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Some results for reproducing kernel Hilbert space operators via Berezin symbols

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Abstract. By applying the Berezin symbols method, we investigate the solvability of the Riccati operator equation $XAX+XB-CX-D=0$ on the set of operators of the form Toeplitz + compact on the Bergman space $L_a^2(\mathbb{D})$ of analytic functions in the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$. We also characterize compact truncated operators on the standard reproducing kernel Hilbert space in the sense of Nordgren and Rosenthal. Moreover, we discuss solvability of the equation

$$T_{\varphi_1}X_1+T_{\varphi_2}X_2+...+T_{\varphi_n}X_n=I+K,$$

where $T_{\varphi_i}\left(i=\overline{1,n}\right)$ is the Toeplitz operator on $L_a^2(\mathbb{D})$ and $K:L_a^2(\mathbb{D})\to L_a^2(\mathbb{D})$ is a fixed compact operator.

1. Introduction

In this paper, we discuss the solvability of the Riccati operator equation

$$XAX+XB-CX-D=0 \tag{1}$$

and the equation

$$T_{\varphi_1}X_1+T_{\varphi_2}X_2+...+T_{\varphi_n}X_n=I+K \tag{2}$$

with given Toeplitz operators T_{φ_i}, $i=1,2,...,n$, and compact operator K. We also characterise compact truncated operator on the standard reproducing kernel Hilbert space in the sense of Nordgren and Rosenthal. Our discussion based on the Berezin symbols technique.

Recall that the classical Hardy space $H^2=H^2(\mathbb{D})$ is the Hilbert space of analytic functions on the unit disc $\mathbb{D}=\{z\in\mathbb{C}:|z|<1\}$ satisfying

$$\|f\|_2:=\left(\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\left|f\left(re^{it}\right)\right|^2dt\right)^{1/2}<+\infty.$$

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For $\lambda \in \mathbb{D}$, the reproducing kernel of H^2 is the function $k_\lambda \in H^2$ such that

$$f(\lambda) = \langle f, k_\lambda \rangle$$

for all $f \in H^2$ and $\lambda \in \mathbb{D}$. The normalized reproducing kernel $\widehat{k_\lambda}$ is the function $\frac{k_\lambda}{\|k_\lambda\|_2}$. Since $\{z^n\}_{n\geq 0}$ is an orthonormal basis for H^2, it is easy to verify that $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$, and hence $\widehat{k_\lambda} = \frac{(1-|\lambda|^2)^{1/2}}{1-\overline{\lambda}z}$. The Banach algebra of all bounded analytic functions on $\mathbb{D}$ is denoted by $H^\infty := H^\infty(\mathbb{D})$. The norm in H^∞ is defined by

$$\|f\|_\infty := \sup_{z\in\mathbb{D}} |f(z)| < +\infty.$$

Clearly, $H^\infty \subset H^2$.

Recall also that the Bergman space $L_a^2 = L_a^2(\mathbb{D}, dA)$ is the Hilbert space consisting of analytic functions on $\mathbb{D}$ that are also in the Lebesgue space $L^2(\mathbb{D}, dA)$, where dA is the Lebesgue area measure of $\mathbb{D}$, normalized so that the measure of $\mathbb{D}$ equal 1, i.e., $dA = \frac{dxdy}{\pi}$. The normalized reproducing kernel of the Bergman space L_a^2 is the function

$$\widehat{k_\lambda}(z) := \frac{k_{\lambda,a}}{\|k_{\lambda,a}\|_{L_a^2}} = \frac{1-|\lambda|^2}{\left(1-\overline{\lambda}z\right)^2},\ \lambda, z \in \mathbb{D}.$$

For $T \in \mathcal{B}\left(L_a^2\right)$, its Berezin symbol (transform) is the function $\widetilde{T}$ defined by $\widetilde{T}(\lambda) := \left\langle T\widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a}\right\rangle$, $\lambda \in \mathbb{D}$. Clearly, $\widetilde{T}$ is the bounded function on $\mathbb{D}$. Often the behavior of the Berezin symbol of an operator provides important information about the operator itself. For instance, it is known that (see, Ahern, Flores and Rudin [2], Engliš [6], Fricain [7] and Yang [19], Zhu [20]) in most of the reproducing kernel Hilbert spaces, including Hardy, Bergman, Dirichlet, Fock and some model spaces, the Berezin symbol uniquely determines the operator, that is $A = 0$ if and only if $\widetilde{A} = 0$. Also, Nordgren and Rosenthal [16] proved that compact operators on the so-called standard reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ are completely characterized by the boundary behavior of Berezin symbols of their unitary orbits, namely, $T \in \mathcal{B}(\mathcal{H})$ is compact if and only if

