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Periodicity, transitivity and distality of real projective transformations

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Abstract. This study investigates the dynamical properties of real projective transformations from a topological viewpoint. We study properties like periodicity, topological mixing, topological transitivity, distality and proximality. Regarding periodicity, we give a complete characterisation of the sets of periods. We show that projective transformations are not topologically mixing and that it is only the isometries among them that are distal.

1. Introduction

The dynamics of real projective transformations is the main topic of this paper. By definition, a *topological discrete dynamical system* (briefly, a dynamical system) is a pair (X, f), where X is a topological space and f is a continuous self map of X. The *trajectory* of $x \in X$ is defined as the sequence $(x, f(x), f^2(x), f^3(x), ...)$, where $f^k(x) = f \circ f \circ ... \circ f(x)$ (k times) for $k \in \mathbb{N}$ and $f^0(x) = x$. The *forward orbit* of x is defined as the set $\{f^k(x) : k$ is a non-negative integer$\}$. The study of dynamics is primarily concerned with the behavior of trajectories in the long run. If (X, f) and (Y, g) are two dynamical systems and $\phi : X \to Y$ is a surjective continuous map such that $\phi \circ f = g \circ \phi$, then ϕ is called a *topological semiconjugacy* from f to g and (Y, g) is called a *factor* of (X, f). If ϕ is a homeomorphism, then ϕ is called a *topological conjugacy*; in this case (X, f) and (Y, g) are said to be *topologically conjugate*.

In this paper, our dynamical system is $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$, where $\mathbb{P}_n(\mathbb{R})$ and $\widetilde{T}$ are defined as follows. Let $n \in \mathbb{N}$ and for $x, y \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$, if there exists a non-zero $\lambda \in \mathbb{R}$ such that $x = \lambda y$, then define $x \sim y$. Then the quotient space $\mathbb{R}^{n+1} \setminus \{\bar{0}\} /\sim$, denoted by $\mathbb{P}_n(\mathbb{R})$ is called the n-dimensional *real projective space*. The quotient map is denoted by π and for an $x \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$, $\pi(x)$ is also denoted as $[x]$. It is well known that $\mathbb{P}_n(\mathbb{R})$ is compact and connected. Besides, note that any open subset of $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ is open in $\mathbb{R}^{n+1}$ as well. Given a linear map $T \in GL_{n+1}(\mathbb{R})$, its associated projective transformation denoted by $\widetilde{T}$, is defined as $\widetilde{T}(\pi(x)) = \pi(Tx)$, for every $x \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$. It can be easily observed that $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$ is a factor of $\left(\mathbb{R}^{n+1} \setminus \{\bar{0}\}, T\right)$.

The literature on the dynamics of projective transformations is extensive. See for example [5], [8] and [11]. In the present article, we investigate some dynamical properties of projective transformations. In the

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next section, we define these properties followed by related known results in some cases and then our main results. We refer to [3] for most of the definitions.

Before proceeding to the results, we will now define a metric d_p on $\mathbb{P}_n(\mathbb{R})$. A metric on $\mathbb{P}_n(\mathbb{R})$ may be already well known but we will define a metric that is convenient for our calculations and show that it does induce the topology of $\mathbb{P}_n(\mathbb{R})$. We finally mention some notations and terms that we are going to use. The cardinality of any set A is denoted by $|A|$. T denotes an invertible linear transformation of $\mathbb{R}^{n+1}$ and $\widetilde{T}$, its associated projective transformation on $\mathbb{P}_n(\mathbb{R})$ for any non-negative integer n. We also identify T with the matrix associated to it. By an eigenvector of T, we mean an eigenvector corresponding to a real eigenvalue, unless otherwise mentioned. We use $\|x\|$ to denote the Euclidean norm of $x \in \mathbb{R}^m$ for any $m \in \mathbb{N}$.

Definition 1.1. *For any* $[x],[y] \in \mathbb{P}_n(\mathbb{R})$*, define* $d_p([x],[y]) = min\left\{\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\|, \left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\|\right\}$.

Proposition 1.2. d_p *is a metric on* $\mathbb{P}_n(\mathbb{R})$.

Proof. If $d_p([x],[y]) = 0$, then either $\frac{x}{\|x\|} = \frac{y}{\|y\|}$ or $\frac{x}{\|x\|} = -\frac{y}{\|y\|}$ and in either case $[x] = [y]$. Also, if $[x] = [y]$, then $x = \lambda y$ for some non-zero $\lambda \in \mathbb{R}$ and thus $\frac{x}{\|x\|} = \pm\frac{y}{\|y\|}$; hence $d_p([x],[y]) = 0$. Obviously, for any $[x],[y] \in \mathbb{P}_n(\mathbb{R})$, we have $d_p([x],[y]) = d_p([y],[x])$. So, it remains to verify the triangle inequality. For any $[x],[y],[z] \in \mathbb{P}_n(\mathbb{R})$, since there are two possibilities for each of the values of $d_p([x],[y])$ and $d_p([y],[z])$, we have four possibilities for the sum $d_p([x],[y]) + d_p([y],[z])$. It can be easily verified that each of them is atleast the value of either $\left\|\frac{x}{\|x\|} - \frac{z}{\|z\|}\right\|$, or $\left\|\frac{x}{\|x\|} + \frac{z}{\|z\|}\right\|$ and hence $d_p([x],[z]) \le d_p([x],[y]) + d_p([y],[z])$. □

Proposition 1.3. d_p *induces the topology of* $\mathbb{P}_n(\mathbb{R})$.

