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Odd primary homotopy types of $Spin(n)$-gauge groups over S^8

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In memory of Professor Mohammad Ali Asadi-Golmankhaneh

Abstract. Let G be one of $Sp(3)$, $Spin(7)$ or $Spin(8)$. Also, let P_k be the principal G-bundle over S^8 and $\mathcal{G}_k(G)$ be the gauge group of P_k classified by $k\varepsilon$, where ε a generator of $\pi_8(B(G)) \cong \mathbb{Z}$. In this article, localized at an odd prime p, we partially classify the homotopy types of $\mathcal{G}_k(G)$.

1. Introduction

Let G be a simply-connected, simple compact Lie group and B be a connected finite complex. Also let $P \to B$ be a principal G-bundle over B. We denote by $\mathcal{G}(P)$ the gauge group of this principal G-bundle, which is the group of G-equivariant automorphisms of P which fix B.

While there are countably many inequivalent principal G-bundles, Crabb and Sutherland [4] showed that their gauge groups have only finitely many distinct homotopy types. Let $P_k \to S^4$ represent the equivalence class of principal G-bundle whose second Chern class is k and $\mathcal{G}_k(G)$ be the gauge group of this principal G-bundle. In recent years there has been considerable interest in determining the precise number of homotopy types of these gauge groups and explicit classification results have been obtained. Also, many classifications for gauge groups of principal bundles with various base spaces and different structure groups have been done in these years. (for example, see [7], [14], [20], [24]).

For two integers a and b, let (a, b) be the their greatest common divisor. T. Cutler in [2] was studied the homotopy types of $\mathcal{G}_k(Sp(3))$. He showed that when localized at an odd prime p there is a homotopy equivalence $\mathcal{G}_k(Sp(3)) \simeq \mathcal{G}_{k'}(Sp(3))$ if and only if $(21, k) = (21, k')$. Also, he gave a lower bound and an upper bound on the number of 2-local homotopy types of $\mathcal{G}_k(Sp(3))$ and proved two following cases.
(i) if there is a homotopy equivalence $\mathcal{G}_k(Sp(3)) \simeq \mathcal{G}_{k'}(Sp(3))$ then $(84, k) = (84, k')$,
(ii) if $(336, k) = (336, k')$ then there is a local homotopy equivalence $\mathcal{G}_k(Sp(3)) \simeq \mathcal{G}_{k'}(Sp(3))$ after rationalisation or localisation at any prime.

B. Harris in [6] showed that for odd primes p there is a homotopy equivalence

$$Spin(2n+1) \simeq_{(p)} Sp(n).$$

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In 1975, E. M. Friedlander in [5] improved this result and showed that localized at odd primes there is a homotopy equivalence between corresponding classifying spaces, that is

$$BSpin(2n+1) \simeq_{(p)} BSp(n).$$

Therefore to study the homotopy type of $\mathcal{G}_k(Spin(7))$ it suffices to study $\mathcal{G}_k(Sp(3))$. Recently, S. Rea in [18] studied the homotopy type of $\mathcal{G}_k(Spin(n))$ when $n = 7$ and 8. He for $n = 7$ and 8 partially classified the homotopy types of $\mathcal{G}_k(Spin(n))$ by showing the following cases.
(i) if there is a homotopy equivalence $\mathcal{G}_k(Spin(7)) \simeq \mathcal{G}_{k'}(Spin(7))$ then $(84, k) = (84, k')$,
(ii) if $(168, k) = (168, k')$ then there is a local homotopy equivalence

$$\mathcal{G}_k(Spin(7)) \simeq \mathcal{G}_{k'}(Spin(7))$$

after rationalisation or localisation at any prime,
(iii) if there is a homotopy equivalence $\mathcal{G}_k(Spin(8)) \simeq \mathcal{G}_{k'}(Spin(8))$ then $(28, k) = (28, k')$,
(iv) if $(168, k) = (168, k')$ then there is a local homotopy equivalence

$$\mathcal{G}_k(Spin(8)) \simeq \mathcal{G}_{k'}(Spin(8))$$

after rationalisation or localisation at any prime.

