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Rational interpolation with parameters based on Chebyshev nodes

Zhinan Yan[a], Yi Zhao[a,*]

[a] *School of Mathematics, Hangzhou Normal University*

Abstract. A type of rational interpolation operator (RIO) with parameter λ is studied in this paper. We give the specific construction of RIO based on Chebyshev nodes of the fourth kind, and calculate the approximation order of the RIO to functions when $1 < \lambda < 2$ and $\lambda > 2$ respectively. Furthermore, we analyze the relation between the parameter λ and the approximate error, while the latter is between the operator and the function. By selecting the sampling function and adjusting the parameter, the theoretical results are visualized and verified from the images and error data tables.

1. Introduction

Rational interpolation is a method with important theoretical and applied value in function approximation theory. Rational fractional functions are a class of simple functions, which are more complex than polynomial functions, but provide a better approximation than polynomials when they are used to represent functions(see [8], for example). They can better reflect the characteristics of the function, or have better numerical stability([6], [13]), so they have a wide range of applications in numerical computation, function approximation, etc.

The approximation and saturation of rational interpolation operators have been of interest to many scholars, such as [5, 9], [1, 2, 7, 11, 14, 15] and so on. In [11], the authors studied the rational interpolation operator based on Chebyshev nodes of the second kind, with the form as follows:

$$L_n^{\lambda}(f,x) = \frac{\sum_{k=0}^{n+1} f(x_k)|q_k(x)|^{\lambda}}{\sum_{k=0}^{n+1} |q_k(x)|^{\lambda}}, \quad x \in [-1,1], \quad \lambda > 2. \tag{1}$$

Here, $x_k = \cos\theta_k = \cos\frac{k\pi}{n+1}$ $(k = 0, 1, \ldots, n+1)$ are the zeros of $(1-x^2)U_n(x)$, and $U_n(x)$, the second kind of Chebyshev polynomial is defined by:

$$U_n(x) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad x = \cos\theta,$$

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* Corresponding author: Yi Zhao
Email address: zhaoyi@hznu.edu.cn (Yi Zhao)

while the fundamental functions are defined by:

$$q_0(x) = \frac{(1+x)U_n(x)}{2(n+1)}, \quad q_{n+1}(x) = \frac{(-1)^n(1-x)U_n(x)}{2(n+1)},$$

$$q_k(x) = \frac{(-1)^{k+1}(1-x^2)U_n(x)}{(n+1)(x-x_k)}, \ k = 1, \dots, n.$$

The authors obtained the following conclusion in the same paper ([11]):

Theorem 1.1. *Assume $f(x) \in C[-1,1]$, $L_n^\lambda(f,x)$ is defined as in* (1), *then for $\lambda > 2$,*

$$L_n^\lambda(f,x) - f(x) = O\left(\omega\left(f, \frac{(1-x^2)|U_n(x)|}{n}\right)\right).$$

In [14], the authors generalized the results based on [11]. Specifically, by extending the value of λ to $\lambda > 1$ and subdividing the range of λ, they further discuss the approximation of $L_n^\lambda(f,x)$ to $f(x)$ when $1 < \lambda < 2$ and $\lambda > 2$. The conclusion is as follows:

Theorem 1.2. *Assume $f(x) \in C[-1,1]$, $L_n^\lambda(f,x)$ is defined by* (1), *then for $1 < \lambda < 2$, we have*

$$L_n^\lambda(f,x) - f(x) = O\left(\omega\left(f, \frac{(1-x^2)|U_n(x)|}{n^{\lambda-1}}\right)\right).$$

While for $\lambda > 2$,

$$L_n^\lambda(f,x) - f(x) = O\left(\omega\left(f, \frac{(1-x^2)|U_n(x)|}{n}\right)\right).$$

Due to the special and good properties of Chebyshev polynomials, different kinds of Chebyshev polynomials have come into the attention of researchers (see [4], [10], [15], for example), and Chebyshev nodes are often a good choice of interpolation nodes. In this paper, we investigate the approximation of a type of rational interpolation operator to continuous function based on Chebyshev nodes of the fourth kind.

Denote Chebyshev polynomial of the fourth kind by:

$$S_n(x) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{\theta}{2}}, \quad x = \cos\theta,$$

then we define the RIO with paramater λ as follows:

$$R_n^\lambda(f,x) = \frac{\sum\limits_{k=1}^{n} f(x_k)|l_k(x)|^\lambda}{\sum\limits_{k=1}^{n} |l_k(x)|^\lambda}, \quad \lambda > 0, \tag{2}$$

where $x_k = \cos\theta_k = \cos\dfrac{2k\pi}{2n+1}$ $(k = 1, \dots, n)$ are the zeros of $S_n(x)$, i.e., Chebyshev nodes of the fourth kind, while $l_k(x), \ k = 1, \dots, n,$ are Lagrange interpolation basis functions, defined by:

$$l_k(x) = \frac{\prod\limits_{j=1, j\neq k}^{n} (x - x_j)}{\prod\limits_{j=1, j\neq k}^{n} (x_k - x_j)}.$$

