq K_I |w|, \quad i = 1,2....,n,$$

for all $v \in C([0,1],\mathbb{R})$ and $w \in \mathbb{R}$.

(H6) There exists a constant $K_\psi > 0$, such that

$$|\psi(t,v) - \psi(t,\overline{v})| \leq K_\psi |v - \overline{v}|, \quad \text{for all} v, \ \overline{v} \in \mathbb{R}, \text{ and } t \in [0,1].$$

(H7) There exists a constant $\Delta > 0$, such that

$$|I_i(v) - I_i(\overline{v})| \leq \Delta |v - \overline{v}| \ for \ i = 1,2,.....,n \text{ and } v, \ \overline{v} \in \mathbb{R}.$$

(H8) There exists a constant $D_\phi > 0$, such that

$$|\Phi(v) - \Phi(\overline{v})| \leq D_\phi \|v - \overline{v}\| \text{ for all } v, \ \overline{v} \in C([0,1],\mathbb{R}).$$

Lemma 3.1. *Assume that hypotheses (H0) and (H1) hold. A function $v \in C(J,\mathbb{R})$ is a solution of the integral equation*

$$v(t) = \varphi(t,v(t))\Big(\Phi(v) + \eta(t) \sum_{k=1}^{k=n} \frac{I_k\big(v(t_k^-)\big)}{\varphi(t_k, v(t_k))} + \frac{1}{\Gamma(\delta)} \int_0^t \left(e^t - e^\mu\right)^{\delta-1} \psi(\mu, v(\mu)) e^\mu d\mu\Big), \tag{2}$$

$t \in [t_k, t_{k+1}[$.

Where

$$\eta(t) = \begin{cases} 0 \ , & t \in [t_0, t_1[, \\ 1 \ , & t \notin [t_0, t_1[. \end{cases}$$

If and only if v is the solution to the fractional problem of the following form:

$$\begin{cases} {}^e_c D_0^\delta \Big(\frac{v(t)}{\varphi(t,v(t))} \Big) = \psi(t,v(t)) \ , & t \in [0,1] \ , \ t \neq t_k, \\ v(t_k^+) = v(t_k^-) + I_k(v(t_k^-)), \\ \frac{v(0)}{\varphi(0,v(0))} = \Phi(v). \end{cases} \tag{3}$$

Proof. We assume that v satisfies (3). If $t \in [t_0, t_1[$, hence we get

$$ {}_{c}^{e}D_0^{\delta}\Big(\frac{v(t)}{\varphi(t, v(t))}\Big) = \psi(t, v(t)) , \quad t \in [t_0, t_1[, \tag{4} $$

$$ \frac{v(0)}{\varphi(0, v(0))} = \Phi(v). \tag{5} $$

Applying ${}^{e}J_0^{\delta}$ on both of side the equation (4), we obtain

$$ \frac{v(t)}{\varphi(t, v(t))} = \frac{v(0)}{\varphi(0, v(0))} + \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu. $$

Then,

$$ v(t) = \varphi(t, v(t))\Big(\frac{v(0)}{\varphi(0, v(0))} + \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu\Big). $$

Hence,

$$ v(t) = \varphi(t, v(t))\Big(\Phi(v) + \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu\Big). $$

for $t \in [t_1, t_2[$, we have

$$ {}_{c}^{e}D_0^{\delta}\Big(\frac{v(t)}{\varphi(t, v(t))}\Big) = \psi(t, v(t)), \tag{6} $$

$$ v(t_1^+) = v(t_1^-) + I_1(v(t_1^-)). \tag{7} $$

According to the continuity of $t \longrightarrow \varphi(t, v(t))$, we obtain

$$ \begin{aligned} \frac{v(t)}{\varphi(t, v(t))} &= \frac{v(t_1^+)}{\varphi(t_1, v(t_1))} - \frac{1}{\Gamma(\delta)}\int_0^{t_1} \Big(e^{t_1} - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu \\ &+ \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu \\ &= \frac{v(t_1^-) + I_1(v(t_1^-)))}{\varphi(t_1, v(t_1))} - \frac{1}{\Gamma(\delta)}\int_0^{t_1} \Big(e^{t_1} - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu \\ &+ \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu. \end{aligned} $$