$$\lim_{\lambda\to\partial\Omega} \widetilde{UTU^{-1}}(\lambda) = 0$$

for all unitary operators U on $\mathcal{H}(\Omega)$. Recall that the reproducing kernel Hilbert space $\mathcal{H}(\Omega)$ is called standard if its normalized reproducing kernels $\widehat{k}_{\lambda,\mathcal{H}}$ weakly converge to zero whenever λ tends to the boundary point of Ω. For $\varphi \in L^\infty(\mathbb{D}, dA)$, the Toeplitz operator T_φ is the operator on L_a^2 defined by $T_\varphi f = P(\varphi f)$, where P is the orthogonal projection from $L^2(\mathbb{D}, dA)$ onto L_a^2 defined by the formula

$$(Pf)(z) = \int_{\mathbb{D}} \frac{f(w)}{(1-\overline{w}z)^2} dA(w).$$

The Berezin symbol $\widetilde{\varphi}$ of a function φ in $L^\infty(\mathbb{D}, dA)$ is defined to be the Berezin symbol of the associated Toeplitz operator T_φ, i.e., $\widetilde{\varphi} = \widetilde{T}_\varphi$, and hence

$$\widetilde{\varphi}(\lambda) = \int_{\mathbb{D}} \varphi(z) \left|\widehat{k}_{\lambda,a}(z)\right|^2 dA(z) = \int_{\mathbb{D}} \varphi(z) \frac{\left(1-|\lambda|^2\right)^2}{\left|\left(1-\overline{\lambda}z\right)^4\right|} dA(z).$$

The Berezin symbol of a function in $L^\infty := L^\infty(\mathbb{D}, dA)$ often plays the same important role in the theory of Bergman spaces as the harmonic extension of a function in $L^\infty(\partial\mathbb{D})$ does in the theory of Hardy spaces (see, Engliš [6] and Zhu [19]).

The space of all bounded harmonic functions on $\mathbb{D}$ will be denoted by $h^\infty := h^\infty(\mathbb{D})$.

2. On the solution of Riccati equations via the Berezin symbol

In this section, we study solvability of the Riccati operator equation (1) on the set $\mathcal{F}_c$ of operators of the form $T_\varphi + \mathcal{K}$, where $T_\varphi : L_a^2 \to L_a^2$ is the Toeplitz operator and $\mathcal{K}$ is a compact operator on L_a^2. Notice that the solvability of equation (1) in concrete operator classes is one of the important problems in operator theory. For example, Adamjan, Langer and Tretter [1] prove a criterion for existence of unitary solutions of some Riccati operator equations. In [1], these authors used the Cayley transform to prove the existence of an accretive solution of a Riccati equation. Karaev [10] firstly applied Berezin symbols technique in solvability of equation (1) in the set of Toeplitz operators on the Hardy space H^2. In the sequel, the similar results are obtained in [4, 8, 9, 11, 13, 16, 17, 21, 22]. Our results in this section improves some results in [4, 10, 12, 15]. Namely, we prove the following theorem.

Theorem 2.1. *Let $B = T_u^*$, $C = T_v$ be Toeplitz operators on L_a^2, where $u, v \in H^\infty$ are nonconstant functions, and $A, D \in \mathcal{B}\left(L_a^2\right)$ be two operators in equation (1). Let $T \in \mathcal{F}_c$ be an operator, i.e., $T = T_\varphi + \mathcal{K}$ for some $\varphi \in h^\infty(\mathbb{D})$ and compact operator $\mathcal{K}$ on L_a^2.*

(i) If T is a solution of the Riccati equation (1), then the function

$$\widetilde{A}(\lambda)(\varphi(\lambda))^2 + \left(\overline{u(\lambda)} - v(\lambda)\right)\varphi(\lambda) - \widetilde{D}(\lambda)$$

has nontangential limit 0 almost everywhere in $\partial\mathbb{D}$.

(ii) Suppose that the nonzero nontangential limits $\widetilde{A}_{nt}(\xi) := \lim_{\lambda \to \xi \in \partial\mathbb{D}} \widetilde{A}(\lambda)$ and $\widetilde{D}_{nt}(\xi) := \lim_{\lambda \to \xi \in \partial\mathbb{D}} \widetilde{D}(\lambda)$ exist for almost all $\xi \in \partial\mathbb{D}$ and verify

$$\left(\overline{u(\xi)} - v(\xi)\right)^2 + 4\widetilde{A}_{nt}(\xi)\widetilde{D}_{nt}(\xi) = 0 \tag{3}$$

for almost all $\xi \in \partial\mathbb{D}$. If $T_\varphi + \mathcal{K}$ is a solution of (1), then

$$\varphi(\xi) = \pm i\left(\frac{\widetilde{D}_{nt}(\xi)}{\widetilde{A}_{nt}(\xi)}\right)^{\frac{1}{2}}$$

for almost all $\xi \in \partial\mathbb{D}$.

