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Several inequalities for bounding sums of two (hyperbolic) sine cardinal functions

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Abstract. In the paper, the authors present several inequalities for bounding the sums of two sine cardinal functions and the sums of two hyperbolic sine cardinal functions. These inequalities improve previously-known results.

1. Introduction

For $x \in \mathbb{R}$, the functions

$$\operatorname{sinc} x = \begin{cases} \dfrac{\sin x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} \qquad \operatorname{sinhc} x = \begin{cases} \dfrac{\sinh x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases}$$

$$\operatorname{tanc} x = \begin{cases} \dfrac{\tan x}{x}, & x \neq 0; \\ 1, & x = 0, \end{cases} \qquad \operatorname{tanhc} x = \begin{cases} \dfrac{\tanh x}{x}, & x \neq 0; \\ 1, & x = 0 \end{cases}$$

are called the sinc function, the tanc function, the hyperbolic sinc function, and the hyperbolic tanc function, respectively. The function $\operatorname{sinc} x$ is also called the sine cardinal or sampling function, as well as the function $\operatorname{sinhc} x$ is also called hyperbolic sine cardinal, see the papers [12, 33]. The sinc function $\operatorname{sinc} x$ arises frequently in signal processing, the theory of the Fourier transforms, and other areas in mathematics, physics, and engineering.

Inequalities involving the sinc function, the tanc function, the hyperbolic sinc function, and the hyperbolic tanc function, such as Wilker's inequality, Huygen's inequality, Jordan's inequality, Caus–Huygen's inequality, Becker–Stark's inequality, and so on, arouse great enthusiasm of researchers, see the literatures [1, 5–11, 13–32, 35–37, 39], for examples, and closely-related references therein.

In [2], Bagul and Chesneau proved two double inequalities

$$1 + 2\cos x \le \operatorname{sinc}(2x) + 2\operatorname{sinc} x \le 2 + \cos^2 x, \quad |x| < \frac{\pi}{2} \tag{1.1}$$

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and

$$1+2\cosh x \le \operatorname{sinhc}(2x)+2\operatorname{sinhc} x \le 2+\cosh^2 x, \quad x\in(-\infty,\infty). \tag{1.2}$$

Meanwhile, these two authors also posed two open problems which can be restated as the following four problems:

i) For $x\in\left(0,\frac{\pi}{2}\right)$ and $t\ge 2$, prove

$$t+\cos^t x > \operatorname{sinc}(tx)+t\operatorname{sinc} x. \tag{1.3}$$

ii) For $x\in\left(0,\frac{\pi}{2}\right)$ and $p\in(0,2]$, prove

$$\operatorname{sinc}(px)+p\operatorname{sinc} x > 1+p\cos x. \tag{1.4}$$

iii) For $x\in\mathbb{R}\setminus\{0\}$ and $s\in(0,2]$, prove

$$s+\cosh^s x > \operatorname{sinhc}(sx)+s\operatorname{sinhc} x. \tag{1.5}$$

iv) For $x\in\mathbb{R}\setminus\{0\}$ and $q\ge 2$, prove

$$\operatorname{sinhc}(qx)+q\operatorname{sinhc} x > 1+q\cosh(x). \tag{1.6}$$

The Fourier series technology to approximate inequalities involving the functions $\operatorname{sinc} x$, $\operatorname{tanc} x$, $\operatorname{sinhc} x$, and $\operatorname{tanhc} x$ has attracted the attention of many researchers. In 2019, Bercu [3] used the cosine polynomials for the even functions $\operatorname{sinc} x$ and $\operatorname{tanc} x$ and obtained some new inequalities of the Wilker–Cusa–Huygens type. In 2021, Wu and Bercu [34] used the power series expansions of $\sin x$ and $\cos x$ and employed the Fourier series technology to approximate the function $1-\operatorname{sinc} x$ and $\operatorname{tanc} x-1$. In 2021, Zhu [38] used the power series expansion technique and established two new sharp bound for inequalities involving $\sin x$ and $\tan x$ in terms of the functions $x^2[\operatorname{sinc}(\lambda x)]^\alpha$ and $x^2[\operatorname{tanc}(\mu x)]^\beta$. In 2022, Bercu [4] refined several inequalities of the Huygens–Wilker–Lazarović type by using the hyperbolic cosine polynomials.