Since,

$$ v(t_1^-) = \varphi(t_1, v(t_1))\Big(\Phi(v) + \frac{1}{\Gamma(\delta)}\int_0^{t_1} \Big(e^{t_1} - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu\Big). $$

Then,

$$ \begin{aligned} \frac{v(t)}{\varphi(t, v(t))} &= \Phi(v) + \frac{1}{\Gamma(\delta)}\int_0^{t_1} \Big(e^{t_1} - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu\Big) + \frac{I_1(v(t_1^-)))}{\varphi(t_1, v(t_1))} \\ &- \frac{1}{\Gamma(\delta)}\int_0^{t_1} \Big(e^{t_1} - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu + \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu. \end{aligned} $$

Hence, for $t \in [t_1, t_2[$, we get

$$ v(t) = \varphi(t, v(t))\Big(\Phi(v) + \frac{I_1(v(t_1^-)))}{\varphi(t_1, v(t_1))} + \frac{1}{\Gamma(\delta)}\int_0^t \Big(e^t - e^{\mu}\Big)^{\delta-1}\psi(\mu, v(\mu))e^{\mu}d\mu\Big). $$

For $t \in [t_2, t_3[$, we obtain

$$\begin{aligned}
\frac{v(t)}{\varphi(t, v(t))} &= \frac{v(t_2^+)}{\varphi(t_2, v(t_2))} - \frac{1}{\Gamma(\delta)} \int_0^{t_2} \left(e^{t_2} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \\
&+ \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \\
&= \frac{v(t_2^-) + I_2(v(t_2^-)))}{\varphi(t_2, v(t_2))} - \frac{1}{\Gamma(\delta)} \int_0^{t_2} \left(e^{t_2} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \\
&+ \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu.
\end{aligned}$$

And

$$v(t_2^-) = \varphi(t_2, v(t_2)) \Big(\Phi(v) + \frac{I_1(v(t_1^-)))}{\varphi(t_1, v(t_1))} + \frac{1}{\Gamma(\delta)} \int_0^{t_2} \left(e^{t_2} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big).$$

Therefore, for $t \in [t_2, t_3[$, we have

$$\begin{aligned}
v(t) &= \varphi(t, v(t)) \Big(\Phi(v) + \frac{I_1(v(t_1^-)))}{\varphi(t_1, v(t_1))} + \frac{I_2(v(t_2^-)))}{\varphi(t_2, v(t_2))} + \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big) \\
&= \varphi(t, v(t)) \Big(\Phi(v) + \sum_{k=1}^{k=2} \frac{I_k(v(t_k^-)))}{\varphi(t_k, v(t_k))} + \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big).
\end{aligned}$$

Using the same technique for $t \in [t_k, t_{k+1}[$, $k = 3, 4, ..., n$, we get

$$v(t) = \varphi(t, v(t)) \Big(\Phi(v) + \sum_{i=1}^{i=k} \frac{I_i(v(t_i^-)))}{\varphi(t_i, v(t_i))} + \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big).$$

Inversely, we assume that v satisfies the equation (2), then for $t \in [t_0, t_1[$, we get

$$v(t) = \varphi(t, v(t)) \Big(\Phi(v) + \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big). \tag{8}$$

Afterward, divide by $\varphi(t, v(t))$ and applying ${}^e_cD^{\delta}_0$, we obtain (4).
By replacing $t = 0$ in (8), we get (5).
If $t \in [t_1, t_2[$, we obtain

$$v(t) = \varphi(t, v(t)) \Big(\Phi(v) + \frac{I_1(v(t_1^-)))}{\varphi(t_1, v(t_1))} + \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big). \tag{9}$$

Hence, divide by $\varphi(t, v(t))$ and applying ${}^e_cD^{\delta}_0$, we get (6). By $(H1)$ replacing $t = t_1$ in (8) and using the limite in (9), then (9) minus (8) gives (7). □

