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Composition and Volterra-type inner derivations on the generalized Fock spaces

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Abstract. A classical result of Calkin [3] says that the inner derivation maps the algebra of all bounded operators on a Hilbert space into the ideal of all compact operators if and only if the induced operator is a compact perturbation of the scalar operator. On the generalized Fock spaces, we use the compact intertwing relations to study the range of the inner derivations induced by the composition operators C_φ and the Volterra type operators J_g and I_g.

1. Introduction

Let $\mathscr{A}$ be a Banach algebra over the complex field. A linear map $D : \mathscr{A} \to \mathscr{A}$ is a *derivation* if $D(xy) = xD(y) + D(x)y$ for all $x, y \in \mathscr{A}$. Over the last half century, there are lots of results giving conditions on a derivation of a Banach algebra implying that its range is contained in some ideal. One of the famous results given by Singer and Wermer [14] is that a continuous derivation on a commutative Banach algebra has the range contained in the Jacobson radial of the algebra. In [3], Calkin proved that an *inner derivation* $S \mapsto [T, S] := TS - ST$ maps the algebra of all bounded operators on a Hilbert space to the ideal of all compact operators if and only if T is a compact perturbation of a scalar operator. But this conclusion fails to hold on the Banach spaces in general. See [13] for example. In this paper, we are interested in the composition inner derivations and Volterra inner derivations on the generalized Fock spaces.

Let $\mathbb{C}$ be the complex plane and denote by $H(\mathbb{C})$ the space of all entire functions on $\mathbb{C}$. We consider a class of smooth radial weights that increase faster than the standard Gaussian weight $|z|^2/2$. Precisely, let $\psi : [0, +\infty) \to [0, +\infty)$ be an increasing function such that $\psi(0) = 0$ and $\lim_{r\to\infty} \psi(r) = +\infty$. Extend ψ to $\mathbb{C}$ by setting $\psi(z) = \psi(|z|)$. Moreover, we assume that ψ is twice continuously differentiable such that $\inf \Delta\psi > 0$ and

$$\tau(z) \simeq \begin{cases} 1, & 0 \le |z| < 1 \\ (\triangle\psi(|z|))^{-1/2}, & |z| \ge 1 \end{cases}$$

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Here $\tau(z)$ is a radial positive differentiable function that decreases to zero as $|z| \to \infty$ and $\lim_{r\to\infty} \tau'(r) = 0$. And there exists a constant $C > 0$ such that $\tau(r)r^C$ increases for large r or

$$\lim_{r\to\infty} \tau'(r) \log \frac{1}{\tau(r)} = 0.$$

The above class of rapidly increasing functions ψ will be denoted by $\mathcal{I}$ and τ is called the associated function of ψ. The power functions $\psi(r) = r^\alpha$, $\alpha > 2$, the exponential functions $\psi(r) = e^{\beta r}$, $\beta > 0$ and the super exponential functions $\psi(r) = e^{e^r}$ are all typical examples of such weight functions. Let $\psi_1(z) = \psi(z) + \log(1 + \psi'(z))$ and $\mathcal{I}_1 = \{\psi \in \mathcal{I} : \Delta\psi_1 \simeq \Delta\psi\}$. Notice that $\psi \in \mathcal{I}_1$ implies $\psi_1 \in \mathcal{I}$.

For $0 < p < \infty$, the generalized Fock space induced by ψ is defined by

$$\mathcal{F}^p_\psi = \left\{ f \in H(\mathbb{C}) : \|f\|^p_{p,\psi} = \int_{\mathbb{C}} |f(z)|^p e^{-p\psi(z)} dA(z) < \infty \right\},$$

where dA is the Lebesgue measure on $\mathbb{C}$. Furthermore,

$$\mathcal{F}^\infty_\psi = \left\{ f \in H(\mathbb{C}) : \|f\|_{\infty,\psi} = \sup_{z\in\mathbb{C}} |f(z)|e^{-\psi(z)} < \infty \right\}.$$

For a given $\varphi \in H(\mathbb{C})$, the composition operator C_φ on $H(\mathbb{C})$ is defined by $C_\varphi f = f \circ \varphi$. The boundedness and compactness of composition operators on various holomorphic function spaces have been studied intensively in the past few decades. One can refer to the books [4, 12] for the theory of composition operators on various specific spaces of holomorphic functions. And interested readers may also refer to the recent papers [1, 5, 8–10] and the references therein for properties of (weighted) composition operators on classical or generalized Fock spaces. Let $\mathcal{B}(\mathcal{F}^p_\psi)$ be the Banach algebra of all bounded linear operators on $\mathcal{F}^p_\psi$. The *composition inner derivation* induced by C_φ is defined by

$$D(C_\varphi) : \mathcal{B}(\mathcal{F}^p_\psi) \to \mathcal{B}(\mathcal{F}^p_\psi) \qquad T \mapsto [C_\varphi, T] = C_\varphi T - TC_\varphi.$$

Our first result about composition inner derivation reads as follows.