The purpose of this article is to extend results in the direction of [2] and [18] by considering $B = S^8$ and $G = Sp(3), Spin(7)$ and $Spin(8)$. Let P_k be the principal G-bundle over S^8 classified by $k\varepsilon$, where ε a generator of $\pi_8(B(G)) \cong \mathbb{Z}$. Also, let $\mathcal{G}_k(G)$ be the gauge group of this principal G-bundle. We partially classify the homotopy types of $\mathcal{G}_k(G)$ for $G = Sp(3), Spin(7)$ and $Spin(8)$, when localized at an odd prime p. We don't study the 2-local homotopy types of these gauge groups, which is very hard and realistically out of reach. We will prove the following theorem.

Theorem 1.1. *Let G be one of* $Sp(3)$, $Spin(7)$ *or* $Spin(8)$ *. The following hold:*
(a) *if* $\mathcal{G}_k(G)$ *is homotopy equivalent to* $\mathcal{G}_{k'}(G)$ *then* $(105, k)$ *is equal to* $(105, k')$,
(b) *if* $(2835, k)$ *is equal to* $(2835, k')$ *then* $\mathcal{G}_k(G)$ *is homotopy equivalent to* $\mathcal{G}_{k'}(G)$.

2. Preliminaries

Let BG and $B\mathcal{G}_k(G)$ be the classifying spaces of G and $\mathcal{G}_k(G)$ respectively. Also, let $Map_k(S^8, BG)$ be the component of the space of continuous unbased maps from S^8 to BG which contains the map inducing P, similarly let $Map^*_k(S^8, BG)$ be the component of the space of pointed continuous maps from S^8 and BG which contains the map inducing P. We know that there is a fibration

$$Map^*_k(S^8, BG) \to Map_k(S^8, BG) \xrightarrow{ev} BG,$$

where the map ev is evaluation map at the basepoint of S^8. M. Atiyah and R. Bott in [1] have shown that there is a homotopy equivalence

$$B\mathcal{G}_k(G)) \simeq Map_k(S^8, BG).$$

The evaluation fibration therefore determines a homotopy fibration sequence

$$G \longrightarrow Map^*_k(S^8, BG) \to B\mathcal{G}_k(G) \xrightarrow{ev} BG. \tag{2.1}$$

By [19], it is well known that there is a homotopy equivalence

$$Map^*_k(S^8, BG) \simeq Map^*_0(S^8, BG).$$

We write $\Omega^7_0 G$ for $Map^*_0(S^8, BG)$, so we get the following fiber sequence

$$G \xrightarrow{\alpha_k} \Omega^7_0 G \longrightarrow B\mathcal{G}_k(G) \xrightarrow{ev} BG, \tag{2.2}$$

where α_k is the fibration connecting map.

Let H be a topological group. The commutator of H is the map $C\colon H \times H \to H$ defined by sending (h, h') to $hh'h^{-1}h'^{-1}$. The restriction of C to $H \vee H$ is trivial, so induces a map $c\colon H \wedge H \to H$. The Samelson product of two maps $f\colon X \to H$ and $g\colon Y \to H$ denoted by $\langle f, g\rangle$ is defined to be the composition

$$\langle f, g\rangle\colon X \wedge Y \xrightarrow{f\wedge g} H \wedge H \xrightarrow{c} H.$$

Let $\varepsilon\colon S^7 \to G$ represent a generator of $\pi_8(BG) \cong \pi_7(G)$ and let $1\colon G \to G$ be the identity map on G. For an H-space X, let $k\colon X \to X$ be the k^{th}-power map. By [12], we have the following lemma.

Lemma 2.1. *The adjoint of the connecting map $G \xrightarrow{\alpha_k} \Omega^7_0 G$ is homotopic to the Samelson product $S^7 \wedge G \xrightarrow{\langle k\varepsilon, 1\rangle} G$.* □

The linearity of the Samelson product implies that $\langle k\varepsilon, 1\rangle \simeq k\langle \varepsilon, 1\rangle$. Taking adjoints therefore implies the following.