The goals of this paper are to refine the inequalities (1.1) and (1.2) and to verify the inequality (1.6).

2. Inequalities

Now we are in a position to state and prove our main results.

Theorem 1. *For $|x|<\frac{\pi}{2}$, we have*

$$\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4} \le \operatorname{sinc}(2x)+2\operatorname{sinc} x \le \frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{1}{1260}x^6. \tag{2.1}$$

Proof. For $x\in\left(0,\frac{\pi}{2}\right)$, using the power series expansions

$$\sin x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k+1}}{(2k+1)!} \quad \textit{and} \quad \cos x=\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!},$$

we acquire

$$\begin{aligned}
&\operatorname{sinc}(2x)+2\operatorname{sinc} x-\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2+\frac{3}{4}\\
&=\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}x^{2k}}{(2k+1)!}+2\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k+1)!}-\frac{22}{15}\sum_{k=0}^{\infty}(-1)^k\frac{x^{2k}}{(2k)!}-\frac{2}{15}\sum_{k=0}^{\infty}(-1)^k\frac{2^{2k}x^{2k}}{(2k)!}
\end{aligned}$$

$$= \frac{4}{15}\sum_{k=3}^{\infty}(-1)^{k+1}\frac{11k-2+2^{2k-2}(4k-13)}{(2k+1)!}x^{2k}.$$

Let

$$a_k = \frac{11k-2+2^{2k-2}(4k-13)}{(2k+1)!}, \quad k \ge 3. \tag{2.2}$$

In order to prove Theorem 1, it suffices to prove that the positive sequence a_k is decreasing in $k \ge 3$, that is,

$$\frac{11k-2+2^{2k-2}(4k-13)}{(2k+1)!} > \frac{11k+9+2^{2k}(4k-9)}{(2k+3)!}, \quad k \ge 3.$$

This inequality is equivalent to

$$2^{2k-1}\Big[8(k-4)^3+90(k-4)^2+275(k-4)+151\Big]+44(k-3)^3+498(k-3)^2+1835(k-3)+2190>0,$$

which is clearly valid for $k \ge 3$. The proof of Theorem 1 is complete. □

Remark 1. Since

$$\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}-(1+2\cos x)=\frac{4(1-\cos x)^2}{15}\ge 0$$

and

$$\begin{aligned}
&2+\cos^2 x-\left[\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{x^6}{1260}\right]\\
&=\frac{11}{10}-\frac{x^6}{1260}-\frac{22}{15}\cos x+\frac{11}{30}\cos 2x\\
&=\frac{11}{10}-\frac{x^6}{1260}-\frac{22}{15}\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k}+\frac{11}{30}\sum_{k=0}^{\infty}\frac{(-1)^k 2^{2k}}{(2k)!}x^{2k}\\
&=\frac{x^4}{2520}\left[462-79x^2+924\sum_{k=4}^{\infty}\frac{(-1)^k(4^k-4)}{(2k)!}x^{2k-4}\right]\\
&=\frac{x^4}{2520}\left(462-79x^2+924\sum_{k=4}^{\infty}\lambda_k x^{2k-4}\right),
\end{aligned}$$

where $\lambda_k=\frac{(-1)^k(4^k-4)}{(2k)!}$. For $|x|<\frac{\pi}{2}$ and $k\ge 4$, we have $x^2<\frac{5}{2}$, $462>79x^2$, and

$$\left|\frac{\lambda_{k+1}}{\lambda_k}x^2\right|=\frac{x^2}{(2k+2)(2k+1)}\frac{4^{k+1}-4}{4^k-4}<\frac{5}{(k+1)(2k+1)}\left(1+\frac{12}{4^{k+1}-16}\right)\le\frac{1}{9}\left(1+\frac{12}{4^5-16}\right)<1.$$

This implying that $2+\cos^2 x-\left[\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{x^6}{1260}\right]>0$ for $|x|<\frac{\pi}{2}$. So, the double inequality (2.1) improves the double inequality (1.1).