Proof. (i) If $T = T_\varphi + \mathcal{K}$ satisfyes the equation (1), then we have that

$$\left(\left(T_\varphi + \mathcal{K}\right)A\left(T_\varphi + \mathcal{K}\right) + \left(T_\varphi + \mathcal{K}\right)T_u^* - T_v\left(T_\varphi + \mathcal{K}\right) - D\right)^{\sim}(\lambda) = 0,$$

or equivalently

$$\left\{T_\varphi A T_\varphi + T_\varphi T_u^* - T_v T_\varphi - D + \left[\left(T_\varphi + \mathcal{K}\right)A\mathcal{K} + \mathcal{K}A\left(T_\varphi + \mathcal{K}\right) + \mathcal{K}T_u^* - T_v\mathcal{K}\right]\right\}^{\sim}(\lambda) = 0$$

for all $\lambda \in \mathbb{D}$. Hence

$$\left\langle\left(T_\varphi A T_\varphi + T_\varphi T_u^* - T_v T_\varphi - D\right)\widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a}\right\rangle + \widetilde{\mathcal{K}_1}(\lambda) = 0$$

for all $\lambda \in \mathbb{D}$, where

$$\mathcal{K}_1 := \left[\left(T_\varphi + \mathcal{K}\right)A\mathcal{K} + \mathcal{K}\mathcal{A}\left(T_\varphi + \mathcal{K}\right) + \mathcal{K}T_u^* - T_v\mathcal{K}\right]$$

is the compact operator on L_a^2. Since $\widetilde{T_\varphi}(\lambda) = \widetilde{\varphi}(\lambda) = \varphi(\lambda)$ by Engliš result [5], we have :

$$\begin{aligned}
0 &= \left\langle T_\varphi A T_\varphi \widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a} \right\rangle + \left\langle T_\varphi T_u^* \widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a} \right\rangle - \left\langle T_v T_\varphi \widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a} \right\rangle - \left\langle D \widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a} \right\rangle + \widetilde{\mathcal{K}_1}(\lambda) \\
&= \left\langle A T_\varphi \widehat{k}_{\lambda,a}, T_{\overline{\varphi}} \widehat{k}_{\lambda,a} \right\rangle + \overline{u(\lambda)} \left\langle T_\varphi \widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a} \right\rangle - v(\lambda) \left\langle T_\varphi \widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a} \right\rangle - \widetilde{D}(\lambda) + \widetilde{\mathcal{K}_1}(\lambda) \\
&\text{(since } T_\psi^* \widehat{k}_{\lambda,a} = \overline{\psi(\lambda)} \widehat{k}_{\lambda,a} \text{ for any } \psi \in H^\infty) \\
&= \left\langle A\left(\left(T_\varphi - \widetilde{\varphi}(\lambda)\right) \widehat{k}_{\lambda,a} + \widetilde{\varphi}(\lambda) \widehat{k}_{\lambda,a}\right), T_{\overline{\varphi}} \widehat{k}_{\lambda,a} \right\rangle + \left(\overline{u(\lambda)} - v(\lambda)\right) \widetilde{T}_\varphi(\lambda) - \widetilde{D}(\lambda) + \widetilde{\mathcal{K}_1}(\lambda) \\
&= \left\langle A\left(T_{\varphi - \varphi(\lambda)} \widehat{k}_{\lambda,a}\right), T_{\overline{\varphi}} \widehat{k}_{\lambda,a} \right\rangle + \varphi(\lambda) \left\langle A \widehat{k}_{\lambda,a}, T_{\overline{\varphi}} \widehat{k}_{\lambda,a} - \overline{\varphi(\lambda)} \widehat{k}_{\lambda,a} + \overline{\varphi(\lambda)} \widehat{k}_{\lambda,a} \right\rangle \\
&+ \left(\overline{u(\lambda)} - v(\lambda)\right) \varphi(\lambda) - \widetilde{D}(\lambda) + \widetilde{\mathcal{K}_1}(\lambda) \\
&= \left\langle A\left(T_{\varphi - \varphi(\lambda)} \widehat{k}_{\lambda,a}\right), T_{\overline{\varphi}} \widehat{k}_{\lambda,a} \right\rangle + \varphi(\lambda) \left\langle A \widehat{k}_{\lambda,a}, T_{\overline{\varphi - \varphi(\lambda)}} \widehat{k}_{\lambda,a} \right\rangle + (\varphi(\lambda))^2 \widetilde{A}(\lambda) + \left(\overline{u(\lambda)} - v(\lambda)\right) \varphi(\lambda) - \widetilde{D}(\lambda) + \widetilde{\mathcal{K}_1}(\lambda)
\end{aligned}$$