Now, we have the ability to prove our result concerning the existence of an integral solution for problem (1). This result was established based on the Leray-Schauder alternative theorem. So, we define the operator P by

$$P(v)(t) = \varphi(t, v(t)) \Big(\Phi(v) + \eta(t) \sum_{k=1}^{k=n} \frac{I_k\big(v(t_k^-)\big)}{\varphi(t_k, v(t_k))} + \frac{1}{\Gamma(\delta)} \int_0^{t} \left(e^{t} - e^{\mu}\right)^{\delta-1} \psi(\mu, v(\mu)) e^{\mu} d\mu \Big).$$

Theorem 3.2. *Given that assumptions (H0)-(H4) are satisfied, we further assume that* $\frac{(e^1-1)^{\delta}}{\Gamma(\delta+1)}\beta_1 < 1$*. Then the fractional Cauchy problem* (1) *has at least one solution.*

Proof. The operator $P : Y \longrightarrow Y$ will be demonstrated to satisfy the conditions of the Leray-Schauder alternative theorem.
Step 1: We demonstrate that the operator P is completely continuous. It is evident that the continuity of the functions φ, ϕ, and ψ implies that the operator P is continuous.
Let $B \subset Y$ be bounded, then, we can show that there exists a positive constant K such that $|\psi(t, v(t))| \leq K$. Then, we get

$$\begin{aligned} |P(v)(t)| &\leq l\left(\lambda + \sum_{k=1}^{n} d + K\frac{1}{\Gamma(\delta)}\int_0^t \left(e^t - e^\mu\right)^{\delta-1} e^\mu d\mu\right) \\ &\leq l\left(\lambda + nd + K\frac{(e^t-1)^\delta}{\Gamma(\delta+1)}\right) \\ &\leq l\left(\lambda + nd + K\frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\right). \end{aligned}$$

Then,

$$\|P(v)\| \leq l\left(\lambda + nd + K\frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\right). \tag{10}$$

From the inequality (10), we conclude that P is uniformly bounded.
Now, we prove that the operator P is equicontinuous.
For $\xi_1, \xi_2 \in [0, 1]$ with $\xi_1 < \xi_2$, we obtain

$$\begin{aligned} |P(v)(\xi_2) - P(v)(\xi_1)| &\leq l\Bigg(\left|(\eta(\xi_2) - \eta(\xi_1))\sum_{k=1}^{n}\frac{I_k\left(v(t_k^-)\right)}{\varphi(t_k, v(t_k))}\right| \\ &+ K\left|\frac{1}{\Gamma(\delta)}\int_0^{\xi_2}\left(e^{\xi_2} - e^\mu\right)^{\delta-1} e^\mu d\mu - \frac{1}{\Gamma(\delta)}\int_0^{\xi_1}\left(e^{\xi_1} - e^\mu\right)^{\delta-1} e^\mu d\mu\right|\Bigg) \\ &\leq l\Bigg(\left|\left(\eta(\xi_2) - \eta(\xi_1)\right)\sum_{k=1}^{n}\frac{I_k\left(v(t_k^-)\right)}{\varphi(t_k, v(t_k))}\right| \\ &+ K\left|\int_0^{\xi_1}\frac{\left(e^{\xi_2} - e^\mu\right)^{\delta-1} - \left(e^{\xi_1} - e^\mu\right)^{\delta-1}}{\Gamma(\delta)} e^\mu d\mu + \int_{\xi_1}^{\xi_2}\frac{\left(e^{\xi_2} - e^\mu\right)^{\delta-1}}{\Gamma(\delta)} e^\mu d\mu\right|\Bigg). \end{aligned}$$

Therefore, the operator $P(v)$ is equicontinuous, thus, based on the previous result, we can conclude that the operator $P(v)$ is completely continuous.
Step 2: Now, we prove that the set $M(P) = \{v \in Y : v = \rho P(v), 0 < \rho < 1\}$ is bounded.
For $v \in M$, we obtain $v = \rho P(v)$. Hence, for any $t \in [0, 1]$, we obtain $v(t) = \rho P(v)(t)$. Then, we get

$$\begin{aligned} \|v\| &\leq l\left(\lambda + nd + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\left(\beta_0 + \beta_1\|v\|\right)\right) \\ &\leq l\left(\lambda + nd + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\beta_0\right) + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\beta_1\|v\|\Bigg). \end{aligned}$$