Fixing Chebyshev nodes of the fourth kind as the interpolation nodes of $l_k(x)$, it is easy to deduce that the basis functions for Lagrange interpolation are:

$$l_k(\cos\theta) = \frac{(-1)^{k+1}\sin(n+\frac{1}{2})\theta\sin\frac{\theta_k}{2}\sin\theta_k}{(n+\frac{1}{2})\sin\frac{\theta}{2}(\cos\theta-\cos\theta_k)}, \quad k=1,\dots,n. \tag{3}$$

The main purpose of this paper is to discuss the approximation of the RIO $R_n^\lambda(f,x)$ to the continuous function $f(x)$. The approximation order of $R_n^\lambda(f,x)$ to $f(x)$ are obtained in the cases $1<\lambda<2$ and $\lambda>2$ respectively. Furthermore, the relationship between the parameter λ and the approximation performance is analyzed, and the theoretical results are verified by the presentation of images and error data.

The paper is organized as follows: Chapter 2 shows the relevant notation, basic concepts and lemmas, and proofs of lemmas; Chapter 3 gives the main theorems and their proofs; Chapter 4 shows the images and data simulations, including the graphical representations of the error between the sampling function and the RIO, the data analysis of their error, etc.; the last chapter is the conclusion.

2. Basic concepts and lemmas

The notation "$A\sim B$" below indicates the existence of the constants C_1 and C_2 $(0<C_1<C_2)$, such that $C_1A\le B\le C_2A$, and "$A=O(B)$" indicates the existence of a constant $C>0$, such that $A\le C\cdot B$. Through out the paper, C refers to a positive constant and $C(\lambda)$ denotes a positive constant related only to λ, whose value may vary under different circumstances.

The following definition can be found in [3].

Definition 2.1. *Let $f(x)$ be defined on I. For $t>0$, $\omega(f,t)$ denote the modulus of continuity of $f(x)$, defined by:*

$$\omega(f,t)=\sup_{x,x+h\in I,|h|\le t}|f(x+h)-f(x)|.$$

It is obvious that $\omega(f,t)$ is nondecreasing with respect to t on $[0,+\infty)$. The modulus of continuity $\omega(f,t)$ has some important properties, we list part of them here, which are related to the research presenting in this paper.

Proposition 2.2. *Let $f(x)$ be defined on I, the modulus of continuity $\omega(f,t)$ has the following properties:*
(I) *If $\delta\ge0,\eta\ge0,\delta+\eta\le|I|$ (where $|I|$ represents the length of interval I), then*

$$\omega(f,\delta+\eta)\le\omega(f,\delta)+\omega(f,\eta).$$

(II) *For $\delta\ge0,\lambda>0,\lambda\delta\le|I|$, we have*

$$\omega(f,\lambda\delta)\le(1+\lambda)\omega(f,\delta).$$

(III) *If $t_1\sim t_2$, then $\omega(f,t_1)\sim\omega(f,t_2)$.*

Proof. The proof of Proposition 2.2 (I)-(II) can be found in [12].

For Proposition(III), since $t_1\sim t_2$, there exist constants C_1,C_2 such that $C_1t_2\le t_1\le C_2t_2$ $(0<C_1<C_2)$. By the monotonicity of $\omega(f,t)$ on $[0,+\infty)$ and Proposition 2.2 (II),

$$\omega(f,t_1)\le\omega(f,C_2t_2)\le(1+C_2)\omega(f,t_2),$$

while

$$\omega(f,t_2) \le \omega(f,\frac{1}{C_1}t_1) \le (1+\frac{1}{C_1})\omega(f,t_1),$$

which shows

$$\omega(f,t_1) \ge \frac{C_1}{C_1+1}\omega(f,t_2),$$

therefore, $\omega(f,t_1) \sim \omega(f,t_2)$. □

Lemma 2.3. *Let x_k be the zeros of $S_n(x)$. Denote by $x_j = \cos\theta_j$ the node that brings θ_j closest to $\theta = \arccos x$, and write $i = |k-j|$. We have the following inequalities:*

$$|x-x_k| \sim \left(\frac{i}{2n+1}\sin\theta + \frac{i^2}{(2n+1)^2}\right),\quad k \ne j, \tag{4}$$

$$\frac{\omega(f,|x-x_k|)}{|x-x_k|^\lambda} \sim \frac{\omega\left(f,\dfrac{i}{2n+1}\sin\theta + \dfrac{i^2}{(2n+1)^2}\right)}{\left|\dfrac{i}{2n+1}\sin\theta + \dfrac{i^2}{(2n+1)^2}\right|^\lambda}. \tag{5}$$

Proof. By the definition of x_j, we have $|\theta-\theta_j| \le \dfrac{\pi}{2n+1}$. Consequently,

$$\frac{i\pi}{2n+1} = \frac{1}{2}|\theta_j-\theta_k| \le |\theta-\theta_k| \le 2|\theta_j-\theta_k| = \frac{4i\pi}{2n+1}. \tag{6}$$