Theorem A . *Suppose $\psi(r) = r^m$ with $m \in \mathbb{N}$ and $m > 2$, or $\psi(r) = e^{\beta r^m}$ with $\beta > 0$ and $m \in \mathbb{N}$, or $\psi(r) = e^{e^r}$. Let $0 < p \le \infty$, then the composition inner derivation $D(C_\varphi)$ on $\mathcal{B}(\mathcal{F}^p_\psi)$ ranges into the ideal of compact operators if and only if $\varphi(z) = z$ or $\varphi(z) = az + b$ with $|a| < 1$.*

For a given $g \in H(\mathbb{C})$, the Volterra-type operator J_g and its companion operator I_g with symbol g are defined by

$$J_g f(z) = \int_0^z f(w)g'(w)dw \quad \text{and} \quad I_g f(z) = \int_0^z f'(w)g(w)dw.$$

The discussion of Volterra-type operators first arose in connection with the semigroup of composition operators. One can refer to [15] for more backgrounds. Constantin [6] studied the boundedness and compactness of Volterra-type operators on the classical Fock spaces. Later, Peleàz [7] and Mengestie [11] characterized the boundedness and compactness of Volterra-type operators on generalized Fock spaces. Two classes of *Volterra inner derivation* induced by $g \in H(\mathbb{C})$ are defined as follows.

$$D(J_g) : \mathcal{B}(\mathcal{F}^p_\psi) \to \mathcal{B}(\mathcal{F}^p_\psi) \qquad T \mapsto [J_g, T] = J_g T - TJ_g.$$

and

$$D(I_g) : \mathcal{B}(\mathcal{F}^p_\psi) \to \mathcal{B}(\mathcal{F}^p_\psi) \qquad T \mapsto [I_g, T] = I_g T - TI_g.$$

Our results about Volterra inner derivations read as follows.

Theorem B . *Let $0 < p \leq \infty$ and $\psi \in \mathcal{I}$. Then the Volterra inner derivation $D(J_g)$ on $\mathcal{B}(\mathcal{F}^p_\psi)$ ranges into the ideal of compact operators if and only if*

$$\lim_{|z|\to\infty} \frac{|g'(z)|}{1+\psi'(z)} = 0.$$

Theorem C . *let $0 < p \leq \infty$ and $\psi \in \mathcal{I}_1$. Then the Volterra inner derivation $D(I_g)$ on $\mathcal{B}(\mathcal{F}^p_\psi)$ ranges into the ideal of compact operators if and only if g is a constant.*

Indeed, Theorem C is trivial since the boundedness of I_g implies that g is a constant function by [11] and $I_g - id$ is compact when g is a constant.

Let X and Y be two metric linear spaces, the symbol $\mathcal{B}(X, Y)$ denotes the collection of all continuous linear operators from X to Y. Let $\mathcal{K}(X, Y)$ be the collection of all compact elements of $\mathcal{B}(X, Y)$ and $\mathcal{Q}(X, Y) = \mathcal{B}(X, Y)\backslash\mathcal{K}(X, Y)$.

For $A \in \mathcal{B}(X, X)$, $B \in \mathcal{B}(Y, Y)$ and $T \in \mathcal{B}(X, Y)$, the phrase "T intertwines A and B in $\mathcal{Q}(X, Y)$" (or "T intertwines A and B compactly") means that

$$TA - BT \in \mathcal{K}(X, Y) \quad \text{where} \quad T \neq 0.$$

To be more intuitive, the compact intertwining relation means the following commutative diagram,

$$\begin{array}{ccc} X & \xrightarrow{A} & X \\ \downarrow{\scriptstyle T} & & \downarrow{\scriptstyle T} \\ Y & \xrightarrow{B} & Y \end{array} \qquad \text{mod } \mathcal{K}(X, Y).$$

In the series papers [16–18], Yuan, Tong and Zhou investigated the intertwining relations for Volterra-type operators and composition operators on the Bergman spaces, bounded analytic function spaces and Bloch spaces over the unit disk.

When $X = Y$ and $A = B$ it is easy to see the following two assertions are equivalent:

- T intertwines every $A \in \mathcal{B}(X)$ compactly.
- The inner derivation $D(T)$ on $\mathcal{B}(X)$ ranges into the compact ideal.

In this point of view, we will study the compact intertwing relations for composition operators and Volterra-type integral operators between different generalized Fock spaces. And our main results will follow immediately as direct corollaries.

Throughout this paper, for two non-negative real-valued functions U and V, we write $U \lesssim V$ if there exists a positive constant $C > 0$ independent of the essential argument such that $U \leq CV$. And we write $U \simeq V$ if both $U \lesssim V$ and $V \lesssim U$.

2. Preliminaries

In this section, we collect some basic properties of generalized Fock spaces and auxiliary lemmas which will be used latter.

We say that a positive function τ belongs to $\mathcal{L}$ if there exists a constant $c_\tau > 0$ such that

$$|\tau(z) - \tau(w)| \leq c_\tau |z - w|$$

for all $z, w \in \mathbb{C}$. It is obvious that the associated function τ of ψ belongs to $\mathcal{L}$ as $\psi \in \mathcal{I}$. We will use the notation $m_\tau = \frac{\max\{1, c_\tau^{-1}\}}{4}$. Let $D(w, r)$ be the Euclidean disc centered at w with radius $r > 0$. For simplicity, write $D(\delta\tau(w))$ for the disc $D(w, \delta\tau(w))$ with $\delta > 0$. The following lemmas can be found in [7].

Lemma 2.1. *Suppose* $\tau \in \mathcal{L}$ *and* $0 < \delta \leq m_\tau$. *Then*

$$\frac{1}{2}\tau(w) \leq \tau(z) \leq 2\tau(w)$$

whenever $z \in D(\delta\tau(w))$.