Corollary 2.2. *The connecting map α_k satisfies $\alpha_k \simeq k \circ \alpha_1$.* □

Let Y be an H-space with a homotopy inverse, and let $k\colon Y \to Y$ be the k^{th}-power map. S. Theriault in [23] proved the following lemma that is very important in the determining the number of homotopy types of gauge groups.

Lemma 2.3. *Let X be a space and Y be an H-space with a homotopy inverse. Suppose there is a map $X \xrightarrow{f} Y$ of order m, where m is finite. Let F_k be the homotopy fiber of map $k \circ f$. If $(m, k) = (m, k')$ then F_k and $F_{k'}$ are homotopy equivalent when localized rationally or at any prime.* □

In the following lemmas, we collect some information from [17] and [8] regarding the homotopy groups of $Sp(3)$, respectively. We use these homotopy groups throughout the article.

Lemma 2.4. *The following hold:*

$\pi_7(Sp(3)) \cong \mathbb{Z}$, $\pi_{10}(Sp(3)) \cong 0$, $\pi_{11}(Sp(3)) \cong \mathbb{Z}$,
$\pi_{14}(Sp(3)) \cong \mathbb{Z}_{2\cdot 7!}$, $\pi_{17}(Sp(3)) \cong 0$, $\pi_{18}(Sp(3)) \cong \mathbb{Z}_{3\cdot 7!}$,
$\pi_{21}(Sp(3)) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_{12}$, $\pi_{22}(Sp(3)) \cong \mathbb{Z}_{\frac{11!}{5!}} \oplus \mathbb{Z}_2$. □

Lemma 2.5. *The following hold:*
(i) localized at 3, we have
$\pi_{24}(Sp(3)) \cong \mathbb{Z}_3$, $\pi_{25}(Sp(3)) \cong \mathbb{Z}_3$, $\pi_{28}(Sp(3)) \cong \mathbb{Z}_3$,
(ii) localized at 5, we have $\pi_{25}(Sp(3)) \cong 0$ and $\pi_{28}(Sp(3)) \cong 0$. □

The organization of this article is as follows. In Sections 3 and 4, we will give a lower bound and an upper bound for the number of homotopy types of gauge groups of principal $Sp(3)$-bundles over S^8, respectively. In Section 5, we will study the homotopy types of gauge groups of principal $Spin(8)$-bundles over S^8.

3. Lower bound on the number of homotopy types of $\mathcal{G}_k(Sp(3))$

Recall that the symplectic quasi projective space Q_2 has the cellular structure

$$Q_2 = S^3 \cup_{\nu_1} e^7,$$

where ν_1 is the attaching map for the top cell of Q_2 and represents a generator of $\pi_6(S^3) \cong \mathbb{Z}_{12}$. Our main goal in this section is to study the group $[\Sigma^7 Q_2, Sp(3)]$. Then we get a lower bound for the number of homotopy types of $\mathcal{G}_k(Sp(3))$ over S^8.

Since the dimension of Q_2 is equal to 7, we have

$$[Q_2, BSp(3)] \cong [Q_2, BSp(\infty)] \cong \widetilde{KSp}(Q_2).$$

The cofibration sequence $S^3 \to Q_2 \to S^7$ induces the following exact sequence

$$\to \widetilde{KSp}(S^7) \to \widetilde{KSp}(Q_2) \to \widetilde{KSp}(S^3) \to \cdots.$$

Since $\widetilde{KSp}(S^{4i-1}) = 0$ for all $i \geq 1$, this implies that $\widetilde{KSp}(Q_2) = 0$. Thus we get the following lemma.

Lemma 3.1. *There is an isomorphism* $[Q_2, BSp(3)] \cong 0$. □

Consider the homotopy fibration sequence

$$Sp(3) \xrightarrow{\alpha_k} \Omega_0^7 Sp(3) \longrightarrow B\mathcal{G}_k(Sp(3)) \xrightarrow{ev} BSp(3). \tag{3.1}$$

