Filomat 38:11 (2024), 3957–3970
https://doi.org/10.2298/FIL2411957K

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Almost convergent double sequences of bi-complex numbers

Sujeet Kumar[a,*], **Binod Chandra Tripathy**[a]

[a] *Department of Mathematics, Tripura University, Agartala 799022, India*

Abstract. In this article we have introduced the notion of almost convergence, strongly almost convergence, almost null and strongly almost null of double sequences in Pringsheim sense of bi-complex numbers. We have proved that these are linear spaces and with the help of the Euclidean norm defined on bi-complex numbers, we have proved their different algebraic and topological properties. Suitable examples have been discussed.

1. Introduction

In this section we procure the detailed preliminaries of bi-complex numbers.
Throughout $\mathbb{C}_0, \mathbb{C}_1$ and $\mathbb{C}_2$ denote the set of real, complex and bi-complex numbers.
Segre [18] defined the bi-complex numbers as follows:

$$\begin{aligned}\gamma &= u_1 + i_2 u_2 \\ &= w + i_1 x + i_2 y + i_1 i_2 z,\end{aligned}$$

where $u_1, u_2 \in \mathbb{C}_1; w, x, y, z \in \mathbb{C}_0$, and i_1, i_2 are two distinct imaginary unit whose square is -1, and $i_1 i_2$ is hyperbolic unit whose square is 1.
The set of bi-complex numbers is denoted by $\mathbb{C}_2$ and defined by

$$\mathbb{C}_2 = \gamma = \{w + i_1 x + i_2 y + i_1 i_2 z : w, x, y, z \in \mathbb{C}_0\}.$$

There are three types of conjugations defined by Rochon and Shapiro [17] as follows:
(i) i_1- conjugation of bi-complex number γ is $\gamma^* = \overline{u_1} + i_2 \overline{u_2}$, for all $u_1, u_2 \in \mathbb{C}_1$ and $\overline{u_1}, \overline{u_2}$ are complex conjugates of u_1, u_2 respectively.
(ii) i_2- conjugation of bi-complex number γ is $\tilde{\gamma} = u_1 - i_2 u_2$, for all $u_1, u_2 \in \mathbb{C}_1$.
(iii) $i_1 i_2$- conjugation of bi-complex number γ is $\gamma^{'} = \overline{u_1} - i_2 \overline{u_2}$, for all $u_1, u_2 \in \mathbb{C}_1$ and $\overline{u_1}, \overline{u_2}$ are complex conjugates of u_1, u_2 respectively.
A bi-complex number $\gamma = u_1 + i_2 u_2$ is called hyperbolic if $\gamma' = \gamma$, where γ' is the $i_1 i_2$- conjugation of γ.
The set of all hyperbolic element is denoted by $\mathcal{H}$ and defined by

$$\mathcal{H} = \{w + i_1 i_2 z : w, z \in \mathbb{C}_0\},$$

2020 *Mathematics Subject Classification*. 46A45; 40C05; 46B45; 30G35; 40A05.
Keywords. double sequences, bi-complex numbers, almost convergence, strongly almost convergence.
Received: 18 August 2023; Accepted: 20 September 2023
Communicated by Miodrag Spalević
* Corresponding author: Sujeet Kumar
Email addresses: sksujeetjai@gmail.com (Sujeet Kumar), tripathybc@gmail.com, tripathybc@yahoo.com (Binod Chandra Tripathy)

A bi-complex number $\gamma = u_1 + i_1u_2$ is called singular if $|u_1^2 + u_2^2| = 0$ and otherwise it is called non-singular.
In $\mathbb{C}_2$, there are exactly two non-trivial idempotent elements e_1 and e_2, where

$$e_1 = \frac{\{1 + i_1i_2\}}{2} \text{ and } e_2 = \frac{\{1 - i_1i_2\}}{2}.$$

Obviously, $e_1 + e_2 = 1$ and $e_1 \cdot e_2 = e_2 \cdot e_1 = 0$.
Every bi-complex number $\gamma = u_1 + i_2u_2$ has a unique idempotent representation as
$\gamma = \mu_1e_1 + \mu_2e_2$, where $\mu_1 = u_1 - i_1u_2$ and $\mu_2 = u_1 + i_1u_2$ are called the idempotent components of γ.
Norm (Euclidean Norm) on $\mathbb{C}_2$ is defined by

$$\begin{aligned}||\gamma||_{\mathbb{C}_2} &= \sqrt{w^2 + x^2 + y^2 + z^2} \\ &= \sqrt{|u_1|^2 + |u_2|^2} \\ &= \sqrt{\frac{|\mu_1|^2 + |\mu_2|^2}{2}}.\end{aligned}$$

$\mathbb{C}_2$ becomes a modified Banach algebra with respect to this norm in the sense that

$$||\gamma \cdot s||_{\mathbb{C}_2} \leq \sqrt{2}||\gamma||_{\mathbb{C}_2} \cdot ||s||_{\mathbb{C}_2}.$$

"The set $\mathbb{C}_2$ is a Banach space w.r.t. the Euclidean norm."
A double sequence $x = (x_{lm})$ is said to be bounded if

$$||x|| = \sup_{l,m\in\mathbb{N}} |x_{lm}| < \infty.$$

The symbol ${}_2\ell_\infty$ stands for the set of all bounded double sequences.
A double sequence (x_{lm}) of real and complex terms is said to be covergent in Pringsheim's sense to K, if for each $\varepsilon > 0$ then $\exists$ a natural number N_0 such that $|x_{lm} - K| < \varepsilon$, for all $l, m \geq N_0$, written as

$$P - \lim_{\substack{l\to\infty \\ m\to\infty}} x_{lm} = K.$$

where $l, m \to \infty$ independent to one another.
A double sequence $x = (x_{lm})$ is said to converge regularly (introduced by Hardy [5]) if it converges in the Pringsheim's sense and the following limits exist,

$$\lim_{l\to\infty} x_{lm} = P_m, \text{ exists for each } m \in \mathbb{N},$$

and

$$\lim_{m\to\infty} x_{lm} = Q_l, \text{ exists for each } l \in \mathbb{N}.$$

Throughout ${}_2\omega$, ${}_2\ell_\infty$, ${}_2c$, ${}_2c_0$, ${}_2c^R$, ${}_2c_0^R$, ${}_2c^B$ and ${}_2c_0^B$ denote the classes of all, double bounded, double convergent, double null, double regular, double regular null, double bounded convergent and double bounded null sequence of real or complex terms.
In [4], Čunjalo established the concept of double almost Cauchy sequences.
A double sequence $x = (x_{lm})$ is said to be almost Cauchy if for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\left|\frac{1}{p_1q_1}\sum_{l=0}^{p_1-1}\sum_{m=0}^{q_1-1} x_{l+n_1,m+k_1} - \frac{1}{p_2q_2}\sum_{l=0}^{p_2-1}\sum_{m=0}^{q_2-1} x_{l+n_2,m+k_2}\right| < \varepsilon,$$

for all $p_1, p_2, q_1, q_2 > N_0$ and $(n_1, k_1), (n_2, k_2) \in \mathbb{N} \times \mathbb{N}$.