Lemma 2.2. *Let* $0 < p \leq q < \infty$, $\psi \in \mathcal{I}$ *and* μ *be a finite positive Borel measure on* $\mathbb{C}$. *Then*

(i) *The embedding* $id : \mathcal{F}^p_\psi \to L^q(\mu)$ *is bounded if and only if*

$$K_{\mu,\psi} := \sup_{a\in\mathbb{C}} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} e^{q\psi(z)} d\mu(z) < \infty$$

for some $\delta > 0$. *Moreover, if any of the two equivalent conditions holds, then*

$$K_{\mu,\psi} \simeq \|id\|^q_{\mathcal{F}^p_\psi \to L^q(\mu)}.$$

(ii) *The embedding* $id : \mathcal{F}^p_\psi \to L^q(\mu)$ *is compact if and only if*

$$\lim_{|a|\to\infty} \frac{1}{\tau(a)^{2q/p}} \int_{D(\delta\tau(a))} e^{q\psi(z)} d\mu(z) = 0$$

for some $\delta > 0$.

Lemma 2.2 is known as the Carleson embedding theorem for $\mathcal{F}^p_\psi$. Notice that if $\psi \in \mathcal{I}_1$, then Lemma 2.2 still holds for $\mathcal{F}^p_{\psi_1}$.

By Lemma 7 and Lemma 20 in [7], we have the following pointwise estimate for functions in $\mathcal{F}^p_\psi$, which is an important ingredient in our subsequent consideration. See also [19, Lemma 3.1].

Lemma 2.3. *Let* $\psi \in \mathcal{I}$ *and* τ *be the associate function of* ψ. *Suppose* $0 < p < \infty$ *and* $\alpha, \beta \in \mathbb{R}$, *then*

$$\frac{|f(z)|^p e^{-\beta\psi(z)}}{(1+\psi'(z))^\alpha} \lesssim \frac{1}{\tau(z)^2} \int_{D(\delta\tau(z))} \frac{|f(w)|^p e^{-\beta\psi(w)}}{(1+\psi'(w))^\alpha} dA(w)$$

for all $f \in H(\mathbb{C})$ *and* $z \in \mathbb{C}$.

According to the results in [7, 11], the following Littlewood-Paley type estimates hold for all $0 < p \leq \infty$, providing a natural description of the generalized Fock spaces in terms of the first derivatives.

Lemma 2.4. *Let* $\psi \in \mathcal{I}$, *then for* $0 < p < \infty$,

$$\|f\|^p_{p,\psi} \simeq |f(0)|^p + \int_{\mathbb{C}} |f'(z)|^p e^{-p\psi_1(z)} dA(z)$$

for all $f \in H(\mathbb{C})$. *And when* $p = \infty$,

$$\|f\|_{\infty,\psi} \simeq |f(0)| + \sup_{z\in\mathbb{C}} |f'(z)| e^{-\psi_1(z)}.$$

Lemma 2.3 tells us that the point evaluations are bounded linear functionals on $\mathcal{F}^p_\psi$. In particular, $\mathcal{F}^2_\psi$ is a Hilbert space with the following inner product

$$\langle f, g\rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} e^{-2\psi(z)} dA(z),$$

which is equivalent to

$$\langle f,g\rangle_* = f(0)\overline{g(0)} + \int_{\mathbb{C}} f'(z)\overline{g'(z)}e^{-2\psi_1(z)}dA(z)$$

by Lemma 2.4. It follows the Riesz representation theorem in Hilbert space theory that for each $z \in \mathbb{C}$, there exists a unique function $K_{\psi,z}$ in $\mathcal{F}^2_\psi$ such that

$$f(z) = \langle f, K_{\psi,z}\rangle$$

for all $f \in \mathcal{F}^2_\psi$. $K_{\psi,z}$ is the reproducing kernel function in $\mathcal{F}^2_\psi$ at z. Let $K^{[1]}_{\psi,z} = \partial K_{\psi,z}/\partial\overline{z}$ be the first-order reproducing kernel function in $\mathcal{F}^2_\psi$, then

$$f'(z) = \langle f, K^{[1]}_{\psi,z}\rangle$$

for all $f \in \mathcal{F}^2_\psi$. By the equivalence of the inner product, $(K^{[1]}_{\psi,z})'$ is the reproducing kernel function of $\mathcal{F}^2_{\psi_1}$. Unlike the classical Fock space, the explicit expression for $K_{\psi,z}$ is still unknown. Recently, Yang and Zhou [19] characterized the important pointwise and norm estimate for the reproducing kernel $K_{\psi,z}$.

Lemma 2.5. *Let $\psi \in \mathcal{I}$ and τ be the associated function of ψ, then for $0 < p < \infty$,*

$$\|K_{\psi,z}\|_{p,\psi} \simeq e^{\psi(z)}\tau(z)^{2(1-p)/p}$$

for all $z \in \mathbb{C}$. Moreover, let $k_{p,\psi,z} = \frac{K_{\psi,z}}{\|K_{\psi,z}\|_{p,\psi}}$ be the normalized reproducing kernel in $\mathcal{F}^p_\psi$, then

$$|k_{p,\psi,z}(w)|\tau(z)^{\frac{2}{p}} \simeq |k_{q,\psi,z}(w)|\tau(z)^{\frac{2}{q}}$$

for all $z, w \in \mathbb{C}$. And there exists a small $\delta_0 > 0$ such that

$$|k_{p,\psi,z}(w)|^p e^{-p\psi(w)} \simeq \tau(z)^{-2}$$

whenever $w \in D(\delta\tau(z))$ and $\delta \le \delta_0$.