Apply the functor $[Q_2, -]$ to fibration (3.1) to obtain the exact sequence

$$[Q_2, Sp(3)] \xrightarrow{(\alpha_k)_*} [Q_2, \Omega_0^7 Sp(3)] \to [Q_2, B\mathcal{G}_k(Sp(3))] \to [Q_2, BSp(3)], \tag{3.2}$$

where by Lemma 3.1 we have $[Q_2, BSp(3)] \cong 0$. Note that

$$[Q_2, Sp(3)] \cong [\Sigma Q_2, BSp(3)] \cong \widetilde{KSp}(\Sigma Q_2).$$

Also by adjunction we have $[Q_2, \Omega_0^7 Sp(3)] \cong [\Sigma^7 Q_2, Sp(3)]$. Thus the exact sequence becomes

$$\widetilde{KSp}(\Sigma Q_2) \xrightarrow{(\alpha_k)_*} [\Sigma^7 Q_2, Sp(3)] \to [Q_2, B\mathcal{G}_k(Sp(3))] \to 0. \tag{3.3}$$

Note that localized at an odd prime, Q_2 is a co-H-space at primes $p \geq 3$, but it is known not to be homotopy coassociative at $p = 3$, so $[Q_2, B\mathcal{G}_k(Sp(3))]$ is only a set. Therefore we get the following lemma.

Lemma 3.2. *The set* $[Q_2, B\mathcal{G}_k(Sp(3))]$ *is equal to Coker*$(\alpha_k)_*$. □

The author in [15] has proved the following lemma.

Lemma 3.3. *Localized at an odd prime p, there is an isomorphism of sets*

$$[Q_2, B\mathcal{G}_k(Sp(3))] \cong \mathbb{Z}_{(105,k)}. \quad \square$$

We write $|H|$ for the cardinality of set H. In the following we prove the part (a) of Theorem 1.1 when $G = Sp(3)$.

Proof of part (a) of Theorem 1.1 for G = Sp(3)

Consider the homotopy cofibration sequence

$$S^6 \xrightarrow{\nu_1} S^3 \xrightarrow{i} Q_2 \xrightarrow{q} S^7 \xrightarrow{\Sigma\nu_1} S^4, \qquad (\star)$$

where the map i is the inclusion of the bottom cell and the map q is the pinch map to the top cell. Now apply the functor $[-, B\mathcal{G}_k(Sp(3))]$ to cofibration $(\star)$ to obtain the exact sequence of pointed sets

$$[S^4, B\mathcal{G}_k(Sp(3))] \xrightarrow{(\Sigma\nu_1)^*} [S^7, B\mathcal{G}_k(Sp(3))] \xrightarrow{q^*} [Q_2, B\mathcal{G}_k(Sp(3))] \longrightarrow [S^3, B\mathcal{G}_k(Sp(3))].$$

First, we show that $[S^3, B\mathcal{G}_k(Sp(3))]$ is equal to zero. Apply π_3 to homotopy fibration (3.1). By Lemma 2.4, we have $\pi_3(\Omega_0^7 Sp(3)) \cong \pi_{10}(Sp(3)) \cong 0$. Also, we know that $\pi_3(BSp(3))$ is zero, therefore we obtain $\pi_3(B\mathcal{G}_k(Sp(3))) \cong 0$. So we get the following exact sequence of pointed sets

$$[S^4, B\mathcal{G}_k(Sp(3))] \xrightarrow{(\Sigma\nu_1)^*} [S^7, B\mathcal{G}_k(Sp(3))] \xrightarrow{q^*} [Q_2, B\mathcal{G}_k(Sp(3))] \longrightarrow 0.$$

Note that $[S^4, B\mathcal{G}_k(Sp(3))]$ and $[S^7, B\mathcal{G}_k(Sp(3))]$ are groups and the map $(\Sigma\nu_1)^*$ is a group homomorphism. Therefore we get the following lemma.