1.1. Almost Convergence of Double Sequences

In this subsection we procure the basics of almost convergence and strongly almost convergence of double sequences. For the following, one may refer to Mursaleen and Mohiuddine [10].

Definition 1.1. *A linear functional L on ${}_2\ell_\infty$ is said to be a Banach limit if it has the following properties:*
(i) $L(x) \geq 0$ if $x \geq 0$ (i.e., $x_{lm} \geq 0$, for all l, m),
(ii) $L(E) = 1$, where $E = (a_{lm})$ with $a_{lm} = 1$, for all j, k, and
(iii) $L(D_{11}x) = L(x) = L(D_{10}x) = L(D_{01}x)$, where the shift operators D_{01}, D_{10} and D_{11} are defined by

$$D_{01}x = (x_{l,m+1}), D_{10}x = (x_{l+1,m}), D_{11}x = (x_{l+1,m+1}), \text{ for all } l, m \in \mathbb{N}.$$

Lorentz [6] established the idea of almost convergence for a single sequence. Later, Móricz and Rhoades [9] introduced the notion of almost convergence for double sequences.
Consider, B_2 is the collection of all Banach limits on ${}_2\ell_\infty$. A double sequence $x = (x_{lm})$ is said to be almost convergent to a number K if $L(x) = K$, for all $L \in B_2$.

Definition 1.2. *The sequence space f_2 of almost convergent double sequences was defined in [9] as*

$$f_2 = \left\{ x = (x_{lm}), \lim_{p,q\to\infty} |\tau_{pqij}(x) - K| = 0, \text{ uniformly in } i, j \in \mathbb{N} \right\},$$

where

$$\tau_{pqij} = \frac{1}{(p+1)(q+1)} \sum_{l=0}^{p} \sum_{m=0}^{q} x_{l+i,m+j}, \text{ for all } p, q, i, j \in \mathbb{N}.$$

It may be noted that a convergent double sequence has some unbounded sequences, therefore it need not be almost convergent. However, the inclusion ${}_2c^B \subset f_2 \subset {}_2\ell_\infty$ are strict.
Maddox [7] introduced the notion of strong almost convergence for single sequences. Later on, Başarir [3] introduced the concept of strong almost double sequences.

Definition 1.3. *A double sequence (x_{lm}) is said to be strongly almost convergent to a number K if*

$$P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} |x_{lm} - K| = 0, \text{ uniformly in } l, m \in \mathbb{N}.$$

1.2. Almost Convergent Double Sequence of Bi-complex Numbers

In this subsection we introduce the notion of almost convergent and strongly almost convergent double sequences of bi-complex numbers.

Definition 1.4. *A double sequence (γ_{lm}) of bi-complex numbers is called bounded, if there exists a real number $M > 0$ such that*

$$\|\gamma_{lm}\|_{\mathbb{C}_2} \leq M, \text{ for all } l, m \in \mathbb{N}.$$

Let ${}_2\ell_\infty(\mathbb{C}_2)$ be the set of all bounded double sequences (γ_{lm}) of bi-complex numbers with the norm

$$\|(\gamma_{lm})\|_\infty = \sup_{l,m\in\mathbb{N}} \|\gamma_{lm}\|_{\mathbb{C}_2} < \infty.$$

Definition 1.5. *A double sequence (γ_{lm}) of bi-complex numbers is said to be convergent to $\gamma \in \mathbb{C}_2$ if for each $\varepsilon > 0$ there corresponds an $N_0(\varepsilon) \in \mathbb{N}$ such that*

$$\|\gamma_{lm} - \gamma\|_{\mathbb{C}_2} < \varepsilon, \text{ for all } l, m \geq N_0(\varepsilon).$$

It is written as

$$\lim_{\substack{l\to\infty \\ m\to\infty}} \gamma_{lm} = \gamma.$$

where l and m tend to ∞ independent of each others.

Definition 1.6. *A double sequence $\gamma = (\gamma_{lm})$ of bi-complex numbers is said to converge regularly if it converges in the Pringsheim's sense and the following limits exist,*

$$\lim_{l\to\infty}\gamma_{lm} = P_m, \text{ exists for each } m \in \mathbb{N},$$

and

$$\lim_{m\to\infty}\gamma_{lm} = Q_l, \text{ exists for each } l \in \mathbb{N}.$$

Definition 1.7. *A linear functional L on ${}_2\ell_\infty(\mathbb{C}_2)$ is said to be Banach limit if it has the following properties:*
(i) $L(\gamma_{lm}) \geq 0$, if $\|\gamma_{lm}\|_{\mathbb{C}_2} \geq 0$, for all $l, m \in \mathbb{N}$,
(ii) $L(E) = 1$, where $E = (a_{lm})$ with $a_{lm} = e_1 + e_2$, for all $l, m \in \mathbb{N}$, and
(iii) $L(D_{11}\gamma_{lm}) = L(\gamma_{lm}) = L(D_{10}\gamma_{lm}) = L(D_{01}\gamma_{lm})$, where the shift operators D_{01}, D_{10} and D_{11} are defined by

$$D_{01}\gamma_{lm} = \gamma_{l,m+1}, D_{10}\gamma_{lm} = \gamma_{l+1,m}, D_{11}\gamma_{lm} = \gamma_{l+1,m+1}, \text{ for all } l, m \in \mathbb{N}.$$

Definition 1.8. *A double sequence (γ_{lm}) of bi-complex numbers is said to be almost convergent to a limit K if*

$$P - \lim_{p,q\to\infty}\left\|\frac{1}{(p+1)(q+1)}\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\gamma_{lm} - K\right\|_{\mathbb{C}_2} = 0, \text{ uniformly in } i, j \in \mathbb{N}.$$

In this case K is said to be the generalized limit of (γ_{lm}).