Through a similar argument as in [1], we obtain the following Berezin-type transform for a positive measure μ.

Lemma 2.6. *Suppose $\psi \in \mathcal{I}$ and τ is the associated function of ψ. Let μ be a positive Borel measure on $\mathbb{C}$. Then for $0 < p \le q < \infty$,*

(i) $\sup_{z\in\mathbb{C}} \frac{\mu(D(\delta\tau(z)))}{\tau(z)^{2q/p}} < \infty$
$\Leftrightarrow \sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |k_{p,\psi,z}(w)|^q e^{-q\psi(w)}d\mu(w) < \infty.$

(ii) $\lim_{|z|\to\infty} \frac{\mu(D(\delta\tau(z)))}{\tau(z)^{2q/p}} = 0$
$\Leftrightarrow \lim_{|z|\to\infty} \int_{\mathbb{C}} |k_{p,\psi,z}(w)|^q e^{-q\psi(w)}d\mu(w) = 0.$

We end this section with a key result which helps us to obtain the characterization of composition inner derivatives.

Proposition 2.7. *Let $s > 0$ and $f, \varphi \in H(\mathbb{C})$ such that $f(0) \neq 0$. Suppose $\psi : [0,+\infty) \to [0,+\infty)$ is a radial differentiable function such that $\frac{\psi'(r)}{r^{1+\varepsilon}}$ increases to $+\infty$ as $r \to \infty$ for some $\varepsilon > 0$. If*

$$\sup_{z\in\mathbb{C}} |f(z)|\frac{e^{\psi(\varphi(z))-\psi(z)}}{(1+\psi'(z))^s} < \infty, \tag{2.1}$$

then $\varphi(z) = az + b$ for some $|a| \le 1$ and $b = 0$ whenever $|a| = 1$.

Proof. Taking logarithms on both sides of (2.1), there exists a $M > 0$ such that

$$\psi(\varphi(z)) - \psi(z) + \log|f(z)| - s\log(1+\psi'(z)) \le M$$

for all $z \in \mathbb{C}$. For any $R > 0$, putting $z = Re^{i\theta}$ and integrating with respect to θ on $[0, 2\pi]$ yields

$$\int_0^{2\pi} \psi(\varphi(Re^{i\theta}))\frac{d\theta}{2\pi} - \psi(R) + \int_0^{2\pi} \log|f(Re^{i\theta})|\frac{d\theta}{2\pi} - s\log(1+\psi'(R)) \le M.$$

By Mean-value property of harmonic function, we have

$$\int_0^{2\pi} \log|f(Re^{i\theta})|\frac{d\theta}{2\pi} = \log|f(0)|.$$

It follows that

$$\int_0^{2\pi} \psi(\varphi(Re^{i\theta}))\frac{d\theta}{2\pi} - \psi(R) - s\log(1+\psi'(R)) \le M$$

for any $R > 0$. Since ψ is radial, there exists a radial function h on $\mathbb{C}$ such that $\psi(|z|) = h(|z|^2)$ and $h'(r^2)/r^\varepsilon$ increases to $+\infty$ as $r \to \infty$ by our assumption. Then Jensen's inequality gives

$$h\left(\int_0^{2\pi} |\varphi(Re^{i\theta})|^2\frac{d\theta}{2\pi}\right) - h(R^2) - s\log(1+2Rh'(R^2)) \le M.$$

Now consider the power expansion $\varphi(z) = \sum_{n=0}^\infty a_n z^n$ for $z \in \mathbb{C}$. Then

$$\int_0^{2\pi} |\varphi(Re^{i\theta})|^2\frac{d\theta}{2\pi} = \sum_{n=0}^\infty |a_n|^2 R^{2n}.$$

Notice that $\log(1+2Rh'(R^2)) \le \log(1+2R) + \log(1+h'(R^2))$. Then by Langrange's Differential Mean Value Theorem, we obtain

$$h'(\zeta_R)\left(\sum_{n=0}^\infty |a_n|^2R^{2n} - R^2\right) - s\log(1+2R) - s\log(1+h'(R^2)) \le M$$

for some ζ_R. If $|a_1| > 1$ or $a_n \ne 0$ for some $n \ge 2$, then $\zeta_R > R^2$ and

$$\lim_{R\to\infty} h'(\zeta_R)\left(\sum_{n=0}^\infty |a_n|^2R^{2n} - R^2\right) - s\log(1+2R) - s\log(1+h'(R^2)) = +\infty,$$

which is a contradiction. Therefore $\varphi(z) = a_0 + a_1 z$ with $|a_1| \le 1$. Moreover, if $|a_1| = 1$ and $a_0 \ne 0$, then we also have

$$\lim_{R\to\infty} |a_0|^2 h'(\zeta_R) - s\log(1+2R) - s\log(1+h'(R^2)) = +\infty,$$

which is also a contradiction. Thus $a_0 = 0$ whenever $|a_1| = 1$. The proof is complete. □

3. $D(C_\varphi)$ and $D(J_g)$

In this section, we give the proof for Theorem A and Theorem B. To this end, we first study the compact intertwing relations for composition operator C_φ and Volterra-type operator J_g between different generalized Fock spaces.