Lemma 3.4. *There is a bijection between the set* $[Q_2, B\mathcal{G}_k(Sp(3))]$ *and the group* $Coker(\Sigma v_1)^*$. □

Suppose that $\mathcal{G}_k(Sp(3))$ is homotopy equivalent to $\mathcal{G}_{k'}(Sp(3))$. Consider the commutative diagram

$$\begin{array}{ccccccc} [S^4, B\mathcal{G}_k(Sp(3))] & \xrightarrow{(\Sigma v_1)^*} & [S^7, B\mathcal{G}_k(Sp(3))] & \xrightarrow{q^*} & [Q_2, B\mathcal{G}_k(Sp(3))] & \longrightarrow & 0 \\ \cong\downarrow & & \cong\downarrow & & & & \\ [S^4, B\mathcal{G}_{k'}(Sp(3))] & \xrightarrow{(\Sigma v_1)^*} & [S^7, B\mathcal{G}_{k'}(Sp(3))] & \xrightarrow{q^*} & [Q_2, B\mathcal{G}_{k'}(Sp(3))] & \longrightarrow & 0 \end{array}$$

where the top and bottom rows are exact sequences of pointed sets and the vertical isomorphisms are induced by adjunction. In the top and bottom rows, by Lemma 3.4 we have the set $[Q_2, B\mathcal{G}_k(Sp(3))]$ bijects with $Coker(\Sigma v_1)^*$ and the set $[Q_2, B\mathcal{G}_{k'}(Sp(3))]$ bijects with $Coker(\Sigma v_1)^*$, respectively. Therefore there is a bijection of sets

$$[Q_2, B\mathcal{G}_k(Sp(3))] \cong [Q_2, B\mathcal{G}_{k'}(Sp(3))].$$

Thus we have $|[Q_2, B\mathcal{G}_k(Sp(3))]| = |[Q_2, B\mathcal{G}_{k'}(Sp(3))]|$. Now by Lemma 3.3 we have that the set $[Q_2, B\mathcal{G}_k(Sp(3))]$ is isomorphic to $\mathbb{Z}_{(105,k)}$. Similarly, we have $[Q_2, B\mathcal{G}_{k'}(Sp(3))]$ is isomorphic to $\mathbb{Z}_{(105,k')}$. Therefore the bijection between $[Q_2, B\mathcal{G}_k(Sp(3))]$ and $[Q_2, B\mathcal{G}_{k'}(Sp(3))]$ implies that $(105, k) = (105, k')$. □

4. Upper bound on the number of homotopy types of $\mathcal{G}_k(Sp(3))$

In this section, localized at an odd prime p, we will study the order of Samelson product $S^7 \wedge Sp(3) \to Sp(3)$. This helps us to obtain an upper bound for the number of homotopy types of $\mathcal{G}_k(Sp(3))$. We denote the free abelian group with a generator e by $\mathbb{Z}\{e\}$. In the following, we obtain the p-local order of Samelson product $S^7 \wedge Sp(3) \to Sp(3)$, for $p = 3, 5$ and 7, respectively.

3-primary

In this part, all spaces and maps are to be localized at 3. We will use Toda notations $\alpha_1(n)$ and $\alpha_2(n)$ for the nontrivial maps $S^{n+3} \to S^n$ and $S^{n+7} \to S^n$, respectively. Consider the following cofibration sequence

$$(S^7 \wedge Q_3) \vee S^{17} \xrightarrow{f_2} S^7 \wedge Sp(3) \xrightarrow{f_3} (S^{21} \cup_{\alpha_1(21)} e^{25}) \vee S^{28}. \tag{4.1}$$

Note that this cofibration comes from including the 17-skeleton into $S^7 \wedge Sp(3)$ and by cellular structure of $S^7 \wedge Sp(3)$ the 17-skeleton is homotopy equivalent to $(S^7 \wedge Q_3) \vee S^{17}$. Now, apply $[-, Sp(3)]$ to cofibration (4.1), we get the following long exact sequence

$$\xrightarrow{f_4^*} [(S^{21} \cup_{\alpha_1(21)} e^{25}) \vee S^{28}, Sp(3)] \xrightarrow{f_3^*} [S^7 \wedge Sp(3), Sp(3)] \xrightarrow{f_2^*} [(S^7 \wedge Q_3) \vee S^{17}, Sp(3)] \xrightarrow{f_1^*} .(\star\star)$$

First, we calculate the groups $[(S^7 \wedge Q_3) \vee S^{17}, Sp(3)]$ and $[(S^{21} \cup_{\alpha_1(21)} e^{25}) \vee S^{28}, Sp(3)]$.