A simple calculation combined with (6) yields:

$$\begin{aligned} |x-x_k| &= |2\sin^2\frac{1}{2}(\theta-\theta_k)\cos\theta - \sin(\theta-\theta_k)\sin\theta| \\ &\le C(\sin\theta|\theta-\theta_k| + |\theta-\theta_k|^2) \\ &\le C(\sin\theta\frac{i}{2n+1} + \frac{i^2}{(2n+1)^2}). \end{aligned} \tag{7}$$

On the other side, noticing that $\sin\theta \sim \sin\frac{1}{2}(\theta+\theta_k)$, and by (6) we have:

$$\begin{aligned} |x-x_k| &= \left|2\sin\frac{1}{2}(\theta+\theta_k)\sin\frac{1}{2}(\theta-\theta_k)\right| \\ &\ge C\sin\theta\left|\sin\frac{1}{2}(\theta-\theta_k)\right| \\ &\ge C\sin\theta\frac{i}{2n+1}. \end{aligned} \tag{8}$$

Then by (6) and the following inequality

$$\left|\sin\frac{1}{2}(\theta-\theta_k)\right| \le \sin\frac{1}{2}(\theta+\theta_k),$$

we have

$$\begin{aligned} |x-x_k| &= \left|2\sin\frac{1}{2}(\theta+\theta_k)\sin\frac{1}{2}(\theta-\theta_k)\right| \\ &\ge 2\sin^2\frac{1}{2}(\theta-\theta_k) \\ &\ge C\frac{i^2}{(2n+1)^2}. \end{aligned} \tag{9}$$

By (8) and (9):

$$|x - x_k| \geq C\left(\sin\theta\frac{i}{2n+1} + \frac{i^2}{(2n+1)^2}\right). \tag{10}$$

Combining (9) and (10), we obtain (4). Then (5) holds obviously by (4) and Proposition 2.2(III). □

Lemma 2.4. *Let $\lambda > 1$, then for every $\zeta \in (0, \pi)$, we have*

$$\sum_{k=1}^{n} \frac{1}{|\cos\zeta - \cos\theta_k|^\lambda} \sim \frac{1}{|\cos\zeta - \cos\theta_j|^\lambda}. \tag{11}$$

Proof. Refer to [14]. □

Lemma 2.5. *Let $-1 < \alpha < 0$, $n \in \mathbb{Z}^+$, we have the following asymptotic equation:*

$$\sum_{k=1}^{n} k^\alpha \sim \frac{n^{\alpha+1}}{\alpha+1}.$$

Proof. We first prove a general situation. Assume $f(x)$ is positive, bounded, and decreasing for $x \geq 1$. Write

$$F(x) = \int_1^x f(t)\,dt, \quad F_n = f(1) + f(2) + \cdots + f(n),$$

we can assert that $\lim\limits_{n\to+\infty}(F(n) - F_n)$ exists.

In fact, by the monotonicity of $f(x)$ and the definition of $F(x)$, we have

$$f(k+1) \leq F(k+1) - F(k) \leq f(k),$$

or

$$0 \leq F(k+1) - F(k) - f(k+1) \leq f(k) - f(k+1), \quad (k = 1, 2, \ldots).$$

Now adding the above inequality for k from 1 to $n-1$,

$$0 \leq \sum_{k=1}^{n-1}(F(k+1) - F(k) - f(k+1)) \leq \sum_{k=1}^{n-1}(f(k) - f(k+1)),$$

the convergence of the series $\sum\limits_{k=1}^{\infty}(F(k+1) - F(k) - f(k+1))$ can be deduced by $\sum\limits_{k=1}^{\infty}(f(k) - f(k+1))$ since f is decreasing and bounded on $[1, \infty)$.

Now the partial sum can be written by

$$\begin{aligned}
&\sum_{k=1}^{n-1}(F(k+1) - F(k) - f(k+1)) \\
&= F(2) - F(1) - f(2) + F(3) - F(2) - f(3) + \cdots + F(n) - F(n-1) - f(n) \\
&= F(n) - F(1) - f(2) - f(3) - \cdots - f(n) \\
&= F(n) - F_n + f(1),
\end{aligned}$$

which shows $(F(n) - F_n)$ tends to a finite limit.

Let $f(x) = x^\alpha, -1 < \alpha < 0$, then $f(x)$ satisfies all the assumptions as in the general case. Therefore,

$$\sum_{k=1}^{n} k^\alpha \sim \frac{n^{\alpha+1}}{\alpha+1}.$$

□

3. Main results

In this section, we give the main results of this paper and their proofs. First, the following theorem is established.