For $\varphi, g \in H(\mathbb{C})$, let

$$T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi = [C_\varphi, J_g].$$

Suppose $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is bounded, then it is obvious that $g \circ \varphi - g = T_{\varphi,g}1 \in \mathcal{F}^q_\psi$.

Theorem 3.1. *Let $\psi \in \mathcal{I}$, $0 < p \leq q \leq \infty$ and $g \circ \varphi - g \in \mathcal{F}^q_\psi$, then*

(i) If $0 < p \leq q < \infty$, then $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is bounded if and only if

$$\sup_{z\in\mathbb{C}} \int_{\mathbb{C}} |k_{p,\psi,z}(\varphi(w))|^q |(g\circ\varphi - g)'(w)|^q e^{-q\psi_1(w)} dA(w) < \infty. \tag{3.1}$$

(ii) If $0 < p < q = \infty$, then $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$ is bounded if and only if

$$\sup_{z\in\mathbb{C}} |(g\circ\varphi - g)'(z)| \Delta\psi(\varphi(z))^{\frac{1}{p}} e^{\psi(\varphi(z))-\psi_1(z)} < \infty. \tag{3.2}$$

(iii) If $p = q = \infty$, then $T_{\varphi,g} : \mathcal{F}^\infty_\psi \to \mathcal{F}^\infty_\psi$ is bounded if and only if

$$\sup_{z\in\mathbb{C}} |(g\circ\varphi - g)'(z)| e^{\psi(\varphi(z))-\psi_1(z)} < \infty.$$

Proof. We begin with the proof of (i). By Lemma 2.4, we have

$$\begin{aligned} \|T_{\varphi,g}f\|^q_{q,\psi} &\simeq |T_{\varphi,g}f(0)|^q + \int_{\mathbb{C}} |(T_{\varphi,g}f)'(w)|^q e^{-q\psi_1(w)} dA(w) \\ &= |T_{\varphi,g}f(0)|^q + \int_{\mathbb{C}} |f\circ\varphi(w)|^q |(g\circ\varphi - g)'(w)|^q e^{-q\psi_1(w)} dA(w) \end{aligned}$$

for all $f \in \mathcal{F}^p_\psi$. Taking $f = k_{p,\psi,z}$, the "only if part" of (i) follows immediately. To prove the "if part", we consider the weighted pullback measure $\mu_{\varphi,g}$ defined by

$$\mu_{\varphi,g}(E) = \int_{\varphi^{-1}(E)} |(g\circ\varphi - g)'(z)|^q e^{-q\psi_1(z)} dA(z),$$

where E is any Borel subset of $\mathbb{C}$. Then $\|T_{\varphi,g}f - T_{\varphi,g}f(0)\|_{q,\psi} \simeq \|f\|_{L^q(\mu_{\varphi,g})}$ for any $f \in \mathcal{F}^p_\psi$. Let $d\nu_{\varphi,g} = e^{q\psi(z)} d\mu_{\varphi,g}$, by Lemma 2.6, condition (3.1) implies that

$$\sup_{z\in\mathbb{C}} \frac{\nu_{\varphi,g}(D(\delta(\tau(z)))}{\tau(z)^{2q/p}} < \infty.$$

This, together with Lemma 2.2, shows that $id : \mathcal{F}^p_\psi \to L^q(\mu_{\varphi,g})$ is bounded. Thus, $\|T_{\varphi,g}f - T_{\varphi,g}f(0)\|_{q,\psi} \lesssim \|f\|_{p,\varphi}$ for all $f \in \mathcal{F}^p_\psi$. On the other hand, by Lemma 2.3,

$$\begin{aligned} |T_{\varphi,g}f(0)| = \left| \int_0^{\varphi(0)} f(w)g'(w)dw \right| &\leq |\varphi(0)| \sup_{|w|\leq|\varphi(0)|} |f(w)||g'(w)| \\ &\lesssim |\varphi(0)| \sup_{|w|\leq|\varphi(0)|} |g'(w)| \frac{e^{\psi(\varphi(0))}}{\tau(\varphi(0))^{\frac{2}{p}}} \|f\|_{p,\varphi}. \end{aligned} \tag{3.3}$$

Therefore, the boundness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is established.

Now we proceed to prove (ii). If $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$ is bounded, then by Lemma 2.4, we have

$$\begin{aligned} \|T_{\varphi,g}\| \geq \|T_{\varphi,g}k_{p,\psi,z}\|_{\infty,\psi} &\gtrsim |(T_{\varphi,g}k_{p,\psi,z})'(w)| e^{-\psi_1(w)} \\ &\simeq |(g\circ\varphi - g)'(w)||k_{p,\psi,z}(\varphi(w))| e^{-\psi_1(w)} \end{aligned}$$

for all $z, w \in \mathbb{C}$. In particular, taking $z = \varphi(w)$ and by Lemma 2.5, we have

$$\sup_{w\in\mathbb{C}} |(g\circ\varphi - g)'(w)|\tau(\varphi(w))^{-\frac{2}{p}} e^{\psi(\varphi(w))-\psi_1(w)} < \infty.$$