The space ${}_2c_f(\mathbb{C}_2)$ of almost convergent double sequences of bi-complex numbers is defined by,

$${}_2c_f(\mathbb{C}_2) = \left\{\gamma = (\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2) : P - \lim_{p,q\to\infty}\|\tau_{pqij}(\gamma_{lm}) - K\|_{\mathbb{C}_2} = 0, \text{ uniformly in } i, j \in \mathbb{N}\right\},$$

and for the set of almost bounded and almost null double sequence of bi-complex numbers are defined as follows:

$${}_2\ell_{\infty f}(\mathbb{C}_2) = \left\{\gamma = (\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2) : \sup_{l,m}\|\tau_{pqij}(\gamma_{lm})\|_{\mathbb{C}_2} < \infty, \text{ uniformly in } i, j \in \mathbb{N}\right\},$$

$${}_2c_{0f}(\mathbb{C}_2) = \left\{\gamma = (\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2) : P - \lim_{p,q\to\infty}\|\tau_{pqij}(\gamma_{lm})\|_{\mathbb{C}_2} = 0, \text{ uniformly in } i, j \in \mathbb{N}\right\},$$

where

$$\tau_{pqij} = \frac{1}{(p+1)(q+1)}\sum_{l=0}^{p}\sum_{m=0}^{q}\gamma_{l+i,m+j}, \text{ for all } p, q, i, j \in \mathbb{N}.$$

Definition 1.9. *A double sequence (γ_{lm}) of bi-complex numbers is said to be strongly almost convergent to a limit K if*

$$P - \lim_{p,q\to\infty}\frac{1}{(p+1)(q+1)}\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\|\gamma_{lm} - K\|_{\mathbb{C}_2} = 0, \text{ uniformly in } i, j \in \mathbb{N}.$$

We denote the space of strongly almost convergent double sequence of bi-complex numbers by $[{}_2c_f(\mathbb{C}_2)]$.

The space $[{}_2c_f(\mathbb{C}_2)]$ of *strongly almost convergent double sequences of bi-complex numbers* is defined by,

$$[{}_2c_f(\mathbb{C}_2)] = \left\{\gamma = (\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2) : P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=0}^{p}\sum_{m=0}^{q} \|\gamma_{l+i,m+j} - K\|_{\mathbb{C}_2} = 0, \text{ uniformly in } i,j \in \mathbb{N}\right\},$$

and for the set of strongly almost bounded and strongly almost null double sequence of bi-complex numbers are defined as follows:

$$[{}_2\ell_{\infty f}(\mathbb{C}_2)] = \left\{\gamma = (\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2) : \sup_{l,m} \frac{1}{(p+1)(q+1)} \sum_{l=0}^{p}\sum_{m=0}^{q} \|\gamma_{l+i,m+j}\|_{\mathbb{C}_2} < \infty, \text{ uniformly in } i,j \in \mathbb{N}\right\},$$

$$[{}_2c_{0f}(\mathbb{C}_2)] = \left\{\gamma = (\gamma_{lm}) \in {}_2\ell_\infty(\mathbb{C}_2) : P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=0}^{p}\sum_{m=0}^{q} \|\gamma_{l+i,m+j}\|_{\mathbb{C}_2} = 0, \text{ uniformly in } i,j \in \mathbb{N}\right\}.$$

Definition 1.10. *A double sequence $\gamma = (\gamma_{lm})$ of bi-complex numbers is called almost Cauchy if for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that*

$$\left\| \frac{1}{p_1q_1} \sum_{l=0}^{p_1-1}\sum_{m=0}^{q_1-1} \gamma_{l+n_1,m+k_1} - \frac{1}{p_2q_2} \sum_{l=0}^{p_2-1}\sum_{m=0}^{q_2-1} \gamma_{l+n_2,m+k_2} \right\|_{\mathbb{C}_2} < \varepsilon,$$

for all $p_1, p_2, q_1, q_2 > N_0$ and $(n_1,k_1),(n_2,k_2) \in \mathbb{N}\times\mathbb{N}$.

"Note that every almost convergent double sequence of bi-complex numbers is bounded."

Definition 1.11. *A double sequence space E of bi-complex number is said to be solid if $(\alpha_{lm}\gamma_{lm}) \in E$, whenever $(\gamma_{lm}) \in E$, for all double sequences (α_{lm}) of scalars with $|\alpha_{lm}| \leq 1$, for all $l, m \in \mathbb{N}$.*

Definition 1.12. *A double sequence space E of bi-complex number is said to be symmetric if $(\gamma_{lm}) \in E \implies (\gamma_{\pi(l,m)}) \in E$, where π is the permutation of $\mathbb{N}\times\mathbb{N}$.*

Definition 1.13. *Let $K = \{(l_i, m_j) : i,j \in \mathbb{N}; l_1 < l_2 < .... \text{ and } m_1 < m_2 < ....\} \subseteq \mathbb{N}\times\mathbb{N}$ and E be a double sequence space. A K-step space of E is a sequence space*

$$\lambda_K^E = \{(\gamma_{l_i m_j}) \in {}_2\omega(\mathbb{C}_2) : (\gamma_{lm}) \in E\}.$$

A canonical pre-image of a sequence $(\gamma_{lm}) \in E$ is defined as follows:

$$t_{lm} = \begin{cases} \gamma_{lm}, & \text{if } (l,m) \in K; \\ 0, & \text{otherwise.} \end{cases}$$

A canonical pre-image of a step space λ_K^E is a set of canonical pre-images of all elements in λ_K^E.

Definition 1.14. *A double sequence space E is said to be monotone if it contains the canonical pre-images of all its step spaces.*

Remark 1.15. *It follows that, "If a sequence space E is solid, then sequence space E is monotone."*

Definition 1.16. *A double sequence space E is said to be convergence free if $(t_{lm}) \in E$ whenever $(\gamma_{lm}) \in E$ and $t_{lm} = \theta$, whenever $\gamma_{lm} = \theta$, where θ is the zero element of $\mathbb{C}_2$.*

Definition 1.17. *A double sequence spaces E is said to be sequence algebra if $(\gamma_{lm}) \star (t_{lm}) = (\gamma_{lm}t_{lm}) \in E$, whenever $(\gamma_{lm}), (t_{lm}) \in E$.*

In this article we consider the termwise product of the sequences.

Definition 1.18. *Let E be a subset of a linear spaces X. Then E is said to be convex if $(1-\lambda)(\gamma_{lm}) + \lambda(t_{lm}) \in E$ for all $(\gamma_{lm}), (t_{lm}) \in E$ and scalar $\lambda \in [0,1]$.*

2. Main Result

This section discusses the article's results. The following results are stated without proof since they can be established using standard methods.