Proof. To avoid ambiguity, we refer to the topology of $\mathbb{P}_n(\mathbb{R})$ as the quotient topology, as it is a quotient space of $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ and the topology induced by d_p as metric topology. Let U be an open set in $\mathbb{P}_n(\mathbb{R})$ with respect to the quotient topology and $[x] \in U$. Then $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ and $\{\lambda x \mid \lambda \in \mathbb{R} \setminus \{0\}\} \subset \pi^{-1}(U)$; in particular, $\frac{x}{\|x\|} \in \pi^{-1}(U)$. Choose $\epsilon > 0$ such that the Euclidean open ball $B_E(\frac{x}{\|x\|}, \epsilon)$ centered at $\frac{x}{\|x\|}$ with radius ϵ is contained in $\pi^{-1}(U)$. Now, consider $B_{d_p}([x],\epsilon)$, the open ball in $\mathbb{P}_n(\mathbb{R})$, centered at $[x]$ and radius ϵ with respect to the metric d_p. If $[y] \in B_{d_p}([x],\epsilon)$, then either $\left\|\frac{x}{\|x\|} - \frac{y}{\|y\|}\right\| < \epsilon$ or $\left\|\frac{x}{\|x\|} + \frac{y}{\|y\|}\right\| < \epsilon$. Then $\frac{y}{\|y\|} \in B_{d_p}(\frac{x}{\|x\|}, \epsilon) \subset \pi^{-1}(U)$ or $-\frac{y}{\|y\|} \in B_{d_p}(\frac{x}{\|x\|}, \epsilon) \subset \pi^{-1}(U)$ and in either case $[y] \in U$. Hence U is open in metric topology.

Conversely, consider $B_{d_p}([x],\epsilon)$, the open ball in $\mathbb{P}_n(\mathbb{R})$ centered at $[x]$ with radius ϵ. Now, $\pi^{-1}\left(B_{d_p}([x],\epsilon\right) = \phi^{-1}\left(B_E(\frac{x}{\|x\|},\epsilon)\right) \cup \phi^{-1}\left(B_E(\frac{-x}{\|x\|},\epsilon)\right)$, where $\phi : \mathbb{R}^{n+1} \setminus \{\bar{0}\} \to S^n$ is the map given by $\phi(z) = \frac{z}{\|z\|}$. Since ϕ is continuous, the set $\pi^{-1}\left(B_{d_p}([x],\epsilon)\right)$ is open in $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$ and thus $B_{d_p}([x],\epsilon)$ is open in the quotient topology. □

2. Main Results

2.1. Periodicity

A point $x \in X$ is said to be *periodic* if there is a $k \in \mathbb{N}$ such that $f^k(x) = x$; the least such k is termed as the *period* of x. A periodic point of period one is called a *fixed point*. There have been several papers that study various aspects of periodic points of dynamical systems. In this paper, we are concerned with two of them, namely the characterization of the sets of periodic points and the sets of periods, i.e. we try to describe $\{Per(f) : f \in \mathfrak{F}\}$, where $Per(f) = \{n \in \mathbb{N} : f$ has a periodic point in X of period $n\}$ and $\{P(f) : f \in \mathfrak{F}\}$, where $P(f) = \{x \in X : x$ is a periodic point of $f\}$ for a family $\mathfrak{F}$ of continuous maps on a space X. The problems of characterizing these sets have been well-studied in the literature. The articles [2],[4],[6],[7] and [13] are some such papers and [10] is a nice survey of these results.

In the present case, i.e. $(\mathbb{P}_n(\mathbb{R}), \widetilde{T})$, $P(\widetilde{T})$ can be easily found as described in one of the following paragraphs and $Per(\widetilde{T})$ as described in Theorem 1. Beside these characterisations, an another well studied notion is a dynamical invariant, called the zeta function. If the number of fixed points of f^k, denoted by $|Fix(f^k)|$ is finite for every $k \in \mathbb{N}$ in a dynamical system (X, f), we define the zeta function $\zeta_f(z)$ of f as the formal power series $\zeta_f(z) = exp(\sum_{k=1}^{\infty} \frac{1}{k}|Fix(f^k)|z^k)$. The dynamical zeta function for a projective transformation was found in [8].