This is exactly (3.2) since $\tau(\varphi(w))^{-2} \simeq \Delta\psi(\varphi(w))$. Conversely, suppose (3.2) holds, then by Lemma 2.4 and (3.3), we have

$$\begin{aligned}
\|T_{\varphi,g}f\|_{\infty,\psi} &\simeq |T_{\varphi,g}f(0)| + \sup_{z\in\mathbb{C}} |(g\circ\varphi - g)'(z)||f(\varphi(z))|e^{-\psi_1(z)} \\
&\lesssim \|f\|_{p,\psi} + \sup_{z\in\mathbb{C}} |(g\circ\varphi - g)'(z)|e^{\psi(\varphi(z))-\psi_1(z)}\tau(\varphi(z))^{-\frac{2}{p}} \left(\int_{D(\delta\tau(\varphi(z)))} |f(w)|^p e^{-p\psi(w)} dA(w)\right)^{\frac{1}{p}} \\
&\lesssim \|f\|_{p,\psi} + \|f\|_{p,\psi} \sup_{z\in\mathbb{C}} |(g\circ\varphi - g)'(z)|\Delta\psi(\varphi(z))^{\frac{1}{p}} e^{\psi(\varphi(z))-\psi_1(z)} \\
&\lesssim \|f\|_{p,\psi}
\end{aligned}$$

for all $f \in \mathcal{F}^p_\psi$, which estalishes the boundedness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$.

The proof for (iii) is similar to that for (ii) and we omit the routine details. □

The condition in (3.1) is difficult to apply. In fact, under the assumption of Proposition 2.7, the boundedness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ will imply that $\varphi(z) = az + b$ with $|a| \le 1$.

Proposition 3.2. *Let $\psi \in \mathcal{I}$ and τ be the associated function of ψ. If $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is bounded, then*

$$\sup_{z\in\mathbb{C}} \frac{\tau(z)^{2/q}}{\tau(\varphi(z))^{2/p}} \frac{|(g\circ\varphi - g)'(z)|}{1+\psi'(z)} e^{\psi(\varphi(z))-\psi(z)} < \infty. \tag{3.4}$$

Furthermore, suppose $\frac{\psi'(r)}{r^{1+\varepsilon}}$ increases to $+\infty$ for some $\varepsilon > 0$ as $r \to \infty$, then $\varphi(z) = az + b$ with $|a| \le 1$ and $b = 0$ whenever $|a| = 1$.

Proof. If $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is bounded, then by Lemma 2.3 and Lemma 2.4, we have

$$\begin{aligned}
\|T_{\varphi,g}\| &\ge \|T_{\varphi,g}k_{p,\psi,z}\|_{q,\psi} \\
&\gtrsim |(T_{\varphi,g}k_{p,\psi,z})'(w)|\tau(w)^{\frac{2}{q}} e^{-\psi_1(w)} \\
&= \tau(w)^{2/q}|(g\circ\varphi - g)'(w)||k_{p,\psi,z}(\varphi(w))|e^{-\psi_1(w)}
\end{aligned}$$

for all $z, w \in \mathbb{C}$. In particular, taking $z = \varphi(w)$ and using Lemma 2.5, we obtain

$$\sup_{w\in\mathbb{C}} \frac{\tau(w)^{2/q}}{\tau(\varphi(w))^{2/p}} \frac{|(g\circ\varphi - g)'(w)|}{1+\psi'(w)} e^{\psi(\varphi(w))-\psi(w)} < \infty.$$

Since $\lim_{r\to\infty} \tau(r)\psi'(r) = +\infty$ by [7, Lemma 18] and $\tau(r) \lesssim 1$ for all r, then (3.4) implies that

$$\sup_{w\in\mathbb{C}} \frac{|(g\circ\varphi - g)'(w)|}{(1+\psi'(w))^{1+\frac{2}{q}}} e^{\psi(\varphi(w))-\psi(w)} < \infty.$$

Thus according to Proposition 2.7, $\varphi(z) = az + b$ with $|a| \le 1$ and $b = 0$ whenever $|a| = 1$. □

Remark 3.3. *Note that if $0 < p \le q = \infty$, then condition (3.4) is exactly sufficient for the boundedness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$ by Theorem 3.1. In fact, if $0 < p \le q < \infty$ and $\psi(r) = r^\alpha$ with $\alpha > 2$, then the condition in Proposition 3.2 is also sufficient for the boundedness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$.*

Proof. Without loss of generality, assume $\varphi(z) = az + b$ with $0 < |a| \leq 1$. Let

$$M = \sup_{z\in\mathbb{C}} \frac{\tau(z)^{2/q}}{\tau(\varphi(z))^{2/p}} \frac{|(g\circ\varphi - g)'(z)|}{1+\psi'(z)} e^{\psi(\varphi(z))-\psi(z)} < \infty$$

and $\nu_{\varphi,g}$ be defined as in Theorem 3.1, then

$$\begin{aligned}\frac{\nu_{\varphi,g}(D(\delta\tau(z)))}{\tau(z)^{2q/p}} &= \frac{\int_{D(\delta\tau(z))} |(g\circ\varphi - g)'(w)|^q e^{q\psi(\varphi(w))-q\psi_1(w)} dA(w)}{\tau(z)^{2q/p}} \\ &\leq M \frac{\int_{D(\delta\tau(z))} \frac{\tau(\varphi(w))^{2q/p}}{\tau(w)^2} dA(w)}{\tau(z)^{2q/p}} \\ &\lesssim \frac{\tau(\varphi(z))^{2q/p}}{\tau(z)^{2q/p}},\end{aligned}$$

where the last inequality follows from Lemma 2.1. Since $\psi(r) = r^\alpha$ with $\alpha > 2$ and $\varphi(z) = az + b$, we have

$$\lim_{|z|\to\infty} \frac{\tau(\varphi(z))}{\tau(z)} = a^{\alpha-2}.$$

Thus,

$$\sup_{z\in\mathbb{C}} \frac{\nu_{\varphi,g}(D(\delta\tau(z)))}{\tau(z)^{2q/p}} < \infty.$$