Result 2.1. *The following inclusions are strict.*

(*i*) ${}_2c^B(\mathbb{C}_2) \subset {}_2c_{0f}(\mathbb{C}_2) \subset {}_2c_f(\mathbb{C}_2) \subset {}_2\ell_\infty(\mathbb{C}_2)$.

(*ii*) ${}_2c^B(\mathbb{C}_2) \subset [{}_2c_{0f}(\mathbb{C}_2)] \subset [{}_2c_f(\mathbb{C}_2)] \subset {}_2\ell_\infty(\mathbb{C}_2)$.

(*iii*) $[{}_2c_{0f}(\mathbb{C}_2)] \subset {}_2c_{0f}(\mathbb{C}_2)$.

We discuss the following example to show that the inclusion (iii) is strict. Similar examples can be constructed to show that the other inclusions are strict.

Example 2.2. *Consider the double sequences (γ_{lm}) of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} (e_1 + e_2), & \text{if } l + m \text{ is even ;} \\ -(e_1 + e_2), & \text{otherwise .} \end{cases}$$

Then $(\gamma_{lm}) \in {}_2c_{0f}(\mathbb{C}_2)$, but $(\gamma_{lm}) \notin [{}_2c_{0f}(\mathbb{C}_2)]$.

Result 2.3. *A P-convergent double sequence of bi-complex numbers may not be almost convergent.*

Example 2.4. *Let us consider the double sequence (γ_{lm}) of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} (e_1 + e_2)l^2, & \text{if} \quad m = 1 \text{ for all } l \in \mathbb{N}, \\ (e_1 + e_2)m, & \text{if} \quad l = 1 \text{ for all } m \in \mathbb{N}, \\ \frac{(e_1+e_2)}{l+m}, & \text{if} \quad l > 1 \text{ and } m > 1. \end{cases}$$

$$P - \lim_{m\to\infty} \gamma_{1m} = \infty.$$

In this case,

$$\sup_{l,m\in\mathbb{N}} \|\gamma_{1m}\|_{\mathbb{C}_2} = \infty.$$

Therefore, the double sequence (γ_{lm}) of bi-complex numbers is unbounded.
Hence, the double sequence (γ_{lm}) of bi-complex numbers is not almost convergent double sequence of bi-complex numbers.

In view of the above example, we state the following result.

Result 2.5. *A P-convergent double sequence of bi-complex numbers may not be strongly almost convergent.*

We present the following two results without proof, these can be established using standard technique.

Theorem 2.6. *Every bounded convergent double sequence of bi-complex numbers is almost convergent double sequence of bi-complex numbers.*

Theorem 2.7. *Every regular convergent double sequence of bi-complex numbers is almost convergent double sequence of bi-complex numbers.*

Result 2.8. *Almost convergent double sequences (γ_{lm}) of bi-complex numbers may not be P-convergent in general.*

Example 2.9. *Consider the double sequences (γ_{lm}) of bi-complex numbers defined by*

$$\gamma_{lm} = \begin{cases} e_1 + e_2, & \text{if } l = m \text{ odd}, \\ -(e_1 + e_2), & \text{if } l = m \text{ even}, \\ e_1 e_2, & \text{otherwise}. \end{cases}$$

It is clear that (γ_{lm}) is almost null but not P-converget, that is, ${}_2c^B(\mathbb{C}_2) \subset {}_2c_f(\mathbb{C}_2)$.

We state the following theorem without a proof that can be established by standard techniques.

Theorem 2.10. *The class of double sequences of bi-complex numbers ${}_2c_{0f}(\mathbb{C}_2)$, ${}_2c_f(\mathbb{C}_2)$, $[{}_2c_{0f}(\mathbb{C}_2)]$ and $[{}_2c_f(\mathbb{C}_2)]$ are linear space under the coordinatewise addition and coordinatewise scalar multiplication.*

Theorem 2.11. *Every double sequence $\gamma = (\gamma_{lm})$ of bi-complex numbers is almost convergence if and only if it is almost Cauchy.*

Proof. Consider $\gamma = (\gamma_{lm})$ be almost convergent double sequence of bi-complex numbers. Then, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\left\| \frac{1}{pq} \sum_{l=0}^{p-1} \sum_{m=0}^{q-1} \gamma_{l+n,m+k} - K \right\|_{\mathbb{C}_2} < \frac{\varepsilon}{2}, \text{ for all } p, q > N_0 \text{ and } (n,k) \in \mathbb{N} \times \mathbb{N}.$$

Therefore,

$$\begin{aligned} &\left\| \frac{1}{p_1 q_1} \sum_{l=0}^{p_1-1} \sum_{m=0}^{q_1-1} \gamma_{l+n_1,m+k_1} - \frac{1}{p_2 q_2} \sum_{l=0}^{p_2-1} \sum_{m=0}^{q_2-1} \gamma_{l+n_2,m+k_2} \right\|_{\mathbb{C}_2} \\ &\leq \left\| \frac{1}{p_1 q_1} \sum_{l=0}^{p_1-1} \sum_{m=0}^{q_1-1} \gamma_{l+n_1,m+k_1} - K \right\|_{\mathbb{C}_2} + \left\| \frac{1}{p_2 q_2} \sum_{l=0}^{p_2-1} \sum_{m=0}^{q_2-1} \gamma_{l+n_2,m+k_2} - K \right\|_{\mathbb{C}_2} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

for all $p_1, p_2, q_1, q_2 > N_0$ and $(n_1, k_1), (n_2, k_2) \in \mathbb{N} \times \mathbb{N}$. Hence, the sequence $\gamma = (\gamma_{lm})$ is almost Cauchy. Conversely, let $\gamma = (\gamma_{lm})$ be almost Cauchy double sequence of bi-complex numbers. Then, for every $\varepsilon > 0$, there exists $N_0 \in \mathbb{N}$ such that

$$\left\| \frac{1}{p_1 q_1} \sum_{l=0}^{p_1-1} \sum_{m=0}^{q_1-1} \gamma_{l+n_1,m+k_1} - \frac{1}{p_2 q_2} \sum_{l=0}^{p_2-1} \sum_{m=0}^{q_2-1} \gamma_{l+n_2,m+k_2} \right\|_{\mathbb{C}_2} < \frac{\varepsilon}{2}, \tag{1}$$

for all $p_1, p_2, q_1, q_2 > N_0$ and $(n_1, k_1), (n_2, k_2) \in \mathbb{N} \times \mathbb{N}$.