for all $\lambda \in \mathbb{D}$, and hence

$$\begin{aligned}
&\widetilde{A}(\lambda)(\varphi(\lambda))^2 + \left(\overline{u(\lambda)} - v(\lambda)\right) \varphi(\lambda) - \widetilde{D}(\lambda) \\
&= -\left\langle A\left(T_{\varphi - \varphi(\lambda)} \widehat{k}_{\lambda,a}\right), T_{\overline{\varphi}} \widehat{k}_{\lambda,a} \right\rangle - \varphi(\lambda) \left\langle A \widehat{k}_{\lambda,a}, T_{\overline{\varphi - \varphi(\lambda)}} \widehat{k}_{\lambda,a} \right\rangle - \widetilde{\mathcal{K}_1}(\lambda)
\end{aligned} \tag{4}$$

for all $\lambda \in \mathbb{D}$. It follows from Axler and Zheng result [3] that the functions $\left\| T_{\varphi - \varphi(\lambda)} \widehat{k}_{\lambda,a} \right\|$ and $\left\| T_{\overline{\varphi - \varphi(\lambda)}} \widehat{k}_\lambda \right\|$ have nontangential limits 0 at almost all point of $\partial\mathbb{D}$. Then, by using Cauchy-Schwarz inequality, we have from equality (4) that

$$\begin{aligned}
\left| \widetilde{A}(\lambda)(\varphi(\lambda))^2 + \left(\overline{u(\lambda)} - vIn(\lambda)\, the\right) \varphi(\lambda) - \widetilde{D}(\lambda) \right| &\le \|A\| \left\| T_{\overline{\varphi}} \right\| \left\| T_{\varphi - \varphi(\lambda)} \widehat{k}_{\lambda,a} \right\| \\
&+ \|\varphi\|_{h^\infty} \|A\| \left\| T_{\overline{\varphi - \varphi(\lambda)}} \widehat{k}_{\lambda,a} \right\| + \left| \widetilde{\mathcal{K}_1}(\lambda) \right| \to 0
\end{aligned}$$

as $\lambda \to \xi \in \partial\mathbb{D}$ for almost all $\xi \in \partial\mathbb{D}$; here we used the fact that L_a^2 is a standard reproducing kernel Hilbert space and $\mathcal{K}_1$ is compact, and hence $\widetilde{\mathcal{K}_1}(\lambda) \to 0$ as $\lambda \to \xi \in \partial\mathbb{D}$. This proves (i).

(ii) If $T = T_\varphi + \mathcal{K}$ is a solution of the Riccati equation (1), then as in (i), we have that

$$\widetilde{A}_{nt}(\xi)(\varphi(\xi))^2 + \left(\overline{u(\xi)} - v(\xi)\right) \varphi(\xi) - \widetilde{D}_{nt}(\xi) = 0$$

for almost all $\xi \in \mathbb{T}$, which can be written as

$$\widetilde{A}_{nt}(\xi) \left(\varphi(\xi) + \frac{\overline{u(\xi)} - v(\xi)}{2\widetilde{A}_{nt}(\xi)} \right)^2 = \frac{\left(\overline{u(\xi)} - v(\xi)\right)^2 + 4\widetilde{A}_{nt}(\xi)\widetilde{D}_{nt}(\xi)}{4\widetilde{A}_{nt}(\xi)}.$$

Now, by using condition (3) of the theorem, from this we obtain

$$\varphi(\xi) = \frac{v(\xi) - \overline{u(\xi)}}{2\widetilde{A}_{nt}(\xi)}$$

or equivalently,

$$\varphi^2(\xi) = -\frac{\widetilde{D}_{nt}(\xi)}{\widetilde{A}_{nt}(\xi)},$$

which gives that

$$\varphi(\xi) = \pm i \left(\frac{\widetilde{D}_{nt}(\xi)}{\widetilde{A}_{nt}(\xi)} \right)^{\frac{1}{2}}$$

for almost all $\xi \in \partial\mathbb{D}$. This proves (ii). The theorem is proven. □

In the next result, we use the Berezin symbols technique to study solutions of "Toeplitz corona equation" (2).

Theorem 2.2. *Let $\varphi_i \in L^\infty(\mathbb{D})$ $(i = 1, 2, ..., n)$ be functions such that the nontangential limits $\widetilde{\varphi}_{i,nt}(\xi)$ $(i = 1, 2, ..., n)$ exist almost everywhere in $\partial\mathbb{D}$ and $\widetilde{\varphi}_{i,nt}(\xi) \in L^\infty(\partial\mathbb{D})$, $i = 1, 2, ..., n$. Let $\mathcal{K} \in \mathcal{B}\left(L_a^2\right)$ be a compact operator. If there exist $\psi_1, \psi_2, ..., \psi_n \in h^\infty(\mathbb{D})$ satisfying*

$$T_{\varphi_1}T_{\psi_1} + ... + T_{\varphi_n}T_{\psi_n} = I + \mathcal{K}, \tag{5}$$

then

$$\underset{\partial\mathbb{D}}{ess\inf}\left(\left|\varphi_{1,nt}(\xi)\right| + ... + \left|\varphi_{n,nt}(\xi)\right|\right) \geq \frac{1}{\max\left\{\left\|\psi_i\right\|_{h^\infty} : 1 \leq i \leq n\right\}}.$$