We will now describe the periodic points of $\left(\mathbb{P}_n(\mathbb{R}), \widetilde{T}\right)$. If $v \in \mathbb{R}^{n+1} \setminus \{\bar{0}\}$ is an eigenvector of T with eigenvalue λ, then $\widetilde{T}([v]) = [Tv] = [\lambda v] = [v]$, and therefore $[v]$ is a fixed point. Conversely, if $[v]$ is a periodic point with period k, it is a fixed point of $\widetilde{T}^k$, and therefore $[T^k v] = [v]$, i.e. $T^k v = \lambda' v$ for some scalar $\lambda' \in \mathbb{R} \setminus \{0\}$. As a result, v is an eigenvector of T^k. To summarize, $[v]$ is periodic if and only if v is an eigenvector of T^k for some $k \in \mathbb{N}$.

We now state and prove our theorem about the sets of periods. We introduce the following notation to make the statement of theorem simpler. For an $n \in \mathbb{N}$, $\mathfrak{I}_n = \left\{A \subset \mathbb{N} \mid |A| \leq \frac{n}{2}\right\}$, if n is even and $\mathfrak{I}_n = \left\{A \subset \mathbb{N} \mid 1 \in A \text{ and } |A| \leq \frac{n+1}{2}\right\}$, if n is odd.

Theorem 2.1. $\{Per(\widetilde{T}) \mid \widetilde{T}$ *is a projective transformation on* $\mathbb{P}_n(\mathbb{R})\} = \mathfrak{I}_n$, *for any* $n \in \mathbb{N}$.

Proof. If $[x]$ is a periodic point of $\widetilde{T}$ with period k, then x is an eigenvector of T^k. Also, $T^l(x) = \lambda x$ for some non-zero $\lambda \in \mathbb{R}$ will imply that $\widetilde{T}^l([x]) = [x]$. Hence, $k \in Per(\widetilde{T})$ if and only if T^k has an eigenvector x such that x is not an eigenvector of T^l for any $l < k$.

If $\mu \in \mathbb{C}$ is a complex eigenvalue of T and $\mu^k \in \mathbb{R}$ for some $k \in \mathbb{N}$, then denote by k_μ to be the least positive integer such that $\mu^{k_\mu} \in \mathbb{R}$. Note that $k_\mu = 1$ if and only if $\mu \in \mathbb{R}$. By the above argument, it follows that $k_\mu \in Per(\widetilde{T})$. Conversely, if $k \in Per(\widetilde{T})$, then $T^k x = \lambda x$ for some non-zero $\lambda \in \mathbb{R}$. It is very well known that $\sqrt[k]{\lambda}$ is a complex eigenvalue of T and hence $k = k_\mu$, where $\mu = \sqrt[k]{\lambda}$. Therefore, $Per(\widetilde{T}) = \{k_\mu \mid \mu$ is a complex eigenvalue of $T\}$.

Since T has atmost $\frac{n}{2}$ or $\frac{n-1}{2}$ complex eigenvalues which are not conjugates of each other, depending on whether n is even or odd respectively, we have $|Per(\widetilde{T})| \leq \frac{n}{2}$, when n is even and $|Per(\widetilde{T})| \leq \frac{n+1}{2}$, when n is odd. In case n is odd, T has at least one real eigenvalue; so $1 \in Per(\widetilde{T})$. Hence $Per(\widetilde{T}) \in \mathfrak{I}_n$.

Conversely, for any $A \in \mathfrak{I}_n$, say $A \setminus \{1\} = \{m_1, m_2, \cdots, m_l\} \subset \mathbb{N}$. Define $\mu_j = e^{i\frac{\pi}{m_j}}$, where $1 \leq j \leq l$. Let $R_\theta = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$ and T be the block diagonal matrix with the diagonal blocks as $R_{\frac{\pi}{m_1}}, R_{\frac{\pi}{m_2}}, \cdots, R_{\frac{\pi}{m_l}}$ if $l = \frac{n}{2}$ and $R_{\frac{\pi}{m_1}}, R_{\frac{\pi}{m_2}}, \cdots, R_{\frac{\pi}{m_l}}, I_{n-2l}$ if $l < \frac{n}{2}$, where I_{n-2l} is the identity matrix of size $n - 2l$. Then, the set of eigenvalues of T is $\{\mu_1, \overline{\mu_1}, \mu_2, \overline{\mu_2}, \cdots, \mu_l, \overline{\mu_l}\} \cup U$, where $U = \phi$ or $U = \{1\}$. Note that $m_j = k_{\mu_j}$ and hence $Per(\widetilde{T}) \setminus \{1\} = \{k_\mu \mid \mu$ is a non-real eigenvalue of $T\} = \{m_j | 1 \leq j \leq l\}$. Therefore, $Per(\widetilde{T}) = A$. □

2.2. Transitivity and Mixing

In this section, we will consider topological transitivity and topological mixing. A dynamical system (X, f) is said to be *topologically transitive* if for any pair of non-empty open sets U and V in X, there exists a non-negative integer n for which $f^n(U) \cap V \neq \phi$. In addition, if there exists an integer $N > 0$ with $f^n(U) \cap V \neq \phi$ for every $n \geq N$, then (X, f) is called *topologically mixing*. In the contrapositive sense, no topological transitivity ensures no topological mixing. Note that a factor of a mixing system is also mixing (see [3]).