$$\left\| \frac{1}{p_1 q_1} \sum_{l=0}^{p_1-1} \sum_{m=0}^{q_1-1} \gamma_{l+n_1,m+k_1} - \frac{1}{p_1 q_2} \sum_{l=0}^{p_1-1} \sum_{m=0}^{q_2-1} \gamma_{l+n_1,m+k_2} \right\|_{\mathbb{C}_2} < \frac{\varepsilon}{2}, \tag{2}$$

for all $p_1, q_1, q_2 > N_0$ and $(n_1, k_1), (n_1, k_2) \in \mathbb{N} \times \mathbb{N}$.
So $(\gamma_{lm})_q$ is almost Cauchy sequence in $\mathbb{C}_2$ for each $p \geq N_0$. Hence almost convergent. Let

$$\lim_{q \to \infty} \frac{1}{pq} \sum_{l=0}^{p-1} \sum_{m=0}^{q-1} \gamma_{l+n,m+k} = M_p, \text{ for all } p \geq N_0, \text{ uniformly in } l, m \in \mathbb{N}.$$

i.e.,

$$\left\|\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}-M_p\right\|_{\mathbb{C}_2}<\varepsilon,\ \text{for all } p,q\geq N_0 \tag{3}$$

Similarly, Let

$$\lim_{p\to\infty}\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}=L_q,\ \text{for all } q\geq N_0,\ \text{uniformly in } \ l,m\in\mathbb{N}.$$

i.e.,

$$\left\|\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}-M_p\right\|_{\mathbb{C}_2}<\varepsilon,\ \text{for all } p,q\geq N_0 \tag{4}$$

First we show that (M_p) is almost Cauchy sequence. Let $p,i\geq N_0$ then

$$\begin{aligned}\|M_p-M_i\|_{\mathbb{C}_2}&=\left\|M_p-\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}+\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}-M_i\right\|_{\mathbb{C}_2}\\&\leq\left\|M_p-\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}\right\|_{\mathbb{C}_2}+\left\|\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}-M_i\right\|_{\mathbb{C}_2}\end{aligned}$$

$<\varepsilon+\varepsilon=2\varepsilon,$ by equation (3).

Hence, (M_p) is almost Cauchy sequence. Let

$$\lim_{p\to\infty}M_p=K_1,\ \text{uniformly in } \ l,m\in\mathbb{N}. \tag{5}$$

Similarly, we can so that (L_q) is almost Cauchy sequence. Let

$$\lim_{q\to\infty}L_q=K_2,\ \text{uniformly in } \ l,m\in\mathbb{N}. \tag{6}$$

Now we show that $K_1=K_2$.
From equation (5) there exist $N_1\in\mathbb{N}$ such that

$$\|M_p-K_1\|_{\mathbb{C}_2}<\varepsilon,\ \text{for all } p,q\geq N_1. \tag{7}$$

From equation (6) there exists $N_2\in\mathbb{N}$ such that

$$\|L_q-K_2\|_{\mathbb{C}_2}<\varepsilon,\ \text{for all } p,q\geq N_2. \tag{8}$$

Let $N_3=max\{N_0,N_1,N_2\}$. Then for all $p,q>N_3$

$$\begin{aligned}\|K_1-K_2\|_{\mathbb{C}_2}&=\left\|K_1-M_p+M_p-\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}+\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}-L_q+L_q-K_2\right\|_{\mathbb{C}_2}\\&\leq\|K_1-M_p\|_{\mathbb{C}_2}+\left\|M_p-\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}\right\|_{\mathbb{C}_2}+\left\|\frac{1}{pq}\sum_{l=0}^{p-1}\sum_{m=0}^{q-1}\gamma_{l+n,m+k}-L_q\right\|_{\mathbb{C}_2}+\|L_q-K_2\|_{\mathbb{C}_2}\\&<\varepsilon+\varepsilon+\varepsilon+\varepsilon=4\varepsilon,\ \text{By equation (3),(4),(7) and (8).}\end{aligned}$$

Since, ε is arbitrarily small, so $K_1 = K_2 = K$, (say). Therefore, we have

$$\lim_{p,q\to\infty} \frac{1}{pq} \sum_{l=0}^{p-1} \sum_{m=0}^{q-1} \gamma_{l+n,m+k} = K, \text{ uniformly in } l, m \in \mathbb{N}.$$

Hence, the double sequence (γ_{lm}) of bi-complex numbers is almost convergent. □

Theorem 2.12. *The class of double sequences of bi-complex numbers* ${}_2c_{0f}(\mathbb{C}_2)$ *and* ${}_2c_f(\mathbb{C}_2)$ *are Banach spaces under the norm* $\|\cdot\|_\infty$*, defined by*

$$\|(\gamma_{lm})\|_\infty = \sup_{l,m} \|\tau_{pqij}(\gamma_{lm})\|_{\mathbb{C}_2}. \tag{9}$$

Proof. We establish it for the case ${}_2c_f(\mathbb{C}_2)$ and other case can be shown via a similar method.
Let (γ^a_{lm}) be the Cauchy sequence in ${}_2c_f(\mathbb{C}_2)$. Then, for each $l, m, (\gamma^a_{lm})$ is Cauchy in $\mathbb{C}_2$. Therefore $\gamma^a_{lm} \to \gamma_{lm}$ (say). Let $\gamma = (\gamma_{lm})$, then for a given ε there exists an integer $N_0(\varepsilon) = N_0$ say such that $a, b > N_0$.

$$\|\gamma^a - \gamma^b\|_\infty < \frac{\varepsilon}{2},$$

Hence,

$$\sup_{p,q,i,j} \|\tau_{pqij}(\gamma^a_{lm} - \gamma^b_{lm})\|_{\mathbb{C}_2} < \frac{\varepsilon}{2},$$

then for each p, q, i, j and $a, b > N_0$, we have

$$\|\tau_{pqij}(\gamma^a_{lm} - \gamma^b_{lm})\|_{\mathbb{C}_2} < \frac{\varepsilon}{2}.$$

Taking limit as $a \to \infty$, we have for $b > N_0$ and for each p, q, i, j

$$\|\tau_{pqij}(\gamma^a_{lm}) - K\|_{\mathbb{C}_2} < \frac{\varepsilon}{2}. \tag{10}$$