Then, we get

$$\|v\| \leq \frac{l\left(\lambda + nd + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\beta_0\right)}{1 - \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\beta_1}.$$

This prove that the set $M(P)$ is bounded. Therefore, all assumptions of Leray-Schauder alternative theorem are satisfied. Then, the operator P has at least one fixed point, wisch is a solution of fractional Cauchy problem 3. □

In our second result, we discuss the uniqueness of solutions for Cauchy problem (1) by means of Banach point fixe theorem.

Theorem 3.3. *Assume that assumptions (H1)-(H2) and (H5)-(H8) hold and also the function* $\psi : [0,1] \times \mathbb{R} \longrightarrow \mathbb{R}$ *is continuous. Then fractional Cauchy problem* (1) *has an unique integral solution provided that*

$$M = l\left(D_\phi + n\Delta + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}k_\psi\right) < 1.$$

Proof. Let us $\sup_{t\in[0,1]} \psi(t,0) = F < \infty$, we define $\overline{S}$ a closed ball as follows

$$\overline{S} = \{v \in Y \ : \ \|v\| \leq r\}.$$

Where,

$$r \geq \frac{\frac{l\left(e^1-1\right)^\delta F}{\Gamma(\delta+1)}}{1 - l\left(K_\phi + nK_I + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}K_\psi\right)}. \tag{11}$$

We prove that $P(\overline{S}) \subset \overline{S}$ for $v \in \overline{S}$, we obtain

$$\begin{aligned}
|P(v)(t)| &\leq l\left(\left|\Phi(v) + \eta(t)\sum_{k=1}^{n}\frac{I_k\left(v(t_k^-)\right)}{\varphi(t_k, v(t_k))} + \frac{1}{\Gamma(\delta)}\int_0^t \left(e^t - e^\mu\right)^{\delta-1}\psi(\mu, v(\mu))e^\mu d\mu\right|\right) \\
&\leq l\left(k_\phi\|v\| + nk_I\|v\| + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\left(K_\psi\|v\| + F\right)\right) \\
&\leq l\left(k_\phi r + nk_I r + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\left(K_\psi r + F\right)\right).
\end{aligned}$$

Hence,

$$\|P(v)\| \leq l\left(k_\phi r + nk_I r + \frac{\left(e^1-1\right)^\delta}{\Gamma(\delta+1)}\left(K_\psi r + F\right)\right).$$

Then, by using inequality (11), we get $\|P(v)\| \leq r$.
For $(v, \overline{v}) \in Y^2$, for any $t \in [0, 1]$, we get

$$\begin{aligned}
|P(v)(t) - P(\overline{v})(t)| &= |\varphi(t, v(t))\Big(\Phi(v) + \eta(t) \sum_{k=1}^{k=n} \frac{I_k\big(v(t_k^-)\big)}{\varphi(t_k, v(t_k))} \\
&+ \frac{1}{\Gamma(\delta)} \int_0^t \left(e^t - e^\mu\right)^{\delta-1} \psi(\mu, v(\mu)) e^\mu d\mu\Big) \\
&- \varphi(t, \overline{v}(t))\Big(\Phi(\overline{v}) + \eta(t) \sum_{k=1}^{k=n} \frac{I_k\big(\overline{v}(t_k^-)\big)}{\varphi(t_k, \overline{v}(t_k))} \\
&+ \frac{1}{\Gamma(\delta)} \int_0^t \left(e^t - e^\mu\right)^{\delta-1} \psi(\mu, \overline{v}(\mu)) e^\mu d\mu\Big)| \\
&\leq l\left(D_\phi \|v - \overline{v}\| + n\Delta |v - \overline{v}| + \frac{\left(e^1 - 1\right)^\delta}{\Gamma(\delta + 1)} k_\psi |v - \overline{v}|\right).
\end{aligned}$$