On the other hand, (X, f) is said to have *point transitivity* if it has an element whose forward orbit is dense in X. It is known that if X has no isolated point, then any point transitive system (X, f) is topologically transitive (see [1]) and they are equivalent under some conditions on X; for instance, if X is locally compact perfect Hausdorff (see [3]). In fact, the authors in [3] consider the latter notion (i.e., point transitivity) as the definition of topological transitivity. Since $\mathbb{P}_n(\mathbb{R})$ is a connected, compact Hausdorff space, we will use these two notions without any distinction under the name transitivity.

There are several papers in literature on transitivity and mixing, particularly [9] and [12] are related to the current problem. In fact, the author in [12] hinted that the methods in that paper may help in discussing topological transitivity for projective transformations. Though the paper [9] does not mention the term transitivity explicitly, the concept of supercyclic vectors discussed in it is closely related to the transitivity of a projective transformation. We will be using that here and hence quote the necessary results. Let X be a real Banach space and $B(X)$ be the set of linear continuous mappings from X onto itself. A vector $x \in X$ is called a supercyclic vector of $T \in B(X)$ if $\overline{\{\lambda T^k(x) \mid \lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}_0\}} = X$. It is proved in Theorem 1 of [9] that there exist operators in $B(X)$ having supercyclic vectors if and only if $\dim X \in \{0, 1, 2\}$ or $\dim X = \infty$. We now state and prove our result about the relation between the existence of supercyclic vectors for T and the transitivity of $\widetilde{T}$.

Proposition 2.2. *Let $T \in GL_{n+1}(\mathbb{R})$. T has a supercyclic vector if and only if $\widetilde{T}$ is transitive on $\mathbb{P}_n(\mathbb{R})$.*

Proof. Assume that T has a supercyclic vector, say x i.e., $\overline{\{\lambda T^k(x) | \lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}_0\}} = \mathbb{R}^{n+1}$. Let U be a non-empty open set in $\mathbb{P}_n(\mathbb{R})$. Then, $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1} \setminus \{\bar{0}\}$. So, $\lambda T^k(x) \in \pi^{-1}(U)$ for some $\lambda \in \mathbb{R}$ and for some $k \in \mathbb{N}_0$. Thus, $\widetilde{T}^k([x]) \in U$.

For the converse, let $[x] \in \mathbb{P}_n(\mathbb{R})$ whose forward orbit is dense in $\mathbb{P}_n(\mathbb{R})$ and let V be a non-empty open set in $\mathbb{R}^{n+1}$. Choose $y \in V$ and an Euclidean ball $B_1 = B_E(y, \epsilon)$ such that $B_1 \subset V$. Define $W = \{tz | t \in \mathbb{R} \setminus \{0\}, z \in B_1\}$. The map $\phi_t : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ for any $t \neq 0$, defined by $\phi_t(u) = \frac{1}{t}u$ is continuous and thus the set $W_t := \{tz | z \in B_1\}$, being the pre-image of B_1 under ϕ_t is open. Since $W = \cup_{t \neq 0} W_t$, W is open. Also, W is saturated with respect to the map π i.e if $\pi^{-1}([u]) \cap W \neq \phi$ for some $[u] \in \mathbb{P}_n(\mathbb{R})$ then $\pi^{-1}([u]) \subset W$. Hence, $\pi(W)$ is open in $\mathbb{P}_n(\mathbb{R})$. Then, $\widetilde{T}^k([x]) \in \pi(W)$ for some k, implying that $\lambda T^k(x) \in W$ for every non-zero $\lambda \in \mathbb{R}$; in particular $T^k(x) \in W$ and thus $T^k(x) = tz$ for some non-zero $t \in \mathbb{R}$ and $z \in B_1$. It then follows that $\frac{1}{t}T^k(x) = z \in B_1 \subset V$ and hence $\{\lambda T^k(x) | \lambda \in \mathbb{R} \text{ and } k \in \mathbb{N}_0\}$ is dense in $\mathbb{R}^{n+1}$. □

Corollary 2.3. *$\mathbb{P}_n(\mathbb{R})$ admits a transitive projective transformation if and only if $n = 1$.*

The proof of the corollary follows from the above Proposition and Theorem 1 of [9].

Since every topologically mixing system is topologically transitive, it is enough to check the existence of topological mixing maps only on $\mathbb{P}_1(\mathbb{R})$. We prove in Theorem 2 that there exist no projective transformations on $\mathbb{P}_1(\mathbb{R})$ that are topologically mixing; hence $\mathbb{P}_n(\mathbb{R})$ does not admit a topologically mixing projective transformation for any $n \in \mathbb{N}$. However, Example 1 is of some interest, because it is a continous map of $\mathbb{P}_1(\mathbb{R})$ which is mixing; but is not a projective transformation i.e., not induced by a linear transformation of $\mathbb{R}^2$.