Proof. Let T_{ψ_i} $\left(i = \overline{1, n}\right)$ satisfy equation (2). Then, we have from (5) that

$$1 = \widetilde{T_{\varphi_1}T_{\psi_1}}(\lambda) + ... + \widetilde{T_{\varphi_n}T_{\psi_n}}(\lambda) - \widetilde{\mathcal{K}}(\lambda)$$

for all $\lambda \in \mathbb{D}$, or equivalently

$$\begin{aligned}
1 &= \left\langle T_{\varphi_1}T_{\psi_1}\widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a}\right\rangle + ... + \left\langle T_{\varphi_n}T_{\psi_n}\widehat{k}_{\lambda,a}, \widehat{k}_{\lambda,a}\right\rangle - \widetilde{\mathcal{K}}(\lambda)\\
&= \left\langle T_{\psi_1}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_1}\widehat{k}_{\lambda,a}\right\rangle + ... + \left\langle T_{\psi_n}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_n}\widehat{k}_{\lambda,a}\right\rangle - \widetilde{\mathcal{K}}_1(\lambda)\\
&= \left\langle \widetilde{\psi}_1(\lambda)\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_1}\widehat{k}_{\lambda,a}\right\rangle + ... + \left\langle \widetilde{\psi}_n(\lambda)\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_n}\widehat{k}_{\lambda,a}\right\rangle\\
&+ \left\langle T_{\psi_1-\widetilde{\psi}_1}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_1}\widehat{k}_{\lambda,a}\right\rangle + ... + \left\langle T_{\psi_n-\widetilde{\psi}_n}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_n}\widehat{k}_{\lambda,a}\right\rangle - \widetilde{\mathcal{K}}_1(\lambda)\\
&= \widetilde{\psi}_1(\lambda)\widetilde{\varphi}_1(\lambda) + ... + \widetilde{\psi}_n(\lambda)\widetilde{\varphi}_n(\lambda) + \left\langle T_{\psi_1-\widetilde{\psi}_1}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_1}\widehat{k}_{\lambda,a}\right\rangle + ... + \left\langle T_{\psi_n-\widetilde{\psi}_n}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_n}\widehat{k}_{\lambda,a}\right\rangle - \widetilde{\mathcal{K}}_1(\lambda)\\
&= \psi_1(\lambda)\widetilde{\varphi}_1(\lambda) + ... + \psi_n(\lambda)\widetilde{\varphi}_n(\lambda) + \left\langle T_{\psi_1-\psi_1(\lambda)}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_1}\widehat{k}_{\lambda,a}\right\rangle + ... + \left\langle T_{\psi_n-\psi_n(\lambda)}\widehat{k}_{\lambda,a}, T_{\overline{\varphi}_n}\widehat{k}_{\lambda,a}\right\rangle - \widetilde{\mathcal{K}}_1(\lambda).
\end{aligned}$$

Hence

$$\begin{aligned}
1 &\leq \left|\psi_1(\lambda)\right|\left|\widetilde{\varphi}_1(\lambda)\right| + ... + \left|\psi_n(\lambda)\right|\left|\widetilde{\varphi}_n(\lambda)\right| + \left\|T_{\overline{\varphi}_1}\right\|\left\|T_{\psi_1-\psi_1(\lambda)}\widehat{k}_{\lambda,a}\right\|\\
&+ ... + \left\|T_{\overline{\varphi}_n}\right\|\left\|T_{\psi_n-\psi_n(\lambda)}\widehat{k}_{\lambda,a}\right\| + \left|\widetilde{\mathcal{K}}_1(\lambda)\right|\\
&\leq \left\|\psi_1\right\|_{h^\infty}\left|\widetilde{\varphi}_1(\lambda)\right| + ... + \left\|\psi_n\right\|_{h^\infty}\left|\widetilde{\varphi}_n(\lambda)\right| + \left\|\varphi_1\right\|_{L^\infty(\mathbb{D})}\left\|T_{\psi_1-\psi_1(\lambda)}\widehat{k}_{\lambda,a}\right\|\\
&+ ... + \left\|\varphi_n\right\|_{L^\infty(\mathbb{D})}\left\|T_{\psi_n-\psi_n(\lambda)}\widehat{k}_{\lambda,a}\right\| + \left|\widetilde{\mathcal{K}}_1(\lambda)\right|.
\end{aligned}$$