Theorem 3.1. *Assume $f(x) \in C[-1,1]$, $R_n^\lambda(f,x)$ is defined by* (2), *where $1 < \lambda < 2$. Then we have the following estimate:*

$$R_n^\lambda(f,x) - f(x) = O\left(\omega\left(f, \frac{(1-x)(1+x)^{\frac{1}{2}}|S_n(x)|}{(2n+1)^{\lambda-1}}\right)\right), \quad 1 < \lambda < 2. \tag{12}$$

Proof. Let $x_k = \cos\theta_k = \cos\dfrac{2k\pi}{2n+1}(k = 1, \ldots, n)$ be the zeros of $S_n(x)$, $l_k(\cos\theta)$ is defined by (3), $x = \cos\theta$. Then substitute (3) into (2) and simplify $R_n^\lambda(f,x)$, we obtain:

$$R_n^\lambda(f,\cos\theta) = \frac{\sum\limits_{k=1}^{n} f(\cos\theta_k)\,|l_k(\cos\theta)|^\lambda}{\sum\limits_{k=1}^{n} |l_k(\cos\theta)|^\lambda} = \frac{\sum\limits_{k=1}^{n} f(\cos\theta_k)\left|\dfrac{\sin\frac{\theta_k}{2}\sin\theta_k}{\cos\theta - \cos\theta_k}\right|^\lambda}{\sum\limits_{k=1}^{n}\left|\dfrac{\sin\frac{\theta_k}{2}\sin\theta_k}{\cos\theta - \cos\theta_k}\right|^\lambda}.$$

Or

$$R_n^\lambda(f,x) = \frac{\sum\limits_{k=1}^{n} f(x_k)\,|x - x_k|^{-\lambda}}{\sum\limits_{k=1}^{n} |x - x_k|^{-\lambda}}. \tag{13}$$

Then by Lemma 2.4,

$$\sum_{k=1}^{n} |x - x_k|^{-\lambda} \sim |x - x_j|^{-\lambda}. \tag{14}$$

Without loss of generality, we assume $0 \le x \le 1$, then $0 \le \theta \le \dfrac{\pi}{2}$, therefore $0 \le \theta_j \le \dfrac{\pi}{2}$, so we have

$$|x - x_j| = |\cos\theta - \cos\theta_j| \sim |\theta - \theta_j|(\theta + \theta_j) \sim |\theta - \theta_j|\theta \sim |\theta - \theta_j|\sin\theta. \tag{15}$$

By (13) – (15) and Lemma 2.3,

$$\begin{aligned}
|R_n^\lambda(f,x) - f(x)| &\le \frac{\sum\limits_{k=1}^{n} |f(x) - f(x_k)||x - x_k|^{-\lambda}}{\sum\limits_{k=1}^{n} |x - x_k|^{-\lambda}} \\
&\le C(\lambda)\omega(f, |x - x_j|) + C(\lambda)\frac{\sum\limits_{k=1, k\ne j}^{n} \omega(f, |x - x_k|)|x - x_k|^{-\lambda}}{|x - x_j|^{-\lambda}} \\
&\le C(\lambda)\omega(f, |\theta - \theta_j|\sin\theta) + J,
\end{aligned} \tag{16}$$

where

$$J := C(\lambda)\frac{\sum\limits_{k=1}^{n} \omega\left(f, \dfrac{k}{2n+1}\sin\theta + \dfrac{k^2}{(2n+1)^2}\right)\left(\dfrac{k}{2n+1}\sin\theta + \dfrac{k^2}{(2n+1)^2}\right)^{-\lambda}}{|\theta - \theta_j|^{-\lambda}\sin^{-\lambda}\theta}.$$

Next, we estimate J by dividing into two cases:
Case 1. $\sin\theta < \dfrac{1}{n}$.

By Proposition 2.2(II), direct calculations yield that

$$\begin{aligned}
J &= C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta \sum_{k=1}^{n} \omega\left(f, \frac{k}{2n+1}\sin\theta + \frac{k^2}{(2n+1)^2}\right)\left(\frac{k}{2n+1}\sin\theta + \frac{k^2}{(2n+1)^2}\right)^{-\lambda} \\
&\le C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta \sum_{k=1}^{n} \omega\left(f, \frac{k^2}{(2n+1)^2}\right)\frac{(2n+1)^{2\lambda}}{k^{2\lambda}} \\
&= C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta \sum_{k=1}^{n} \omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\frac{(2n+1)^{\lambda-2}k^2}{|\theta-\theta_j|\sin\theta(2n+1)^2}\right)\frac{(2n+1)^{2\lambda}}{k^{2\lambda}} \\
&\le C(\lambda)\omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right)\sum_{k=1}^{n}\left(1+\frac{(2n+1)^{\lambda-2}k^2}{|\theta-\theta_j|\sin\theta(2n+1)^2}\right)|\theta-\theta_j|^\lambda \sin^\lambda\theta\frac{(2n+1)^{2\lambda}}{k^{2\lambda}} \\
&\le C(\lambda)\omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right)\sum_{k=1}^{n}\left(\frac{1}{k^{2\lambda}}+\frac{(2n+1)^{\lambda-2}}{k^{2\lambda-2}}\right) \\
&\le C(\lambda)\omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right).
\end{aligned}$$