Lemma 4.1. *The order of the group* $[(S^7 \wedge Q_3) \vee S^{17}, Sp(3)]$ *is at most* 27.

Proof. Note that $[(S^7 \wedge Q_3) \vee S^{17}, Sp(3)] \cong [(S^7 \wedge Q_3), Sp(3)] \oplus \pi_{17}(Sp(3))$, where by Lemma 2.4 we have $\pi_{17}(Sp(3))$ is zero. By [2], there is a cofibration sequence

$$S^3 \xrightarrow{j} Q_3 \xrightarrow{q} S^7 \vee S^{11} \xrightarrow{(\alpha_1(4), \alpha_2(4))} S^4, \tag{4.2}$$

where the map j is the inclusion of the bottom cell. Now by applying $[S^7 \wedge -, Sp(3)]$ to cofibration (4.2), we get the exact sequence

$$\pi_{11}(Sp(3)) \xrightarrow{\alpha_1(11)^* \oplus \alpha_2(11)^*} \pi_{14}(Sp(3)) \oplus \pi_{18}(Sp(3)) \xrightarrow{q^*} [S^7 \wedge Q_3, Sp(3)] \xrightarrow{j^*} \pi_{10}(Sp(3)),$$

thus by Lemma 2.4 we obtain the exact sequence

$$\mathbb{Z} \xrightarrow{\alpha_1(11)^* \oplus \alpha_2(11)^*} \mathbb{Z}_9 \oplus \mathbb{Z}_{27} \xrightarrow{q^*} [S^7 \wedge Q_3, Sp(3)] \to 0.$$

Therefore we can conclude that the image of q^* has order at most 27, implying that the order of $[S^7 \wedge Q_3, Sp(3)]$ is at most 27. □

Lemma 4.2. *The group* $[(S^{21} \cup_{\alpha_1(21)} e^{25}) \vee S^{28}, Sp(3)]$ *is annihilated by multiplication by* 3.

Proof. We have $[(S^{21} \cup_{\alpha_1(21)} e^{25}) \vee S^{28}, Sp(3)] \cong [(S^{21} \cup_{\alpha_1(21)} e^{25}), Sp(3)] \oplus \pi_{28}(Sp(3))$, where by Lemma 2.5 we have $\pi_{28}(Sp(3))$ is isomorphic to $\mathbb{Z}_3$. Put $A = S^{21} \cup_{\alpha_1(21)} e^{25}$. By applying $[-, Sp(3)]$ to the cofibration $S^{21} \to A \to S^{25}$, we get the exact sequence

$$\pi_{22}(Sp(3)) \xrightarrow{\alpha_1(22)^*} \pi_{25}(Sp(3)) \to [A, Sp(3)] \to \pi_{21}(Sp(3)) \xrightarrow{\alpha_1(21)^*} \pi_{24}(Sp(3)).$$

Therefore by Lemmas 2.4 and 2.5 we get the exact sequence

$$\mathbb{Z}_{27} \xrightarrow{\alpha_1(22)^*} \mathbb{Z}_3 \to [A, Sp(3)] \to \mathbb{Z}_3 \xrightarrow{\alpha_1(21)^*} \mathbb{Z}_3.$$

We can by short exact sequences of homotopy groups of spheres and Symplectic groups show that the composition $\varepsilon^3_{21} \circ \alpha_1(21)$ is nontrivial. By applying π_{21} to the homotopy fibration $Sp(2) \to Sp(3) \xrightarrow{q} S^{11}$, we get the following short exact sequence