Since $\widetilde{\varphi}_{i,nt}(\xi)$ exists for almost all $\xi \in \partial\mathbb{D}$ and for all $i = 1, ..., n$, $\widetilde{\mathcal{K}}(\lambda) \to 0$ as $\lambda \to \partial\mathbb{D}$ and $\left\|T_{\psi_i-\psi_i(\lambda)}\widehat{k}_{\lambda,a}\right\| \to 0$ as $\lambda \to \xi \in \partial\mathbb{D}$ nontangentially for almost all $\xi \in \partial\mathbb{D}$, we deduce that

$$\left\|\psi_1\right\|_{h^\infty}\left|\widetilde{\varphi}_{1,nt}(\xi)\right| + ... + \left\|\psi_n\right\|_{h^\infty}\left|\widetilde{\varphi}_{n,nt}(\xi)\right| \geq 1$$

for almost all $\xi \in \partial\mathbb{D}$. Thus

$$\underset{\partial\mathbb{D}}{ess\inf}\left(\left|\widetilde{\varphi}_{1,nt}(\xi)\right| + ... + \left|\widetilde{\varphi}_{n,nt}(\xi)\right|\right) \geq \frac{1}{\max\left\{\left\|\psi_i\right\|_{L^\infty(\mathbb{D})} : \ i = \overline{1, n}\right\}},$$

as desired. The theorem is proved. □

3. Characterization of compact truncated operators via the Berezin symbol

Let $\mathcal{H} = \mathcal{H}(\Omega)$ be a reproducing kernel Hilbert space on some set Ω with the reproducing kernel $\mathcal{K}_{\mathcal{H},\lambda} \in \mathcal{H}$, that is $\left\langle f, \mathcal{K}_{\mathcal{H},\lambda}\right\rangle_{\mathcal{H}} = f(\lambda)$ for all $f \in \mathcal{H}$ and $\lambda \in \Omega$. $\widehat{\mathcal{K}}_{\mathcal{H},\lambda} := \mathcal{K}_{\mathcal{H},\lambda} / \left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}$ is the normalized reproducing kernel for $\mathcal{H}$ (of course, we suppose that for any $\lambda \in \Omega$, there exists $f_\lambda \in \mathcal{H}$ such that $f_\lambda(\lambda) \neq 0$). The Berezin symbol of operator $A \in \mathcal{B}(\mathcal{H}(\Omega))$ is the complex-valued bounded function

$$\widetilde{A}(\lambda) := \left\langle A\widehat{\mathcal{K}}_{\mathcal{H},\lambda}, \widehat{\mathcal{K}}_{\mathcal{H},\lambda}\right\rangle, \ \lambda \in \Omega,$$

since $\left|\widetilde{A}(\lambda)\right| \leq \|A\|$ for all $\lambda \in \Omega$ by Cauchy-Schwarz inequality. In the sequel, we will use the notation

$$\widetilde{A}^M(\lambda) := \left\langle A\widehat{\mathcal{K}}_{M,\lambda}, \widehat{\mathcal{K}}_{M,\lambda}\right\rangle,$$

where $\mathcal{K}_{M,\lambda} = P_M \mathcal{K}_{H,\lambda}$ is the reproducing kernel of the (closed) subspace $M \subset \mathcal{H}$ and $\widehat{\mathcal{K}}_{M,\lambda} = \frac{\mathcal{K}_{M,\lambda}}{\|\mathcal{K}_{M,\lambda}\|_{\mathcal{H}}}$ is the normalized reproducing kernel of the subspace M. Let E be a closed subspace of the standard reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$, and let $A \in \mathcal{B}(\mathcal{H})$ be an operator. We consider the truncated operator $T : E^\perp \to E^\perp$ defined by

$$T = P_{E^\perp} A | E^\perp,$$

where $P_{E^\perp} : \mathcal{H} \to E^\perp$ is the orthogonal projection onto the orthogonal complement $E^\perp := \mathcal{H} \Theta E$ of the subspace E. In this section, we study compactness property of operator T in terms of Berezin symbols. Before stating our result, note that the closed subspace of the standard reproducing kernel Hilbert space, in general, is not standard (see, [16] and [14]). By this reason, we can not use directly Nordgren-Rosenthal's characterization [16] of compact operators on the subspace $E^\perp$. However, we prove in that situation the following theorem, which essentially improves a result in [15, Theorem 7].

Theorem 3.1. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded linear operator on the standard (in the sense of Nordgren and Rosenthal) reproducing kernel Hilbert space $\mathcal{H} = \mathcal{H}(\Omega)$, and let $E \subset \mathcal{H}$ be a closed subspace. Let*

$$T := P_{E^\perp} A \mid E^\perp$$

be a truncated operator on $E^\perp$. Then, T is compact if and only if

$$\lim_{\lambda \to \partial\Omega} \left(\widetilde{P}_{U^{-1}E^\perp}(\lambda)\, \widetilde{U^{-1}TU}^{U^{-1}E^\perp}(\lambda) \right) = 0$$

for all unitary operators $U : \mathcal{H} \to \mathcal{H}$.