Case 2. $\dfrac{k_0}{n} \le \sin\theta < \dfrac{k_0+1}{n}$, where k_0 is some positive integer. In this case,

$$\begin{aligned}
J &\le C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta \left[\sum_{k=1}^{k_0} \omega\left(f, \frac{k}{2n+1}\sin\theta\right)\left(\frac{k}{2n+1}\sin\theta\right)^{-\lambda}\right. \\
&\quad \left. + \sum_{k=k_0+1}^{n} \omega\left(f, \frac{k^2}{(2n+1)^2}\right)\left(\frac{k^2}{(2n+1)^2}\right)^{-\lambda}\right] \\
&= C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta \left[\sum_{k=1}^{k_0} \omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\frac{k(2n+1)^{\lambda-2}}{(2n+1)|\theta-\theta_j|}\right)\frac{(2n+1)^\lambda}{k^\lambda\sin^\lambda\theta}\right. \\
&\quad \left. + \sum_{k=k_0+1}^{n} \omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\frac{(2n+1)^{\lambda-2}}{|\theta-\theta_j|\sin\theta}\frac{k^2}{(2n+1)^2}\right)\frac{(2n+1)^{2\lambda}}{k^{2\lambda}}\right] \\
&\le C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta\ \ \omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right)\left[\sum_{k=1}^{k_0}\left(1+\frac{k(2n+1)^{\lambda-3}}{|\theta-\theta_j|}\right)\frac{(2n+1)^\lambda}{k^\lambda\sin^\lambda\theta}\right. \\
&\quad \left. + \sum_{k=k_0+1}^{n}\left(1+\frac{(2n+1)^{\lambda-4}k^2}{|\theta-\theta_j|\sin\theta}\right)\frac{(2n+1)^{2\lambda}}{k^{2\lambda}}\right] \\
&\le C(\lambda)\omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right)\left[\sum_{k=1}^{k_0}\left(\frac{1}{k^\lambda}+\frac{n^{\lambda-2}}{k^{\lambda-1}}\right)+\sum_{k=k_0+1}^{n}\left(\frac{1}{k^\lambda}+\frac{n^{\lambda-2}}{k^{\lambda-1}}\right)\right],
\end{aligned}$$

then by Lemma 2.5,

$$J \le C(\lambda)\omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right).$$

Considering the above two cases together we get:

$$J \leq C(\lambda)\omega\left(f, \frac{|\theta-\theta_j|\sin\theta}{(2n+1)^{\lambda-2}}\right), \quad 1<\lambda<2. \tag{17}$$

Furthermore,

$$|\theta-\theta_j| = \frac{\left(n+\frac{1}{2}\right)\left|\theta-\theta_j\right|}{n+\frac{1}{2}} \sim \frac{\left|\sin(n+\frac{1}{2})(\theta-\theta_j)\right|}{n+\frac{1}{2}},$$

while

$$\begin{aligned}\left|\sin\left(n+\frac{1}{2}\right)(\theta-\theta_j)\right| &= \left|\sin\left((n+\frac{1}{2})\theta\right)\cos\left(\left(n+\frac{1}{2}\right)\theta_j\right) - \cos\left(\left(n+\frac{1}{2}\right)\theta\right)\sin\left(\left(n+\frac{1}{2}\right)\theta_j\right)\right| \\ &= \left|\sin\left(n+\frac{1}{2}\right)\theta\right|.\end{aligned}$$

Therefore:

$$|\theta-\theta_j|\sin\theta \sim \frac{\sin\theta\left|\sin(n+\frac{1}{2})\theta\right|}{n+\frac{1}{2}} = \frac{\sin\theta\sin\frac{\theta}{2}\,|S_n(x)|}{n+\frac{1}{2}} = \frac{(1-x)(1+x)^{\frac{1}{2}}\,|S_n(x)|}{n+\frac{1}{2}}, \tag{18}$$

then by (16), (17), and (18),

$$\left|R_n^\lambda(f,x)-f(x)\right| = O\left(\omega\left(f, \frac{(1-x)(1+x)^{\frac{1}{2}}|S_n(x)|}{(2n+1)^{\lambda-1}}\right)\right), \quad 1<\lambda<2.$$

Theorem 3.1 is proved. □

Theorem 3.2. *Assume $f(x)\in C[-1,1]$, $R_n^\lambda(f,x)$ is defined by* (2)*, then for $\lambda>2$, we have the following estimate:*

$$R_n^\lambda(f,x)-f(x) = O\left(\omega\left(f, \frac{(1-x)(1+x)^{\frac{1}{2}}|S_n(x)|}{2n+1}\right)\right), \quad \lambda>2. \tag{19}$$

Proof. Let $x_k=\cos\theta_k=\cos\dfrac{2k\pi}{2n+1}(k=1,\ldots,n)$ be the zeros of $S_n(x)$, $l_k(\cos\theta)$ is defined by (3). For $\lambda>2$, similar to the proof of Theorem 3.1 we still have the inequality (16), then we divide the estimation of J into two cases with different calculation details from Theorem 3.1.

Case 1. $\sin\theta<\dfrac{1}{n}$.