Then via a similar argument as in Theorem 3.1, $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is bounded. □

Theorem 3.4. *Let $\psi \in \mathcal{I}$ and $0 < p \leq q \leq \infty$. Suppose $\varphi(z) = az + b$ with $0 < |a| \leq 1$ and $g\circ\varphi - g \in \mathcal{F}^q_\psi$, then*

(i) If $0 < p \leq q < \infty$, then $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is compact if and only if

$$\lim_{|z|\to\infty} \int_{\mathbb{C}} |k_{p,\psi,z}(\varphi(w))|^q |(g\circ\varphi - g)'(w)|^q e^{-q\psi_1(w)} dA(w) = 0. \tag{3.5}$$

(ii) If $0 < p < q = \infty$, then $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$ is compact if and only if

$$\lim_{|z|\to\infty} |(g\circ\varphi - g)'(z)| \Delta\psi(\varphi(z))^{\frac{1}{p}} e^{\psi(\varphi(z))-\psi_1(z)} = 0. \tag{3.6}$$

(iii) If $p = q = \infty$, then $T_{\varphi,g} : \mathcal{F}^\infty_\psi \to \mathcal{F}^\infty_\psi$ is compact if and only if

$$\lim_{|z|\to\infty} |(g\circ\varphi - g)'(z)| e^{\psi(\varphi(z))-\psi_1(z)} = 0.$$

Proof. We begin with the proof of (i). Assume $0 < p \leq q < \infty$ and $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is compact, then

$$\lim_{|z|\to\infty} \|T_{\varphi,g} k_{p,\psi,z}\|_{q,\psi} = 0. \tag{3.7}$$

By Lemma 2.4,

$$\|T_{\varphi,g} k_{p,\psi,z}\|^q_{q,\psi} \simeq |T_{\varphi,g} k_{p,\psi,z}(0)|^q + \int_{\mathbb{C}} |k_{p,\psi,z}(\varphi(w))|^q |(g\circ\varphi - g)'(w)|^q e^{-q\psi_1(w)} dA(w).$$

Thus the "only if part" follows directly. Conversely, let $\{f_n\}$ be any bounded sequence in $\mathcal{F}^p_\psi$ the converges to 0 uniformly on compact subsets of $\mathbb{C}$, then by (3.3),

$$|T_{\varphi,g}f_n(0)| \le |\varphi(0)| \sup_{|w|\le|\varphi(0)|} |g'(w)||f_n(w)| \to 0 \tag{3.8}$$

as $n \to \infty$. Consider the positive measures $\mu_{\varphi,g}$ and $\nu_{\varphi,g}$ appearing in Theorem 3.1, then by Lemma 2.6 and Lemma 2.2, condition (3.5) implies that

$$\|T_{\varphi,g}f_n - T_{\varphi,g}f_n(0)\|_{q,\psi} \simeq \|f_n\|_{L^q(\mu_{\varphi,g})} \to 0$$

as $n \to \infty$. This, together with (3.8), shows that $\|T_{\varphi,g}f_n\|_{q,\psi} \to 0$, which implies the compactness of $T_{\varphi,g}$.

Now we proceed to prove (ii). Assume $0 < p < q = \infty$ and $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$ is compact, then by (3.7) and the proof of (ii) in Theorem 3.1, we have

$$\lim_{|\varphi(w)|\to\infty} |(g\circ\varphi - g)'(w)|\Delta\psi(\varphi(w))^{\frac{1}{p}} e^{\psi(\varphi(w))-\psi_1(w)} = 0.$$

Since $\varphi(z) = az + b$ with $0 < |a| \le 1$, then $|\varphi(w)| \to \infty$ as $|w| \to \infty$. Thus condition (3.6) holds. Conversely, let

$$M(w) = |(g\circ\varphi - g)'(w)|\Delta\psi(\varphi(w))^{\frac{1}{p}} e^{\psi(\varphi(w))-\psi_1(w)},$$

which converges to 0 as $|w| \to \infty$. Let $\{f_n\}$ be any bounded sequence in $\mathcal{F}^p_\psi$ that converges to 0 uniformly on compact subsets of $\mathbb{C}$, then by Lemma 2.4 and Lemma 2.3, we have

$$\begin{aligned}\|T_{\varphi,g}f_n - T_{\varphi,g}f_n(0)\|_{\infty,\psi} &\lesssim \sup_{w\in\mathbb{C}} M(w)\left(\int_{D(\delta\tau(\varphi(w)))} |f_n(z)|^p e^{-p\psi(z)}dA(z)\right)^{\frac{1}{p}}\\ &\lesssim \sup_{|w|>R} M(w)\|f_n\|_{p,\psi} + \sup_{|w|\le R} M(w)\left(\int_{D(\delta\tau(\varphi(w)))} |f_n(z)|^p e^{-p\psi(z)}dA(z)\right)^{\frac{1}{p}}.\end{aligned}$$

Letting $R \to \infty$ and then $n \to \infty$, and combining (3.8), we have

$$\lim_{n\to\infty} \|T_{\varphi,g}f_n\|_{\infty,\psi} = 0,$$

which establishes the compactness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^\infty_\psi$.