Then,

$$\|P(v) - P(\overline{v})\| \leq l\left(D_\phi + n\Delta + \frac{\left(e^1 - 1\right)^\delta}{\Gamma(\delta + 1)} k_\psi\right) \|v - \overline{v}\| = M \|v - \overline{v}\|.$$

Since $M < 1$, this implies that P is a constraction operator. Then the operator P has an unique fixed point, which is solution of fractional Cauchy problem (1). □

4. Examples

Example 4.1. *Consider the hybrid differential equation with the Caputo's exponential fractional derivative*

$$\begin{cases} {}^e_cD_0^{\frac{1}{2}}\left(\frac{w(\tau)}{\frac{e^{-\tau}+\tau}{40+\tau^2+|w(\tau)|}}\right) = \frac{e^{-\tau}+|\sin(w(\tau))|}{40}, & \tau \in [0,1] \backslash, \{\tau_1\} \\ w(\tau_1^+) = w(\tau_1^-) + (-2w(\tau_1^-)), & \tau_1 \neq 0, 1, \\ \frac{w(0)}{\frac{1}{40+|w(0)|}} = \sum_{k=1}^n \lambda_i w(\tau_i). \end{cases} \tag{12}$$

Here, we get

$$\varphi(\tau, w(\tau)) = \frac{e^{-\tau} + \tau}{40 + \tau^2 + |w(\tau)|},$$

$$\psi(\tau, w(\tau)) = \frac{e^{-\tau} + |\sin(w(\tau))|}{40},$$

$$|\psi(\tau, w_1(\tau)) - \psi(\tau, w_2(\tau))| \leq \frac{1}{40}|w_1 - w_2|, \quad \tau \in [0, 1] \text{ and } w_1, w_2 \in \mathbb{R},$$

$$M = l\left(D_\phi + n\Delta + \frac{\left(e^1 - 1\right)^\delta}{\Gamma(\delta + 1)} k_\psi\right) = 0,2343835178 < 1.$$

Given that all the assumptions of Theorem 3.3 have been met, we can confidently apply our results to the Cauchy problem (12).

Example 4.2. *Consider a different example of hybrid fractional differential equations of the following format*

$$\begin{cases} {}^{e}_{c}D^{\frac{1}{2}}_{0}\left(\frac{w(\tau)}{\frac{e^{\tau}}{72+\tau^2+|w(\tau)|}}\right) = \frac{e^{-3\tau}+\cos^2(w(\tau))}{100} \quad , \qquad \tau \in [0,1]\backslash,\{\tau_1\} \\ w(\tau_1^+) = w(\tau_1^-) + (-2w(\tau_1^-)) \quad , \qquad \tau_1 \neq 0,1, \\ \frac{w(0)}{\frac{1}{72+|w(0)|}} = \sum_{k=1}^{n} \lambda_i w(\tau_i). \end{cases} \tag{13}$$

Here, we get

$$\varphi(\tau, w(\tau)) = \frac{e^{\tau}}{72 + \tau^2 + |w(\tau)|},$$

$$\psi(\tau, w(\tau)) = \frac{e^{-3\tau} + \cos^2(w(\tau))}{100},$$

$$|\psi(\tau, w_1(\tau)) - \psi(\tau, w_2(\tau))| \leq \frac{1}{100}|w_1 - w_2| \quad , \qquad \tau \in [0,1] \;\; and \;\; w_1, \; w_2 \in \mathbb{R},$$

And

$$M = l\left(D_{\phi} + n\Delta + \frac{\left(e^1 - 1\right)^{\delta}}{\Gamma(\delta + 1)} k_{\psi}\right) = 0,0770852262 < 1.$$

Since all the assumptions of Theorem 3.3 are satisfied. We can utilize the results of our analysis to the Cauchy problem (13)

5. Conclusion

In this study the existence and uniqueness of solutions for exponential Caputo fractional differential equations with impulsive boundary conditions are demonstrated . These results are established using fixed point theorems, specifically, the Leray-Schauder alternative fixed point theorem and the Banach fixed point theorem. Finally, an appropriate example is utilized to illustrate the investigation of our theoretical result.

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Conflict of interest

The authors declare that they have no conflict of interest.

Data Availability

The data used to support the findings of this study are included in the references within the article.

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