Remark 2. For $n\ge 1$ and $0<x<\frac{\pi}{2}$, we have

$$\begin{aligned}
\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{4}{15}\sum_{k=2}^{2n}(-1)^{k+1}a_k x^{2k}&<\operatorname{sinc}(2x)+2\operatorname{sinc}x\\
&<\frac{4}{15}\left(\cos x+\frac{11}{4}\right)^2-\frac{3}{4}+\frac{4}{15}\sum_{k=2}^{2n+1}(-1)^{k+1}a_k x^{2k}
\end{aligned}$$

where a_k is defined by (2.2) in the proof of Theorem 1.

Theorem 2. *For $x \in \mathbb{R}$ and $n \ge 2$, we have*

$$1 + 2\cosh x + \sum_{k=2}^{n} b_k x^{2k} \le \operatorname{sinhc}(2x) + 2\operatorname{sinhc} x \le \frac{4}{15}\left(\cosh x + \frac{11}{4}\right)^2 - \frac{3}{4}, \tag{2.3}$$

where $b_k = \frac{2^{2k}-4k}{(2k+1)!}$ for $k \ge 2$.

Proof. Using the power series expansions

$$\sinh x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)!} \quad \textit{and} \quad \cosh x = \sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!}, \tag{2.4}$$

we obtain

$$\begin{aligned}
&\operatorname{sinhc}(2x) + 2\operatorname{sinhc} x - 1 - 2\cosh x - \sum_{k=2}^{n} b_k x^{2k} \\
&= \sum_{k=0}^{\infty} \frac{2^{2k}x^{2k}}{(2k+1)!} + 2\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!} - 1 - 2\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} - \sum_{k=2}^{n} \frac{\left(2^{2k}-4k\right)x^{2k}}{(2k)!} \\
&= \sum_{k=2}^{\infty} \frac{\left(2^{2k}-4k\right)x^{2k}}{(2k+1)!} - \sum_{k=2}^{n} \frac{\left(2^{2k}-4k\right)x^{2k}}{(2k+1)!} \\
&= \sum_{k=n+1}^{\infty} b_k x^{2k}.
\end{aligned}$$

From the positivity $b_k > 0$ for $k \ge 2$, it follows that

$$\operatorname{sinhc}(2x) + 2\operatorname{sinhc} x - 1 + 2\cosh x + \sum_{k=2}^{n} b_k x^{2k} > 0.$$

The left-hand side inequality in (2.3) is thus proved.

Making use of the series expansions in (2.4) once again leads to

$$\begin{aligned}
&\operatorname{sinhc}(2x) + 2\operatorname{sinhc} x - \frac{4}{15}\left(\cosh x + \frac{11}{4}\right)^2 + \frac{3}{4} \\
&= \sum_{k=0}^{\infty} \frac{2^{2k}x^{2k}}{(2k+1)!} + 2\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k+1)!} - \frac{22}{15}\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} - \frac{2}{15}\sum_{k=0}^{\infty} \frac{2^{2k}x^{2k}}{(2k)!} \\
&= -\frac{4}{15}\sum_{k=3}^{\infty} a_k x^{2k},
\end{aligned}$$

where the positive sequence a_k is defined by (2.2). This implies that the right-hand side inequality in (2.3) is valid. The proof of Theorem 2 is complete. □

Remark 3. Since $b_k > 0$ and

$$\cosh^2 x - \frac{4}{15}\left(\cosh x + \frac{11}{4}\right)^2 + \frac{11}{4} = \frac{11(1-\cosh x)^2}{15} \ge 0,$$

the double inequality (2.3) is tighter than the double inequality (1.2).