Proof. We set $B := TP_{E^\perp}$. Obviously $B \in \mathcal{B}(\mathcal{H})$. Then, we have for every unitary operator $U : \mathcal{H} \to \mathcal{H}$ that

$$U^{-1}BU = U^{-1}TP_{E^\perp}U = \left(U^{-1}TU\right)\left(U^{-1}P_{E^\perp}U\right) = U^{-1}TUP_{U^{-1}E^\perp}.$$

Since $\mathcal{K}_{U^{-1}E^\perp,\lambda} = P_{U^{-1}E^\perp}\mathcal{K}_{\mathcal{H},\lambda}$, $\lambda \in \Omega$, we have from the last equality that

$$\begin{aligned}
\widetilde{U^{-1}BU}(\lambda) &= \left\langle U^{-1}BU\widehat{\mathcal{K}}_{\mathcal{H},\lambda}, \widehat{\mathcal{K}}_{\mathcal{H},\lambda}\right\rangle = \left\langle U^{-1}TUP_{U^{-1}E^\perp}\widehat{\mathcal{K}}_{\mathcal{H},\lambda}, \widehat{\mathcal{K}}_{\mathcal{H},\lambda}\right\rangle \\
&= \frac{1}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|^2_{\mathcal{H}}} \left\langle U^{-1}TUP_{U^{-1}E^\perp}\mathcal{K}_{\mathcal{H},\lambda}, \mathcal{K}_{\mathcal{H},\lambda}\right\rangle \\
&= \frac{1}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|^2_{\mathcal{H}}} \left\langle U^{-1}TU\mathcal{K}_{U^{-1}E^\perp,\lambda}, P_{U^{-1}E^\perp}\mathcal{K}_{\mathcal{H},\lambda} + (I - P_{U^{-1}E^\perp})\mathcal{K}_{\mathcal{H},\lambda}\right\rangle \\
&= \frac{1}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|^2_{\mathcal{H}}} \left[\left\langle U^{-1}TU\mathcal{K}_{U^{-1}E^\perp,\lambda}, \mathcal{K}_{U^{-1}E^\perp,\lambda}\right\rangle + \left\langle U^{-1}TU\mathcal{K}_{U^{-1}E^\perp,\lambda}, (I - P_{U^{-1}E^\perp})\mathcal{K}_{\mathcal{H},\lambda}\right\rangle\right].
\end{aligned}$$

It is easy to see that

$$\left\langle U^{-1}TU\mathcal{K}_{U^{-1}E^{\perp},\lambda}, (I - P_{U^{-1}E^{\perp}})\,\mathcal{K}_{\mathcal{H},\lambda}\right\rangle = 0.$$

In fact, by using that

$$\mathcal{K}_{U^{-1}E^{\perp},\lambda} = P_{U^{-1}E^{\perp}}\mathcal{K}_{\mathcal{H},\lambda} = \left(U^{-1}P_{E^{\perp}}U\right)\mathcal{K}_{\mathcal{H},\lambda} \text{ and } U^{-1} = U^{*},$$

we have

$$\begin{aligned}\left\langle U^{-1}TU\mathcal{K}_{U^{-1}E^{\perp},\lambda}, (I - P_{U^{-1}E^{\perp}})\,\mathcal{K}_{\mathcal{H},\lambda}\right\rangle &= \left\langle TU\left(U^{-1}P_{E^{\perp}}U\right)\mathcal{K}_{\mathcal{H},\lambda}, U - U\left(U^{-1}P_{E^{\perp}}U\right)\mathcal{K}_{\mathcal{H},\lambda}\right\rangle \\ &= \left\langle TP_{E^{\perp}}U\mathcal{K}_{\mathcal{H},\lambda}, (U - P_{E^{\perp}}U)\,\mathcal{K}_{\mathcal{H},\lambda}\right\rangle \\ &= \left\langle TP_{E^{\perp}}U\mathcal{K}_{\mathcal{H},\lambda}, (I - P_{E^{\perp}})\,U\mathcal{K}_{\mathcal{H},\lambda}\right\rangle = 0,\end{aligned}$$

since $TP_{E^{\perp}}U\mathcal{K}_{\mathcal{H},\lambda} \in E^{\perp}$ and $(I - P_{E^{\perp}})\,U\mathcal{K}_{\mathcal{H},\lambda} = P_{E}U\mathcal{K}_{\mathcal{H},\lambda} \in E$. Thus, we have