By Proposition 2.2(II) and $|\theta-\theta_j|<\dfrac{\pi}{2n+1}$, $\sin\theta<\dfrac{1}{n}$, we have:

$$\begin{aligned}J &\leq C(\lambda)|\theta-\theta_j|^\lambda\sin^\lambda\theta\sum_{k=1}^n\omega\left(f,\frac{k^2}{(2n+1)^2}\right)\left(\frac{k^2}{(2n+1)^2}\right)^{-\lambda} \\ &\leq C(\lambda)|\theta-\theta_j|^\lambda\sin^\lambda\theta\,\omega\left(f,|\theta-\theta_j|\sin\theta\right)\sum_{k=1}^n\left(1+\frac{k^2}{(2n+1)^2|\theta-\theta_j|\sin\theta}\right)\frac{(2n+1)^{2\lambda}}{k^{2\lambda}} \\ &\leq C(\lambda)\omega\left(f,|\theta-\theta_j|\sin\theta\right)\sum_{k=1}^n\left(\frac{1}{k^{2\lambda}}+\frac{1}{k^{2\lambda-2}}\right) \\ &\leq C(\lambda)\omega\left(f,|\theta-\theta_j|\sin\theta\right).\end{aligned}$$

Case 2. $\frac{k_0}{n} \le \sin\theta < \frac{k_0+1}{n}$, with k_0 a positive integer, then:

$$J \le C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta \left[\sum_{k=1}^{k_0} \omega\left(f, \frac{k}{2n+1}\sin\theta\right)\left(\frac{k}{2n+1}\sin\theta\right)^{-\lambda} + \sum_{k=k_0+1}^{n} \omega\left(f, \frac{k^2}{(2n+1)^2}\right)\left(\frac{k^2}{(2n+1)^2}\right)^{-\lambda}\right]$$

$$\le C(\lambda)|\theta-\theta_j|^\lambda \sin^\lambda\theta\,\omega\left(f, |\theta-\theta_j|\sin\theta\right)\left[\sum_{k=1}^{k_0}\left(1+\frac{k}{(2n+1)|\theta-\theta_j|}\right)\frac{(2n+1)^\lambda}{k^\lambda\sin^\lambda\theta} + \sum_{k=k_0+1}^{n}\left(1+\frac{k^2}{(2n+1)^2|\theta-\theta_j|\sin\theta}\right)\frac{(2n+1)^{2\lambda}}{k^{2\lambda}}\right]$$

$$\le C(\lambda)\omega\left(f, |\theta-\theta_j|\sin\theta\right)\left[\sum_{k=1}^{k_0}\left(\frac{1}{k^\lambda}+\frac{1}{k^{\lambda-1}}\right) + \sum_{k=k_0+1}^{n}\left(\frac{1}{k^\lambda}+\frac{1}{k^{\lambda-1}}\right)\right].$$

Considering the above two cases together with (16), we obtain:

$$\left|R_n^\lambda(f,x) - f(x)\right| \le C(\lambda)\omega\left(f, |\theta-\theta_j|\sin\theta\right).$$

A similar derivation to Theorem 3.1 yields the result of Theorem 3.2. □

4. Graphics and data simulation

In this section, we give some numerical experiments to demonstrate the validity of the obtained results. All figures and computations were done in Matlab2016a, with 15-16 decimal digits of precision.

4.1. Figures of Chebyshev Polynomials and Their Zeros

The images of the fourth kind of Chebyshev polynomials and their zeros are shown in Figure 1. The horizontal axis represents the x coordinates and the vertical axis represents the y coordinates.

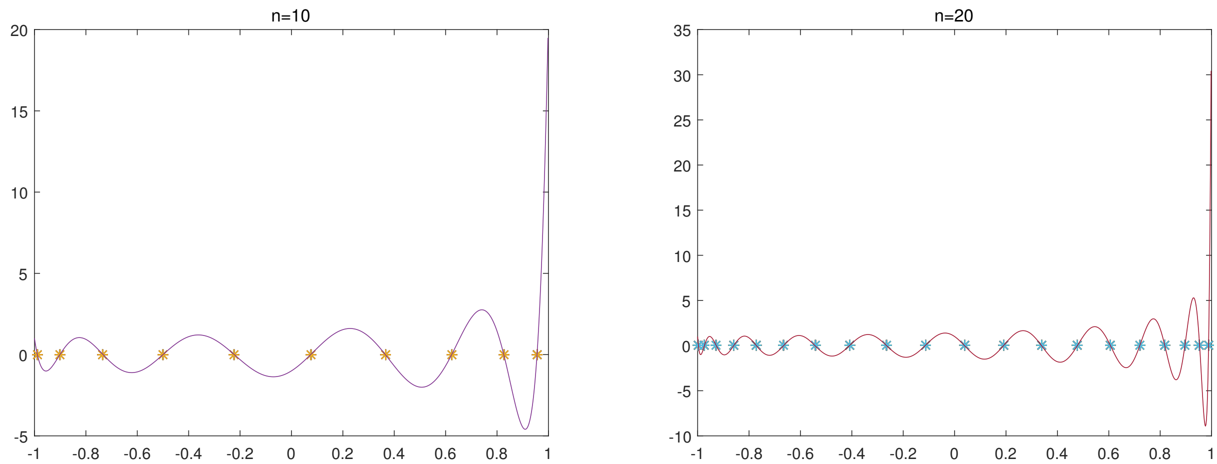

Fig. 1 The image of the fourth kind of Chebyshev polynomial when $n = 10$ and $n = 20$, with zeros marked in the image.