Now for a fixed $b, (\gamma^b_{lm}) \in {}_2c_f(\mathbb{C}_2)$, so we get

$$\lim_{p,q\to\infty} \tau_{pqij}(\gamma^b_{lm}) = K,$$

uniformly in i, j. Then, for a given $\varepsilon > 0$, there exist positive integers p_0, q_0 such that

$$\|\tau_{pqij}(s_{lm}) - K\|_{\mathbb{C}_2} < \frac{\varepsilon}{2}, \tag{11}$$

for $p \geq p_0, q \geq q_0$ and for all i, j. Here p_0, q_0 are independent of i, j but depends on ε. Now by using relation (10) and (11) we get

$$\begin{aligned}\|\tau_{pqij}(\gamma_{lm}) - K\|_{\mathbb{C}_2} =& \|\tau_{pqij}(\gamma_{lm}) - \tau_{pqij}(\gamma_{lm})^b + \tau_{pqij}(\gamma_{lm})^b - K\|_{\mathbb{C}_2} \\ &\leq \|\tau_{pqij}(\gamma_{lm}) - \tau_{pqij}(\gamma_{lm})^b\|_{\mathbb{C}_2} + \|\tau_{pqij}(\gamma_{lm})^b - K\|_{\mathbb{C}_2} \\ &< \varepsilon,\end{aligned}$$

for $p \geq p_0, q \geq q_0$ and for all i, j. Hence, $\gamma = (\gamma_{lm}) \in {}_2c_f(\mathbb{C}_2)$ and ${}_2c_f(\mathbb{C}_2)$ is complete.
This establishes the theorem. □

Theorem 2.13. *The class of double sequences of bi-complex numbers* $[{}_2c_{0f}(\mathbb{C}_2)]$ *and* $[{}_2c_f(\mathbb{C}_2)]$ *are Banach spaces under the norm* $\|\cdot\|_\infty$*, defined by*

$$\|(\gamma_{lm})\|_\infty = \sup_{l,m,i,j\in\mathbb{N}} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm}\|_{\mathbb{C}_2}.$$

Proof. The proof is similar to the proof of Theorem 2.12. □

Result 2.14. *The spaces* ${}_2c_{0f}(\mathbb{C}_2)$ *and* ${}_2c_f(\mathbb{C}_2)$ *are not monotone.*

Example 2.15. *Consider the double sequence* (γ_{lm}) *of bi-complex numbers defined by*

$$\gamma_{lm} = \{-(e_1+e_2)\}^{l+m}, \text{ for all } l,m \in \mathbb{N}$$

Then, we have

$$\frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\gamma_{lm}\right\|_{\mathbb{C}_2} = \frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\{-(e_1+e_2)\}^{l+m}\right\|_{\mathbb{C}_2}$$
$$= \frac{1}{(p+1)(q+1)}\left\|\{-(e_1+e_2)\}^{i+j}\frac{[1+\{-(e_1+e_2)\}^p][1+\{-(e_1+e_2)\}^q]}{4}\right\|_{\mathbb{C}_2}. \tag{12}$$

On apply P-limit as $p,q\to\infty$ *in equation (12), we get* $(\gamma_{lm}) \in {}_2c_{0f}(\mathbb{C}_2)$. *Now consider the double sequence* $t_{lm} \in \{0,1\}^{\mathbb{N}\times\mathbb{N}}$ *defined for all* $l,m\in\mathbb{N}$ *by*

$$t_{lm} = \{-(e_1+e_2)\}^{l+m}, l+m \text{ is even };$$
$$= e_1e_2, \text{ otherwise }.$$

and let $(s_{lm}) = (\gamma_{lm}t_{lm})$. *Hence, when* p,q,i *and* j *are even we have*

$$\frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}s_{lm}\right\|_{\mathbb{C}_2} = \frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\gamma_{lm}t_{lm}\right\|_{\mathbb{C}_2}$$
$$= \frac{1}{(p+1)(q+1)}\left\|\frac{q+1}{2}+\frac{p(q+1)}{2}\right\|_{\mathbb{C}_2},$$

which yields,

$$P-\lim_{p,q\to\infty}\frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}s_{lm}\right\|_{\mathbb{C}_2} = \frac{1}{2}, \text{ uniformly in } l,m\in\mathbb{N}.$$

Hence, $(s_{lm}) \notin {}_2c_{0f}(\mathbb{C}_2)$. *Furthermore, we obtain the same outcome in other case for* p,q,i *and* j. *Define the sequence* $v=(v_k)$ *by*

$$v_k = \begin{cases} (e_1+e_2), & k = 2^{2l},\dots,2^{2l+1}-1; \\ e_1e_2, & k = 2^{2l+1},\dots,2^{2l+2}-1; \end{cases}$$

$(k=0,1,2,3,4,\dots)$ *for all* $k\in\mathbb{N}$. *Now, we contruct the double sequence of bi-complex numbers* $(u_{lm}) \notin {}_2c_f(\mathbb{C}_2)$ *whose rows are all of the sequence* v, *that is*

$$(u_{lm}) = \begin{pmatrix} (e_1+e_2) & e_1e_2 & e_1e_2 & (e_1+e_2) & (e_1+e_2) & (e_1+e_2) & (e_1+e_2) & e_1e_2 & \cdots \\ (e_1+e_2) & e_1e_2 & e_1e_2 & (e_1+e_2) & (e_1+e_2) & (e_1+e_2) & (e_1+e_2) & e_1e_2 & \cdots \\ (e_1+e_2) & e_1e_2 & e_1e_2 & (e_1+e_2) & (e_1+e_2) & (e_1+e_2) & (e_1+e_2) & e_1e_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

and take the double sequence (γ_{lm}) *of bi-complex numbers with* $\gamma_{lm} = (e_1+e_2)$, *for all* $k,l\in\mathbb{N}$. *Therefore, we have*

$$\frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\gamma_{lm}\right\|_{\mathbb{C}_2} = \frac{\|(i+p-i+1)(j+q-j+1)\|_{\mathbb{C}_2}}{(p+1)(q+1)} = 1. \tag{13}$$

If we apply P-limit as $p, q \to \infty$ in the equation (13), then we get

$$P-\lim_{p,q\to\infty}\frac{1}{(p+1)(q+1)}\left\|\sum_{l=i}^{i+p}\sum_{m=j}^{j+q}\gamma_{lm}\right\|_{\mathbb{C}_2}=1,\ \textit{uniformly in}\ \ l,m\in\mathbb{N}.$$

that is $(\gamma_{lm}) \in {}_2c_f(\mathbb{C}_2)$. Obviously, the sequence (u_{lm}) is in $\{0,1\}^{\mathbb{N}\times\mathbb{N}}$ but the multiply sequence $(z_{lm}) = (u_{lm}\gamma_{lm}) = (u_{lm})$ is not in ${}_2c_f(\mathbb{C}_2)$.
Thus, the space ${}_2c_f(\mathbb{C}_2)$ is not monotone.