$$\begin{aligned}\widetilde{U^{-1}BU}(\lambda) &= \frac{1}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}^{2}}\left\langle U^{-1}TU\mathcal{K}_{U^{-1}E^{\perp},\lambda}, \mathcal{K}_{U^{-1}E^{\perp},\lambda}\right\rangle \\ &= \frac{1}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}^{2}}\left\|\mathcal{K}_{U^{-1}E^{\perp},\lambda}\right\|^{2}\left\langle U^{-1}TU\widehat{\mathcal{K}}_{U^{-1}E^{\perp},\lambda}, \widehat{\mathcal{K}}_{U^{-1}E^{\perp},\lambda}\right\rangle \\ &= \left[\frac{\left\|\mathcal{K}_{U^{-1}E^{\perp},\lambda}\right\|_{\mathcal{H}}}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}}\right]^{2}\widetilde{U^{-1}TU}^{U^{-1}E^{\perp}}(\lambda),\end{aligned}$$

hence

$$\widetilde{U^{-1}BU}(\lambda) = \left[\frac{\left\|\mathcal{K}_{U^{-1}E^{\perp},\lambda}\right\|_{\mathcal{H}}}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}}\right]^{2}\widetilde{U^{-1}TU}^{U^{-1}E^{\perp}}(\lambda)$$

for all $\lambda \in \Omega$. On the other hand,

$$\begin{aligned}\left\|\mathcal{K}_{U^{-1}E^{\perp},\lambda}\right\|_{\mathcal{H}}^{2} &= \left\|P_{U^{-1}E^{\perp}}\mathcal{K}_{\mathcal{H},\lambda}\right\|^{2} = \left\langle P_{U^{-1}E^{\perp}}\mathcal{K}_{\mathcal{H},\lambda}, P_{U^{-1}E^{\perp}}\mathcal{K}_{\mathcal{H},\lambda}\right\rangle \\ &= \left\langle P_{U^{-1}E^{\perp}}\mathcal{K}_{\mathcal{H},\lambda}, \mathcal{K}_{\mathcal{H},\lambda}\right\rangle \text{ (since } P_{U^{-1}E^{\perp}}^{2} = P_{U^{-1}E^{\perp}} = P_{U^{-1}E^{\perp}}^{*}) \\ &= \left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}^{2}\left\langle P_{U^{-1}E^{\perp}}\widehat{\mathcal{K}}_{\mathcal{H},\lambda}, \widehat{\mathcal{K}}_{\mathcal{H},\lambda}\right\rangle \\ &= \left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}^{2}\widetilde{P}_{U^{-1}E^{\perp}}(\lambda),\end{aligned}$$

and hence

$$\left[\frac{\left\|\mathcal{K}_{U^{-1}E^{\perp},\lambda}\right\|_{\mathcal{H}}}{\left\|\mathcal{K}_{\mathcal{H},\lambda}\right\|_{\mathcal{H}}}\right]^{2} = \widetilde{P}_{U^{-1}E^{\perp}}(\lambda) \text{ for all } \lambda \in \Omega.$$

So, we have that

$$\widetilde{U^{-1}BU}(\lambda) = \widetilde{P}_{U^{-1}E^{\perp}}(\lambda)^{2}\,\widetilde{U^{-1}TU}^{U^{-1}E^{\perp}}(\lambda) \tag{6}$$

for all unitary operators $U \in \mathcal{B}(\mathcal{H})$ and all $\lambda \in \Omega$. It is not difficult to verify that B is compact on $\mathcal{H}$ if and only if T is compact on $E^{\perp}$. Hence, according to above mentioned result of Nordgren and Rosenthal and formula (6), we deduce that T is compact in $E^{\perp}$ if and only if

$$\lim_{\lambda\to\partial\Omega}\left[\widetilde{P}_{U^{-1}E^{\perp}}(\lambda)\,\widetilde{U^{-1}TU}^{U^{-1}E^{\perp}}(\lambda)\right] = 0$$

for every unitary operator U on $\mathcal{H}$. The proof of theorem is finished. □

The function $\theta \in H^{\infty}$ is called inner if $|\theta(z)| \leq 1$ for all z in $\mathbb{D}$ and $\left|\theta\left(e^{it}\right)\right| = 1$ for almost all t in the segment $[0, 2\pi)$. The model subspace K_{θ} associated by the function θ is defined by

$$K_{\theta} = \left(\theta H^2\right)^{\perp} = H^2 \Theta \theta H^2;$$

we note that since T_{θ} (the analytic Toeplitz operator) is an isometry in H^2, the subspace θH^2 and K_{θ} are closed in H^2.

If we put in Theorem 3.1, $\mathcal{H} = H^2(\mathbb{D})$ and $E = \theta H^2$, where θ is an inner function, then we get Theorem 7 in [15].

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