Theorem 2.4. *$\mathbb{P}_1(\mathbb{R})$ does not admit a topologically mixing projective transformation.*

Proof. Let $T \in GL_2(\mathbb{R})$. We can assume that T is equal to one of the following matrices:

(i) $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, where a and b are distinct real eigenvalues of T.

(ii) $\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ or $\begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$, where a is a real eigenvalue of T.

(iii) aR_θ where $a \in \mathbb{R} \setminus \{0\}$ and $R_\theta = \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ for some $\theta \in \mathbb{R}$.

Case (i): When $T = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, let $U' = \{(x, y) \in \mathbb{R}^2 \mid x > 0 \text{ and } y > 0\}$ and consider the open set $U = \pi(U')$ in $\mathbb{P}_1(\mathbb{R})$. If $ab > 0$, then for any $[(x, y)] \in U$, $\widetilde{T}^k([(x, y)]) = [(a^k x, b^k y)] \in U$ for every $k \in \mathbb{N}$. If $V = \pi(V')$, where $V' = \{(x, y) \in \mathbb{R}^2 \mid x < 0 \text{ and } y > 0\}$, then V is a non-empty open set such that $\widetilde{T}^k(U) \cap V = \phi$ for every $k \in \mathbb{N}$. Thus $\widetilde{T}$ is not mixing. If $ab < 0$, then for any even k, $\widetilde{T}^k([(x, y)]) \in U$ and thus again $\widetilde{T}$ is not mixing.

Case (ii): If $T = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$ then $\widetilde{T}([x]) = [ax] = [x]$, i.e. $\widetilde{T}$ is the identity map and hence not mixing. If

$T = \begin{pmatrix} a & 1 \\ 0 & a \end{pmatrix}$ then $\widetilde{T}^k([(x,y)]) = \left[(a^k x + na^{k-1}y, a^k y)\right]$. Consider the open sets $U = \pi(U')$ and $V = \pi(V')$, where $U' = \{(x,y) \mid x > 0 \text{ and } y > 0\}$ and $V' = \{(x,y) \mid x < 0 \text{ and } y > 0\}$. If $a > 0$, then $\widetilde{T}^k(U) \cap V = \phi$, for any $k \in \mathbb{N}$ and if $a < 0$, then $\widetilde{T}^k(U) \cap V = \phi$ for large enough odd values of k. Hence $\widetilde{T}$ is not mixing.

Case (iii): In this case, $\widetilde{T}$ is an isometry and hence it is not mixing. □

We thus have the following corollary.

Corollary 2.5. $\mathbb{P}_n(\mathbb{R})$ *does not admit a topologically mixing projective transformation for any* $n \in \mathbb{N}$.

Though there are no projective transformations on $\mathbb{P}_1(\mathbb{R})$ that are mixing, we can still have a mixing continuous map, as shown in the following example.

Example 2.6. *Consider the expanding endomorphism* $E_3 : S^1 \to S^1$ *given by* $E_3(e^{i\theta}) = e^{i3\theta}$. *Since* E_3 *is mixing in* S^1 *(refer to [3]),* $\widetilde{E_3}$ *being a factor of* E_3 *is also mixing.*

2.3. Distality and Proximality

We finally consider distality and proximality which are asymptotic dynamical attributes based on the distance between comparable positions on pairs of orbits. They are also dichotomic in nature. Let X be a compact Hausdorff topological space with a homeomorphism $f : X \to X$ and x, y be any two points of X. We define the diagonal set in $X \times X$ as $\Delta = \{(z,z) \in X \times X : z \in X\}$ and the orbit of (x,y) under $f \times f$ is denoted by $O(x,y)$. A pair of points $x, y \in X$ are called proximal if their orbit closure i.e. $\overline{O(x,y)}$ has a non-empty intersection with the diagonal set Δ, else they are known as distal. A homeomorphism on a space X is called distal if any two distinct points $x, y \in X$ are distal. If d is a metric on X, then $x, y \in X$ are proximal if and only if there exists a sequence n_k of integers such that $d(f^{n_k}(x), f^{n_k}(y))$ goes to zero as k tends to infinity. Note that an isometry is distal. We will also need the fact that a factor of a distal homeomorphism of a compact Hausdorff space is also distal (See Corollary 2.7.7, [3]).

Let T be an invertible linear transformation on $\mathbb{R}^{n+1}$. If $\widetilde{T}$ is an isometry on $\mathbb{P}_n(\mathbb{R})$, then it is obviously distal. We now prove in the following theorem that $\widetilde{T}$ is not distal in all other cases (with respect to d_p). We continue to assume that $T \in GL_{n+1}(\mathbb{R})$ and also use the following notations in the next theorem and its proof. A denotes an arbitary matrix of an appropriate order, I_2 stands for the identity matrix of order 2×2.

Theorem 2.7. $\widetilde{T}$ *is distal on* $\mathbb{P}_n(\mathbb{R})$ *if and only if* $\widetilde{T}$ *is an isometry with respect to* d_p.