We state the following result in view of Result 2.14. and Remark 1.15

Result 2.16. *The sequence spaces ${}_2c_{0f}(\mathbb{C}_2)$ and ${}_2c_f(\mathbb{C}_2)$ are not solid.*

Result 2.17. *The sequence spaces ${}_2c_f(\mathbb{C}_2)$ and $[{}_2c_f(\mathbb{C}_2)]$ are not symmtric.*

Example 2.18. *Consider the double sequence (γ_{lm}) of bi-complex numbers defined by*

$$\gamma_{lm}=\begin{cases}(e_1+e_2), & \textit{if } l+m \textit{ is even and } l,m\in\mathbb{N};\\ e_1e_2, & \textit{otherwise}.\end{cases}$$

Then, $(\gamma_{lm}) \in {}_2c_f(\mathbb{C}_2)$.
Consider the rearranged double sequence (t_{lm}) of bi-complex numbers defined by

$$t_{lm}=\begin{cases}(e_1+e_2), & \textit{if } m=k^2, k\in\mathbb{N} \textit{ and for all } l;\\ e_1e_2, & \textit{otherwise}.\end{cases}$$

Then, $(t_{lm}) \notin {}_2c_f(\mathbb{C}_2)$. Hence ${}_2c_f(\mathbb{C}_2)$ is not symmetric.
It is clear from this example that the space $[{}_2c_f(\mathbb{C}_2)]$ is not symmetric.

Result 2.19. *The sequence spaces ${}_2c_{0f}(\mathbb{C}_2)$ and $[{}_2c_{0f}(\mathbb{C}_2)]$ are not symmtric.*

Example 2.20. *Consider the double sequence (γ_{lm}) of bi-complex numbers defined by*

$$\gamma_{lm}=\begin{cases}(e_1+e_2), & \textit{if } l+m \textit{ is even and } l,m\in\mathbb{N};\\ -(e_1+e_2), & \textit{otherwise}.\end{cases}$$

Then, $(\gamma_{lm}) \in {}_2c_{0f}(\mathbb{C}_2)$.
Consider the rearranged double sequence (t_{lm}) of bi-complex numbers defined by

$$t_{lm}=\begin{cases}(e_1+e_2), & \textit{if } m=k^2, k\in\mathbb{N}, \textit{ for all } l;\\ -(e_1+e_2), & \textit{otherwise}.\end{cases}$$

Then, $(t_{lm}) \notin {}_2c_{0f}(\mathbb{C}_2)$. Hence, ${}_2c_{0f}(\mathbb{C}_2)$ is not symmetric.
It is clear from this example that the space $[{}_2c_{0f}(\mathbb{C}_2)]$ is not symmetric.

Result 2.21. *The sequence spaces ${}_2c_{0f}(\mathbb{C}_2)$, ${}_2c_f(\mathbb{C}_2)$, $[{}_2c_{0f}(\mathbb{C}_2)]$ and $[{}_2c_f(\mathbb{C}_2)]$ are not convergence free.*

Example 2.22. *Consider the double sequence (γ_{lm}) of bi-complex numbers defined by*

$$\gamma_{lm}=\begin{cases}(e_1+e_2), & \textit{if } l+m \textit{ even } l,m\in\mathbb{N};\\ -(e_1+e_2), & \textit{otherwise}.\end{cases}$$

Then the sequence $(\gamma_{lm}) \in Z$.
Construct the double sequence (t_{lm}) of bi-complex numbers by

$$t_{lm}=\begin{cases}(e_1+e_2)lm, & \textit{if } l+m \textit{ even } l,m\in\mathbb{N};\\ -(e_1+e_2)lm, & \textit{otherwise}.\end{cases}$$

Clearly, $(t_{lm}) \notin Z$, where $Z = {}_2c_{0f}(\mathbb{C}_2), {}_2c_f(\mathbb{C}_2), [{}_2c_{0f}(\mathbb{C}_2)]$ and $[{}_2c_f(\mathbb{C}_2)]$.
Hence, the spaces ${}_2c_{0f}(\mathbb{C}_2), {}_2c_f(\mathbb{C}_2), [{}_2c_{0f}(\mathbb{C}_2)]$ and $[{}_2c_f(\mathbb{C}_2)]$ are not convergence free.

Result 2.23. *The classes of double sequences* ${}_2c_f(\mathbb{C}_2)$ *and* ${}_2c_{0f}(\mathbb{C}_2)$ *of bi-complex numbers are not sequence algebras in general.*

Example 2.24. *Consider the double sequences* $(\gamma_{lm}), (t_{lm})$ *of bi-complex numbers defined by*

$$\gamma_{lm} = t_{lm} = \begin{cases} (e_1 + e_2), & \text{if } l + m \text{ is even ;} \\ -(e_1 + e_2), & \text{otherwise .} \end{cases}$$

Then

$$P - \lim_{p,q\to\infty} \left\| \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \gamma_{lm} \right\|_{\mathbb{C}_2} = 0, \text{ uniformly in } l, m \in \mathbb{N}$$

and

$$P - \lim_{p,q\to\infty} \left\| \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} t_{lm} \right\|_{\mathbb{C}_2} = 0, \text{ uniformly in } l, m \in \mathbb{N}$$

Now,

$$\begin{aligned} &P - \lim_{p,q\to\infty} \left\| \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \gamma_{lm} \star t_{lm} \right\|_{\mathbb{C}_2} \\ &\leq P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm} \star t_{lm}\|_{\mathbb{C}_2} \\ &\leq P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm} t_{lm}\|_{\mathbb{C}_2} = (e_1 + e_2) \neq 0. \end{aligned}$$

Clearly, $(\gamma_{lm}) \star (t_{lm}) \notin {}_2c_{0f}(\mathbb{C}_2)$. *Hence, the double sequence space* ${}_2c_{0f}(\mathbb{C}_2)$ *is not a sequence algebra.*

Theorem 2.25. *The classes of double sequences* $[{}_2c_{0f}(\mathbb{C}_2)]$ *and* $[{}_2c_f(\mathbb{C}_2)]$ *of bi-complex numbers are sequence algebras.*

Proof. The conclusion comes from the inequality given below,
Let the double sequence $(\gamma_{lm}), (t_{lm}) \in [{}_2c_f(\mathbb{C}_2)]$ of bi-complex numbers. Then

$$P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm}\|_{\mathbb{C}_2} = K_1 \text{ (say) , uniformly in } l, m \in \mathbb{N}.$$

$$P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|t_{lm}\|_{\mathbb{C}_2} = K_2 \text{ (say) , uniformly in } l, m \in \mathbb{N}.$$