4.2. Approximation of Rational Interpolation Operator to Sampling Functions

In this subsection, two types of functions are selected and the approximation of each function by the RIO $R_n^\lambda(f,x)$ based on Chebyshev nodes of the fourth kind is visualized.

The approximation to the first selected function $f(x) = |x|$ is displayed in the first row, while approximation to $f(x) = x^2 \cdot \sin x$ in the second row, where the horizontal axis represents the x coordinates and the vertical axis represents the y coordinates, as shown in Figure 2.

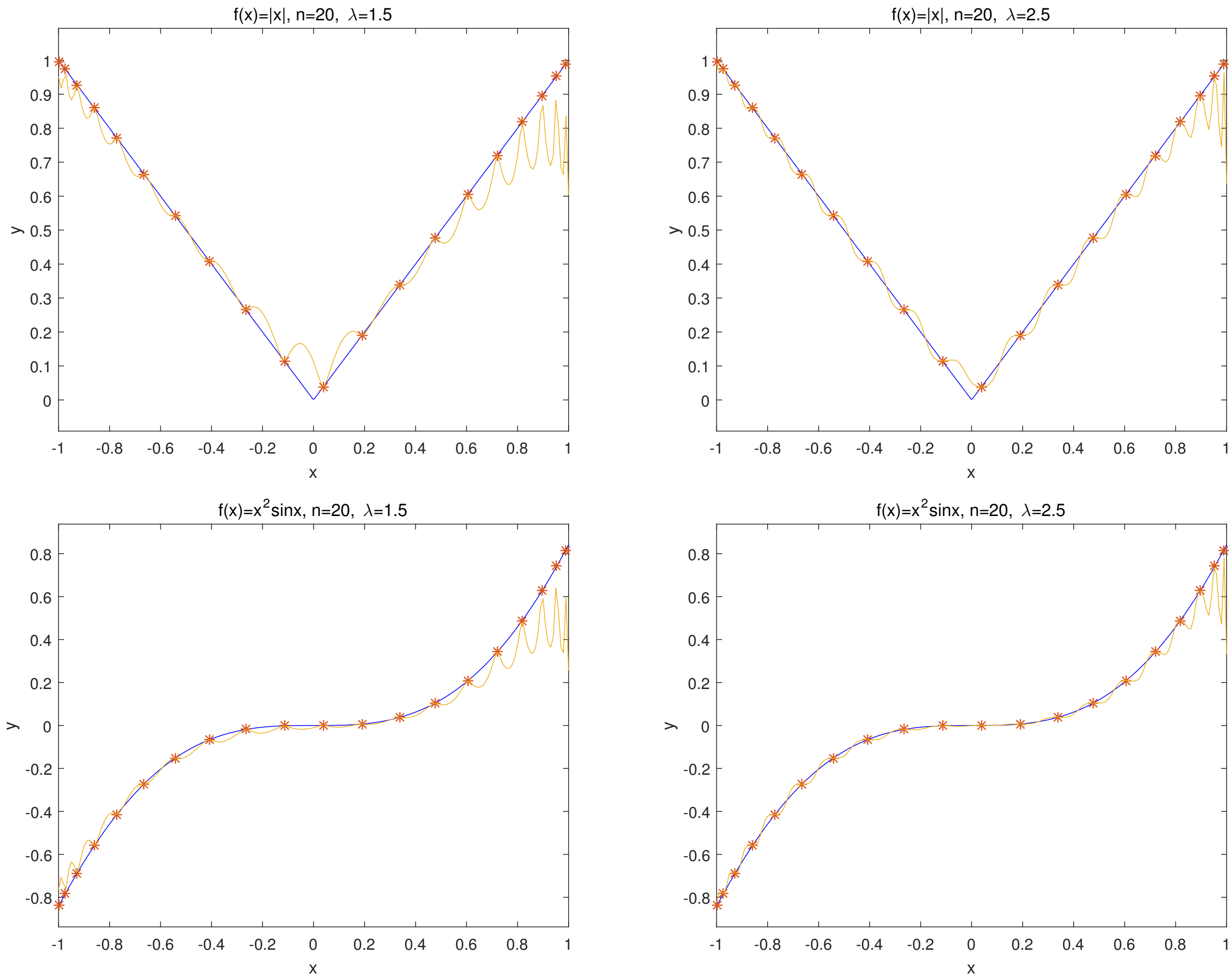

Fig. 2 Approximation of the rational interpolatory operators to two different types of functions.

Theoretically, by comparing the righthand side of (12) in Theorem 3.1 with (19) in Theorem 3.2, we can see that $\lambda = 2$ is actually the cut-off value for the discussion, and the approximation performance of $R_n^\lambda(f,x)$ to f is better in the case $\lambda > 2$ than $1 < \lambda < 2$. These results can be verified by the four images in Figure 2 above. From the comparison of the above four images, it is clear that on the one hand, the smoothness of the function affects the approximation of the function by the rational interpolation operator, and the better the smoothness, the better the approximation (in the vertical direction). On the other hand, as the value of λ increases, the approximation effect becomes better (horizontal). These observations indicate that in the process of approximating the function, the approximation performance of RIO $R_n^\lambda(f,x)$ is closely related to the value of the parameter λ and the smoothness of the function.