Proof. An isometry is obviously distal; so, we now assume that $\widetilde{T}$ is distal and show that it is an isometry. We first claim that T is of the form $T = \bigoplus_{l=1}^{k} \alpha_l T_l$, where each $\alpha_l \in \mathbb{R}$, $|\alpha_i| = \left|\alpha_j\right|$ for any $i, j \in \{1, 2, \cdots, k\}$ and each T_l is an isometry (with respect to Euclidean norm) of either $\mathbb{R}$ or $\mathbb{R}^2$.

In case T is not of this form, we can assume that T is equal to one of the following :

(i) $\begin{pmatrix} J & I_2 & O \\ O & J & \cdots \\ O & O & A \end{pmatrix}$, where $J = \alpha.R_\theta$ for some $\alpha \in \mathbb{R} \setminus \{0\}$ and $\theta \in \mathbb{R}$.

(ii) $\begin{pmatrix} \lambda & 1 & O \\ 0 & \lambda & \cdots \\ O & O & A \end{pmatrix}$, where $\lambda \in \mathbb{R} \setminus \{0\}$.

(iii) $\begin{pmatrix} a & 0 & \cdots \\ 0 & b & \cdots \\ O & O & A \end{pmatrix}$, where $a, b \in \mathbb{R} \setminus \{0\}$, with $|a| \neq |b|$.

(iv) $\begin{pmatrix} J_1 & O & \cdots \\ O & J_2 & \cdots \\ O & O & A \end{pmatrix}$, where each $J_i = \alpha_i.R_{\theta_i}$, $\theta_i \in \mathbb{R}$ and $\alpha_i \in \mathbb{R} \setminus \{0\}$ such that $|\alpha_1| \neq |\alpha_2|$.

(v) $\begin{pmatrix} \lambda & O & \cdots \\ O & J & \cdots \\ O & O & A \end{pmatrix}$, where $\lambda \in \mathbb{R} \setminus \{0\}$, $J = \alpha.R_\theta$ such that $\alpha \in \mathbb{R} \setminus \{0\}$, $\theta \in \mathbb{R}$ and $|\lambda| \neq |\alpha|$.

In the first case, where $T = \begin{pmatrix} J & I_2 & O \\ O & J & \cdots \\ O & O & A \end{pmatrix}$, consider an element $(\overline{x}, \overline{y}, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, such that $\overline{x}, \overline{y} \in \mathbb{R}^2 \setminus \{(0,0)\}$. Note that $T^n(\overline{x}, \overline{y}, 0, \cdots, 0) = \left(\alpha^n R_\theta^n \overline{x} + n\alpha^{n-1} R_\theta^{n-1} \overline{y}, \alpha^n R_\theta^n \overline{y}, 0, \cdots, 0\right)$ and $T^n\left(\frac{R_\theta^{-1}\overline{y}}{\|R_\theta^{-1}\overline{y}\|}, 0, \cdots, 0\right) = \left(\frac{\alpha^n R_\theta^n R_\theta^{-1}\overline{y}}{\|R_\theta^{-1}\overline{y}\|}, 0, \cdots, 0\right)$.

Then, $\widetilde{T}^n\left[(\overline{x}, \overline{y}, 0, \cdots, 0)\right] = \left[\left(\frac{\frac{\alpha}{n}R_\theta^n\overline{x} + R_\theta^{n-1}\overline{y}}{\sqrt{\|\frac{\alpha}{n}R_\theta^n\overline{x} + R_\theta^{n-1}\overline{y}\|^2 + \|\frac{\alpha}{n}R_\theta^n\overline{y}\|^2}}, \frac{\alpha R_\theta^n\overline{y}}{\sqrt{\|\alpha R_\theta^n\overline{x} + nR_\theta^{n-1}\overline{y}\|^2 + \|\alpha R_\theta^n\overline{y}\|^2}}, 0, \cdots, 0\right)\right]$
and $\widetilde{T}^n\left[\left(\frac{R_\theta^{-1}\overline{y}}{\|R_\theta^{-1}\overline{y}\|}, 0, \cdots, 0\right)\right] = \left[\left(\frac{R_\theta^{n-1}\overline{y}}{\|R_\theta^{n-1}\overline{y}\|}, 0, \cdots, 0\right)\right]$.

Note that, as $n \to \infty$,

$$\left\| \frac{\frac{\alpha}{n}R_\theta^n\overline{x} + R_\theta^{n-1}\overline{y}}{\sqrt{\left\|\frac{\alpha}{n}R_\theta^n\overline{x} + R_\theta^{n-1}\overline{y}\right\|^2 + \left\|\frac{\alpha}{n}R_\theta^n\overline{y}\right\|^2}} - \frac{R_\theta^{n-1}\overline{y}}{\left\|R_\theta^{n-1}\overline{y}\right\|} \right\| \to 0$$

and
$$\left\| \frac{\alpha R_\theta^n\overline{y}}{\sqrt{\left\|\alpha R_\theta^n\overline{x} + nR_\theta^{n-1}\overline{y}\right\|^2 + \left\|\alpha R_\theta^n\overline{y}\right\|^2}} \right\| \to 0.$$

Hence, $d_p\left(\widetilde{T}^n\left[(\overline{x}, \overline{y}, 0, \cdots, 0)\right], \widetilde{T}^n\left[\left(\frac{R_\theta^{-1}\overline{y}}{\|R_\theta^{-1}\overline{y}\|}, 0, \cdots, 0\right)\right]\right) \to 0$ and therefore $\widetilde{T}$ is not distal.