Theorem 3. *Let*

$$q_0 = \frac{1}{2} + \frac{\sqrt{33}}{6} = 1.457\cdots$$

and

$$c_k = \frac{3q^{2k} - 2q(1+q)k + q(2-q)}{3(2k+1)!}, \quad k \ge 2.$$

For $x \in \mathbb{R} \setminus \{0\}$, $q > q_0$, *and* $n \ge 2$, *we have*

$$1 + \frac{q(2-q)}{3} + \frac{q(1+q)}{3}\cosh x < 1 + \frac{q(2-q)}{3} + \frac{q(1+q)}{3}\cosh x + \sum_{k=2}^{n} c_k x^{2k} < \operatorname{sinhc}(qx) + q\operatorname{sinhc} x. \tag{2.5}$$

Proof. Let

$$G_q(x) = \operatorname{sinhc}(qx) + q\operatorname{sinhc} x - \frac{q(1+q)}{3}\cosh x - \frac{q(2-q)}{3} - 1.$$

Utilizing the power series expansions in (2.4), we obtain

$$\begin{aligned} G_q(x) &= \sum_{k=0}^{\infty} \frac{q^{2k}x^{2k}}{(2k+1)!} + \sum_{k=0}^{\infty} \frac{qx^{2k}}{(2k+1)!} - \frac{q(1+q)}{3}\sum_{k=0}^{\infty} \frac{x^{2k}}{(2k)!} - \frac{q(2-q)}{3} - 1 \\ &= \frac{1}{3}\sum_{k=2}^{\infty} \frac{3q^{2k} - 2q(q+1)k + q(2-q)}{(2k+1)!} x^{2k}. \end{aligned}$$

For $x \ge 2$ and $q \ge q_0$, let $u(x,q) = 3q^{2x} - 2q(q+1)x + q(2-q)$. Then

$$\begin{aligned} u(2,q) &= q(q+1)\big(3q^2 - 3q - 2\big) \\ &= q(q+1)(q-q_0)\left[q - \left(\frac{1}{2} - \frac{\sqrt{33}}{6}\right)\right] \\ &> 0, \\ u_x'(x,q) &= 6(\ln q)q^{2x} - 2q(q+1), \\ u_{xx}''(x,q) &= 12(\ln q)^2 q^{2x} \\ &> 0. \end{aligned}$$

Thus, the first derivative $u_x'(x,q)$ is increasing in $x \ge 2$. Since

$$\begin{aligned} u_x'(2,q) &= 2q^4\left(3\ln q - \frac{1}{q^2} - \frac{1}{q^3}\right), \\ u_x'(2,q_0) &= \frac{\sqrt{33}+3}{18}\left[\left(5\sqrt{33}+27\right)\ln\frac{\sqrt{33}+3}{6} - \sqrt{33} - 9\right] \\ &= 3.033\cdots, \end{aligned}$$

and

$$\left(3\ln q - \frac{1}{q^2} - \frac{1}{q^3}\right)' = \frac{3q^3 + 2q + 3}{q^4} > 0,$$

we derive $u_x'(x,q) > 0$ for $x \ge 2$ and $q \ge q_0$. Hence, the function $u(x,q)$ is increasing in $x \ge 2$ for $q \ge q_0$. Accordingly, we obtain $u(x,q) > 0$ for $x \ge 2$ and $q \ge q_0$. Consequently, we conclude $G_q(x) > 0$ for $x \ge 2$ for $q \ge q_0$. The proof of Theorem 3 is complete. □

Remark 4. When $q = 2$, the inequality (2.5) can be written as

$$\operatorname{sinhc}(2x) + 2\operatorname{sinhc} x > 1 + 2\cosh x,$$

which is equivalent to the left-hand side inequality in (1.2)

Remark 5. For $x \in \mathbb{R} \setminus \{0\}$ and $q \geq 2$, we have

$$\frac{q(1+q)}{3}\cosh x - q\cosh x + \frac{q(2-q)}{3} = \frac{q(q-2)}{3}(\cosh x - 1) \geq 0.$$

So, the double inequality (2.5) improves the double inequality (1.6).

3. Conclusions

In this paper, we presented our main results in Theorems 1, 2, and 3. These main results are three inequalities (2.1), (2.3), and (2.5). These three inequalities refined the double inequality (1.1), the double inequality (1.2), and the inequality (1.6).

In future, we wish to verify another three inequalities (1.3), (1.4), and (1.5).

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