4.3. Analysis of Interpolation Error

Let $R_n^\lambda(f,x)$ be the RIO based on Chebyshev nodes of the fourth kind, which is used to approximate the selected functions, $f(x) = |x|$, $f(x) = x^2 \cdot \sin x$, respectively. The following tables analyze the error $R_n^\lambda(f,x) - f(x)$, when x is selected between nodes (i.e., different from Chebyshev nodes of the fourth kind), so that the approximation effect can be more clearly demonstrated by the data, as shown in Tables 1 - 4.

Tab. 1 The error between $f(x) = |x|$ and $R_n^\lambda(f,x)$ ($\lambda = 2.5$).

λ	n	x	$f(x)$	$R_n^\lambda(f,x)$	$R_n^\lambda(f,x) - f(x)$
2.5	10	$\cos\frac{5\pi}{21}$	0.733051871829826	0.600365496238415	-0.132686375591411
		$\cos\frac{20\pi}{63}$	0.542546263865759	0.521778837094279	-0.020767426771480
		$\cos\frac{27\pi}{63}$	0.222520933956314	0.194732746837638	-0.027788187118676
		$\cos\frac{33\pi}{63}$	0.074730093586424	0.182582947939112	0.107852854352688
		$\cos\frac{45\pi}{63}$	0.623489801858733	0.600448232503275	-0.023041569355459

Tab. 2 The error between $f(x) = |x|$ and $R_n^\lambda(f,x)$ ($\lambda = 1.5$).

λ	n	x	$f(x)$	$R_n^\lambda(f,x)$	$R_n^\lambda(f,x) - f(x)$
1.5	10	$\cos\frac{5\pi}{21}$	0.733051871829826	0.560882321617258	-0.172169550212569
		$\cos\frac{20\pi}{63}$	0.542546263865759	0.482118411315235	-0.060427852550525
		$\cos\frac{27\pi}{63}$	0.222520933956314	0.255375943873174	0.032855009916859
		$\cos\frac{33\pi}{63}$	0.074730093586424	0.236581535058868	0.161851441472444
		$\cos\frac{45\pi}{63}$	0.623489801858733	0.584350418483810	-0.039139383374924

Tab. 3 The error between $f(x) = x^2 \cdot \sin(x)$ and $R_n^\lambda(f,x)$ ($\lambda = 2.5$).

λ	n	x	$f(x)$	$R_n^\lambda(f,x)$	$R_n^\lambda(f,x) - f(x)$
2.5	10	$\cos\frac{5\pi}{21}$	0.359572824994475	0.233066535345208	-0.126506289649267
		$\cos\frac{20\pi}{63}$	0.151981613757690	0.159672524768084	0.007690911010394
		$\cos\frac{27\pi}{63}$	0.010927545710118	0.013745046625623	0.002817500915505
		$\cos\frac{33\pi}{63}$	-4.169483670118575e-04	-0.013234270677392	-0.012817322310380
		$\cos\frac{45\pi}{63}$	-0.226974086327801	-0.231300011164242	-0.004325924836441

Tab. 4 The error between $f(x) = x^2 \cdot \sin(x)$ and $R_n^\lambda(f,x)$ ($\lambda = 1.5$).

λ	n	x	$f(x)$	$R_n^\lambda(f,x)$	$R_n^\lambda(f,x) - f(x)$
1.5	10	$\cos\frac{5\pi}{21}$	0.359572824994475	0.187658520578135	-0.171914304416340
		$\cos\frac{20\pi}{63}$	0.151981613757690	0.116735092392259	-0.035246521365430
		$\cos\frac{27\pi}{63}$	0.010927545710118	0.003296608632714	-0.007630937077404
		$\cos\frac{33\pi}{63}$	-4.169483670118575e-04	-0.031512348375390	-0.031095400008378
		$\cos\frac{45\pi}{63}$	-0.226974086327801	-0.231958601485729	-0.004984515157929

By comparing the data in Table 1 with Table 2, Table 3 with Table 4, the interpolation error between $R_n^\lambda(f,x)$ and $f(x)$ is smaller when $\lambda = 2.5$; and when comparing the data in Table 1 with Table 3, Table 2 with Table 4, the error between $R_n^\lambda(f,x)$ and $f(x)$ is smaller for the smoother function. i.e., the better the approximation effect.

5. Conclusions

In this paper, we construct the parameterized RIO $R_n^\lambda(f,x)$ based on Chebyshev nodes of the fourth kind. We investigate the approximation order of $R_n^\lambda(f,x)$ to continuous function $f(x)$, with different ranges of parameters. The simulation part, including images and error data, allow us to further discuss the interpolation error of $R_n^\lambda(f,x)$ to different types of functions, also verify the theoretical results of this paper, which is, larger parameters and smoother functions provide better approximation performance.

Conflict of interest

The authors declare that they have no conflicts of interest.

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