For the second case, where $T = \begin{pmatrix} \lambda & 1 & O \\ O & \lambda & \cdots \\ O & O & A \end{pmatrix}$, let $(x, y, 0, \cdots, 0) \in \mathbb{R}^{n+1}$, such that $x, y \in \mathbb{R} \setminus \{0\}$.

Then, $\widetilde{T}^n\left[(x, y, 0, \cdots, 0)\right] = \left[\frac{(\lambda x + ny,\ \lambda y,\ 0, \cdots, 0)}{\sqrt{(\lambda x + ny)^2 + (\lambda y)^2}}\right]$ and $\widetilde{T}^n\left[\left(\frac{y}{\|y\|}, 0, 0, \cdots, 0\right)\right] = [(1, 0, 0, \cdots, 0)]$.

$$\text{Now, } \left\| \widetilde{T}^n\left[(x, y, 0, \cdots, 0)\right] - \widetilde{T}^n\left[\left(\frac{y}{\|y\|}, 0, 0, \cdots, 0\right)\right] \right\|$$
$$= \sqrt{\left(\frac{\lambda x + ny}{\sqrt{(\lambda x + ny)^2 + (\lambda y)^2}} - 1\right)^2 + \left(\frac{\lambda y}{\sqrt{(\lambda x + ny)^2 + (\lambda y)^2}}\right)^2} \to 0 \text{ as } n \to \infty.$$

Hence, $\widetilde{T}$ is not distal in this case also.

In the remaining cases, T is of the form $T = \bigoplus_{l=1}^k \alpha_l T_l$, where each T_l is an isometry (with respect to Euclidean norm) of either $\mathbb{R}$ or $\mathbb{R}^2$ and $|\alpha_i| \neq |\alpha_j|$ for some i and j. Without loss of generality, we assume that $i < j$ and $|\alpha_i| < |\alpha_j|$. Let η_l be the projection of $\mathbb{R}^{n+1}$ on to the domain of T_l for each $l \in \{1, 2, \cdots, k\}$; note that the range of each η_l is either either $\mathbb{R}$ or $\mathbb{R}^2$.

Take two elements $x, x' \in \mathbb{R}^{n+1}$ such that $\eta_l(x) = 0$ for every $l \notin \{i, j\}$, $\eta_l(x') = 0$ for every $l \neq j$, $\eta_i(x) \neq 0 \neq \eta_j(x)$ and finally $\eta_j(x') = \eta_j(x)$. Say $\eta_i(x) = \overline{x_i}$ and $\eta_j(x) = \eta_j(x') = \overline{x_j}$.

$$\text{Then, } \widetilde{T}^n([x]) = \left[\frac{\left(0, \cdots, 0,\ \alpha_i^n T_i^n \overline{x_i},\ 0, \cdots, 0,\ \alpha_j^n T_j^n \overline{x_j}, 0,\ \cdots, 0\right)}{\sqrt{\left\|\alpha_i^n T_i^n \overline{x_i}\right\|^2 + \left\|\alpha_j^n T_j^n \overline{x_j}\right\|^2}}\right]$$
$$\text{and } \widetilde{T}^n([x']) = \left[\left(0, \cdots, 0,\ \frac{T_j^n \overline{x_j}}{\left\|T_j^n \overline{x_j}\right\|}, 0, \cdots, 0\right)\right].$$

Note that, as $n \to \infty$, $\left\| \frac{\alpha_i^n T_i^n \overline{x_i}}{\sqrt{\|\alpha_i^n T_i^n \overline{x_i}\|^2 + \left\|\alpha_j^n T_j^n \overline{x_j}\right\|^2}} \right\| \to 0$ and $\left\| \frac{\alpha_j^n T_j^n \overline{x_j}}{\sqrt{\|\alpha_i^n T_i^n \overline{x_i}\|^2 + \left\|\alpha_j^n T_j^n \overline{x_j}\right\|^2}} - \frac{T_j^n \overline{x_j}}{\left\|T_j^n \overline{x_j}\right\|} \right\| \to 0$.

Hence $d_p(\widetilde{T}^n([x]), \widetilde{T}^n([x'])) \to 0$ and thus $\widetilde{T}$ is not distal.