The proof for (iii) is similar to that for (ii) and we omit the routain details. □

Proposition 3.5. *Let $\psi \in \mathcal{I}$ and τ be the associated function of ψ. If $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$ is compact, then*

$$\lim_{|z|\to\infty} \frac{\tau(z)^{2/q}}{\tau(\varphi(z))^{2/p}} \frac{|(g\circ\varphi - g)'(z)|}{1 + \psi'(z)} e^{\psi(\varphi(z))-\psi(z)} = 0.$$

Furthermore, if $0 < p \le q < \infty$ and $\psi(r) = r^\alpha$ with $\alpha > 2$, or $0 < p < q = \infty$, or $p = q = \infty$, then the above condition is also sufficient for the compactness of $T_{\varphi,g} : \mathcal{F}^p_\psi \to \mathcal{F}^q_\psi$.

Proof. The proof is just a modification of Proposition 3.2 and Remark 3.3, the details are left to interested readers. □

Now we are ready to prove Theorem A and Theorem B.

Proof of Theorem A . The "if part" is trivial since $C_\varphi = id$ if $\varphi(z) = z$ and C_φ is compact if $\varphi(z) = az + b$ with $|a| < 1$ according to [9, Theorem 2.1] and [10, Theorem 2.2].

Now we proceed to prove the "only if part". It suffices to show that C_φ compactly interwines all bounded Volterra-type operators J_g only if $\varphi(z) = z$ or $\varphi(z) = az + b$ with $|a| < 1$.

Let $T_{\varphi,g} = C_\varphi J_g - J_g C_\varphi$. By Proposition 3.2, the compactness (boundedness) of $T_{\varphi,g}$ implies that $\varphi(z) = az+b$ with $|a| \le 1$ and $b = 0$ whenever $|a| = 1$. In the latter case, we only need to show that $a = 1$.

Suppose $T_{\varphi,g}$ is compact and $|a| = 1$, then by Proposition 3.5, we have

$$\lim_{|z|\to\infty} \frac{|ag'(az) - g'(z)|}{1+\psi'(z)} = 0. \tag{3.9}$$

Choose $g \in H(\mathbb{C})$ satisfying that

$$\lim_{r\to+\infty} \frac{|g'(r)|}{1+\psi'(r)} = 1 \quad \text{and} \quad \lim_{r\to+\infty} \frac{|g'(re^{i\theta})|}{1+\psi'(r)} < 1$$

for any $0 < \theta < 2\pi$. Then by [7, Theorem 3], J_g is bounded on $\mathcal{F}^p_\psi$ but is not compact. For such g, (3.9) holds only if $a = 1$. The proof is complete.

Proof of Theorem B . The "if part" is trivial since J_g is compact on $\mathcal{F}^p_\psi$ if $\frac{|g'(z)|}{1+\psi'(z)} \to 0$ as $|z| \to \infty$ by [7, Theorem 3]. To prove the "only if part", we assume that J_g compactly intertwines all bounded composition operators C_φ.

Let $\varphi(z) = e^{i\theta}z$, by [9] and [10], C_φ is bounded but not compact on $\mathcal{F}^p_\psi$ for any $\theta \in [0, 2\pi]$. Then Proposition 3.5 tells us that

$$\lim_{|z|\to\infty} \frac{|e^{i\theta}g'(e^{i\theta}z) - g'(z)|}{1+\psi'(z)} = 0.$$

Suppose $g(z) = \sum_{n=0}^{\infty} a_n z^n$ and integrating with respect to θ from 0 to 2π, we get

$$\begin{aligned}\int_0^{2\pi} \frac{|e^{i\theta}g'(e^{i\theta}z) - g'(z)|}{1+\psi'(z)} d\theta &= \frac{1}{1+\psi'(z)} \int_0^{2\pi} \left|\sum_{n=1}^{+\infty} n a_n z^{n-1}(e^{in\theta} - 1)\right| d\theta \\ &\ge \frac{1}{1+\psi'(z)} \left|\sum_{n=1}^{+\infty} n a_n z^{n-1} \int_0^{2\pi} (e^{in\theta} - 1) d\theta\right| \\ &= \frac{2\pi|g'(z)|}{1+\psi'(z)}.\end{aligned}$$

Thus the "only if part" follows immediately. The proof is complete.

In fact, Theroem B tells us that $D(J_g)$ on $\mathcal{B}(\mathcal{F}^p_\psi)$ ranges into the ideal of compact operators if and only if J_g is compact. However, there exist non-compact C_φ and J_g such that $T_{\varphi,g}$ is compact.

Example 3.6. *Let $\varphi(z) = iz$, $g(z) = z^4$ and $\psi(z) = |z|^4$, then neither C_φ nor J_g is compact on $\mathcal{F}^p_\psi$ but $T_{\varphi,g}$ is compact according to Theorem 3.4.*

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