Thus, we have for the product (termwise) of the two sequences

$$P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm} t_{lm}\|_{\mathbb{C}_2} \leq K_1 K_2, \text{ uniformly in } l, m \in \mathbb{N},$$

where $K_1, K_2 \in \mathbb{C}_2$. Clearly, $(\gamma_{lm}) \star (t_{lm}) \in [{}_2c_f(\mathbb{C}_2)]$. Hence, the double sequence space $[{}_2c_f(\mathbb{C}_2)]$ is a sequence algebra.
Similarly, we have the space $[{}_2c_{0f}(\mathbb{C}_2)]$ is a sequence algebra. □

Theorem 2.26. *The spaces* ${}_2c_{0f}(\mathbb{C}_2), {}_2c_f(\mathbb{C}_2), [{}_2c_{0f}(\mathbb{C}_2)]$ *and* $[{}_2c_f(\mathbb{C}_2)]$ *are convex.*

Proof. Let the double sequences $(\gamma_{lm}), (t_{lm}) \in [{}_2c_f(\mathbb{C}_2)]$ and $\lambda \in \mathbb{C}_0$ satisfying $0 \le \lambda \le 1$. Then

$$P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm}\|_{\mathbb{C}_2}, \text{ uniformly in } l, m \in \mathbb{N}$$

and

$$P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|t_{lm}\|_{\mathbb{C}_2}, \text{ uniformly in } l, m \in \mathbb{N}$$

are strongly almost convergent to K_1 and K_2 respectively.

$$\begin{aligned}
&P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\{\lambda\gamma_{lm} + (1-\lambda)t_{lm}\}\|_{\mathbb{C}_2} \\
&\le P - \lim_{p,q\to\infty} \frac{\lambda}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm}\|_{\mathbb{C}_2} + P - \lim_{p,q\to\infty} \frac{(1-\lambda)}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|t_{lm}\|_{\mathbb{C}_2} \\
&= \lambda\left\{P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|\gamma_{lm}\|_{\mathbb{C}_2}\right\} + (1-\lambda)\left\{P - \lim_{p,q\to\infty} \frac{1}{(p+1)(q+1)} \sum_{l=i}^{i+p} \sum_{m=j}^{j+q} \|t_{lm}\|_{\mathbb{C}_2}\right\} \\
&= \lambda K_1 + (1-\lambda)K_2.
\end{aligned}$$

Which implies, $\lambda(\gamma_{lm}) + (1-\lambda)(t_{lm}) \in [{}_2c_f(\mathbb{C}_2)]$.
Use the similar technique for the spaces ${}_2c_{0f}(\mathbb{C}_2), {}_2c_f(\mathbb{C}_2)$ and $[{}_2c_{0f}(\mathbb{C}_2)]$ are convex. □

Conclusions. In this article, we have introduced and investigated properties of the almost convergent and strongly almost convergent double sequences of bi-complex numbers. We have examined its various algebraic and topological properties, discussed some examples.

Declarations

Funding. Not Applicable.

Conflicts of interest/Competing interests (include appropriate disclosures). We declare that the article is free from Conflicts of interest and Competing interests.

Availability of data and material (data transparency). Not Applicable.

Code availability (software application or custom code). Not Applicable.

Authors' contributions. Both the authors have equal contribution in the preparation of this article.

References

[1] R.P. Agarwal, D.O. Regan and D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*. Springer, New York, 2009.
[2] T. J. Bromwich, *An Introduction to the Theory of Infinite Series*, Macmillan, New York, NY, USA, 1965.
[3] M. Başarir, *On the strong almost convergence of double sequences*. Period. Math. Hung., **30**(3) (1995), 177–181.
[4] F. Čunjalo, *Almost convergence of double sequences some analogies between measure and category*. Math. Maced., **5** (2007), 21–24.

[5] G.H. Hardy, *On the convergence of certain multiple series*, Proc. Camb. Phil. Soc. (1917) 86–95.
[6] G.G. Lorentz, *A contribution to theory of divergent sequences*. Acta Math., **80** (1948), 167–190.
[7] I.J. Maddox, *A new type of convergence*. Math. Proc. Camb. Phil. Soc., **83** (1978), 61–64.
[8] I.J. Maddox, *On strong almost convergence*. Math. Proc. Camb. Phil. Soc., **85** (1979), 345–350.
[9] F. Móricz and B.E. Rhoades, *Almost convergence of double sequences and strong regularity of summability matrices*. Math. Proc. Camb. Philos. Soc., **104** (1988), 283–294.
[10] M. Mursaleen and S.A. Mohiuddine, *Convergence Methods for Double Sequences and Applications*, Springer India, 2014.
[11] M. Mursaleen and O.H.H. Edely, *Almost convergence and a core theorem for double sequences*. J. Math. Anal. Appl., **293** (2004), 532–540.
[12] M. Mursaleen and S.A. Mohiuddine, *Banach limit and some new spaces of double sequences*. Turk. J. Math., **36** (2012), 121–130.
[13] M. Mursaleen and S.A. Mohiuddine, *Invariant mean and some core theorems for double sequences*. Taiwanese J. Math., **14** (2010), 21–33.
[14] Md. Nasiruzzaman and M. Mursaleen, *The multicomplex numbers and their properties on some elementary functions*. Thai J. Math., **21**(1) (2023), 77–100.
[15] G. B. Price, *An Introduction to Multicomplex Spaces and Function*, M. Dekker 1991.
[16] A. Pringsheim, *Zur theotie der zweifach unendlichen zahlenfolgen*, Mathematische Annalen, **53**(3) (1900), 289–321.
[17] D. Rochon, M. Shapiro, *On algebraic properties of bi-complex and hyperbolic numbers*, Anal. Univ. Oradea, fasc. Math. **11** (2004), 71–110.
[18] C. Segre, *Le rappresenttazioni reali delle forme complessee gli enti iperal-gebrici (The real representation of complex elements and the hyperralgebraic entities)*, Math. Ann. **40** (1892), 413–467.
[19] N. Spampinato, *Estensione nel campo bicomplesso di due teoremi, del levi-Civita e del severi, per le funzione olomorfe di due variablili complesse*. I, II, Atti Reale Accad. Naz. Rend, **22**(6) (1935), 96–102.
[20] O. Tuğ, *The spaces of B(r,s,t,u) strongly almost convergent double sequences and matrix transformations*, Bull. Sci. Math., **169** (2021), 1–20.
[21] M. Yeşilkayagil and F. Başar, *Some topological properties of the spaces of almost null and almost convergent double sequences*, Turkish J. Math., **40** (3) (2016), 624–630.