Therefore, $T = \bigoplus_{l=1}^{k} \alpha_l T_l$, where each $\alpha_l \in \mathbb{R}$, each T_l is an isometry (with respect to Euclidean distance) of either $\mathbb{R}$ or $\mathbb{R}^2$ and $|\alpha_i| = \left|\alpha_j\right| = |\alpha|$ (say) for every $i, j \in \{1, 2, \cdots, k\}$. If $x = (\overline{x_1}, \overline{x_2}, \cdots, \overline{x_k})$ and $y = (\overline{y_1}, \overline{y_2}, \cdots, \overline{y_k})$ are in $\mathbb{R}^{n+1}$, with $\|x\| = \left\|y\right\| = 1$ and $\overline{x_l}, \overline{y_l}$ belong to the domain of T_l for each l, then $\frac{Tx}{\|Tx\|} = \frac{1}{|\alpha|}\left(\alpha_1 T_1 \overline{x_1}, \alpha_2 T_2 \overline{x_2}, \cdots, \alpha_k T_k \overline{x_k}\right)$ and $\frac{Ty}{\|Ty\|} = \frac{1}{|\alpha|}\left(\alpha_1 T_1 \overline{y_1}, \alpha_2 T_2 \overline{y_2}, \cdots, \alpha_k T_k \overline{y_k}\right)$.

$$\begin{aligned} \text{Thus}, \left\| \frac{Tx}{\|Tx\|} \pm \frac{Ty}{\left\|Ty\right\|} \right\| &= \frac{1}{|\alpha|} \left\| \alpha_1 T_1 \left(\overline{x_1} \pm \overline{y_1}\right), \alpha_2 T_2 \left(\overline{x_1} \pm \overline{y_2}\right), \cdots, \alpha_k T_k \left(\overline{x_k} \pm \overline{y_k}\right) \right\| \\ &= \frac{1}{|\alpha|} \sqrt{|\alpha_1|^2 \left\|\overline{x_1} \pm \overline{y_1}\right\|^2 + |\alpha_2|^2 \left\|\overline{x_2} \pm \overline{y_2}\right\|^2 + \cdots + |\alpha_k|^2 \left\|\overline{x_k} \pm \overline{y_k}\right\|^2} \\ &= \left\|x \pm y\right\|. \end{aligned}$$

Hence $d_p\left(\widetilde{T}[x], \widetilde{T}[y]\right) = d_p([x], [y])$ and therefore $\widetilde{T}$ is an isometry with respect to d_p. □

The above result is not true for all equivalent metrics i.e., a distal projective transformation need not be an isometry with respect to every metric that is equivalent to d_p. We now give an example to show this. As mentioned in the introduction, let $\|.\|$ denote the Euclidean metric on $\mathbb{R}^m$ for any m. On $\mathbb{R}^{n+1}$, define $\|(x_1, x_2, ..., x_{n+1})\|_1 = |x_1| + |x_2| + ... + |x_{n+1}|$ and for any $[x], [y] \in \mathbb{P}_n(\mathbb{R})$, define $d_t([x], [y]) = min\left\{ \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|_1, \left\| \frac{x}{\|x\|} + \frac{y}{\|y\|} \right\|_1 \right\}$. The map $(x, y) \mapsto \|x - y\|_1$ for every $x, y \in \mathbb{R}^{n+1}$ gives the well-known *taxi cab* metric on $\mathbb{R}^{n+1}$ and thus d_t can be shown to be a metric on $\mathbb{P}_n(\mathbb{R})$ using the ideas of the proof given for Proposition 1.2.

Note that $\|x - y\| \leq \|x - y\|_1 \leq \sqrt{n+1}.\|x - y\|$ for every $x, y \in \mathbb{R}^{n+1}$. Now, for any $[x], [y] \in \mathbb{P}_n(\mathbb{R})$, without loss of generality, we may assume that $\|x\| = \|y\| = 1$. Therefore $\|x - y\| \leq \|x - y\|_1 \leq \sqrt{n+1}.\|x - y\|$ and $\|x + y\| \leq \|x + y\|_1 \leq \sqrt{n+1}.\|x + y\|$ imply that $\min\{\|x - y\|, \|x + y\|\} \leq \min\{\|x - y\|_1, \|x + y\|_1\} \leq \sqrt{n+1}. \min\{\|x - y\|, \|x + y\|\}$. Hence $d_p([x], [y]) \leq d_t([x], [y]) \leq \sqrt{n+1}.d_p([x], [y])$. This shows that d_p and d_t are two equivalent metrics on $\mathbb{P}_n(\mathbb{R})$.

Consider $T = R_{\frac{\pi}{4}}$ and take $[(1, 0)], [(0, 1)] \in \mathbb{P}_1(\mathbb{R})$. Then $d_t(\widetilde{T}[(1, 0)], \widetilde{T}[(0, 1)]) = \sqrt{2}$ but $d_t([(1, 0)], [(0, 1)]) = 2$. Therefore, $\widetilde{T}$ is not an isometry with respect to d_t. On the other hand, it can be shown through usual arguments that it is an isometry with respect to d_p and hence distal. Since d_p and d_t are equivalent, it follows that $\widetilde{T}$ is distal with respect to d_t also.

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