$$0 \to \mathbb{Z}_3 \xrightarrow{q_*} \mathbb{Z}_3 \to 0.$$

Therefore we obtain the composition $q \circ \varepsilon^3_{21}$ is nontrivial. On the other hand, by stable homotopy groups of spheres we have that the composition $S^{24} \xrightarrow{\alpha_1(21)} S^{21} \xrightarrow{\beta_1} S^{11}$ is nontrivial, where $\pi_{24}(S^{21}) \cong \mathbb{Z}_3\{\alpha_1(21)\}$ and $\pi_{21}(S^{11}) \cong \mathbb{Z}_3\{\beta_1\}$. Therefore as $\beta_1 \circ \alpha_1(21)$ is nontrivial stably, it must be the case that the composition $\varepsilon^3_{21} \circ \alpha_1(21)$ is nontrivial. Hence, as $\alpha_1(21)$ has order 3, so do $\varepsilon^3_{21} \circ \alpha_1(21)$. Thus $\varepsilon^3_{21} \circ \alpha_1(21)$ generate $\pi_{24}(Sp(3)) \cong \mathbb{Z}_3\{\varepsilon^3_{24}\}$. Since the map $\alpha_1(21)^*$ sends ε^3_{21} to ε^3_{24}, so $\alpha_1(21)^*$ is injective. Now, ignoring the image of $\alpha_1(22)^*$, this leaves an exact sequence

$$\mathbb{Z}_3 \to [A, Sp(3)] \to 0,$$

implying that $[A, Sp(3)]$ has order at most 3. As $\pi_{28}(Sp(3))$ is isomorphic to $\mathbb{Z}_3$, this implies that $[A \vee S^{28}, Sp(3)]$ is annihilated by multiplication by 3. □

Thus we rewrite ($\star\star$) as the following exact sequence

$$H \longrightarrow [S^7 \wedge Sp(3), Sp(3)] \longrightarrow G, \quad (\star\star\star)$$

where H is annilhilated by multiplication by 3 and G has order at most 27. Therefore we obtain the following proposition.

Proposition 4.3. *The* 3*-local order of Samelson product* $S^7 \wedge Sp(3) \to Sp(3)$ *is at most* 81.

Proof. The proof follows immediately from exactness in ($\star\star\star$). □

Here, we study the 5-local order of Samelson product $S^7 \wedge Sp(3) \to Sp(3)$.

5-primary

In this part, all spaces and maps are to be localized at 5. M. Mimura, G. Nishida and H. Toda in [16] showed that there is a following 5-local homotopy equivalence

$$Sp(3) \simeq B^2_1 \times S^7,$$

where B^2_1 is an S^3-bundle over S^{11}. Therefore we have the following isomorphism

$$\begin{aligned}[\Sigma^7 Sp(3), Sp(3)] &\cong [\Sigma^7(B^2_1 \times S^7), Sp(3)] \\ &\cong [S^{14} \vee \Sigma^7 B^2_1 \vee \Sigma^{14} B^2_1, Sp(3)] \\ &\cong \pi_{14}(Sp(3)) \oplus [\Sigma^7 B^2_1, Sp(3)] \oplus [\Sigma^{14} B^2_1, Sp(3)].\end{aligned}$$

Thus, to calculate the 5-local order of the Samelson product $S^7 \wedge Sp(3) \to Sp(3)$, we need the following lemma.

Lemma 4.4. *For $k = 7$ and 14, the group $[\Sigma^k B_1^2, Sp(3)]$ is equal to zero.*

Proof. According to the [9], it is known that B_1^2 has a cell decomposition

$$B_1^2 \simeq S^3 \cup e^{11} \cup e^{14}.$$

By method in [3], we have that the top cell splits off after a single suspension. This then gives us $\Sigma^7 B_1^2 \simeq (S^{10} \cup e^{18}) \vee S^{21}$ and $\Sigma^{14} B_1^2 \simeq (S^{17} \cup e^{25}) \vee S^{28}$. Put $A_1 = S^{10} \cup e^{18}$ and $A_2 = S^{17} \cup e^{25}$, thus we get

$$[\Sigma^7 B_1^2, Sp(3)] \cong [A_1 \vee S^{21}, Sp(3)] \cong [A_1, Sp(3)] \oplus \pi_{21}(Sp(3)),$$
$$[\Sigma^{14} B_1^2, Sp(3)] \cong [A_2 \vee S^{28}, Sp(3)] \cong [A_2, Sp(3)] \oplus \pi_{28}(Sp(3)).$$

Now apply $[-, Sp(3)]$ to the cofibrations

$$S^{10} \to A_1 \to S^{18} \xrightarrow{\alpha_2(11)} S^{11},$$
$$S^{17} \to A_2 \to S^{25} \xrightarrow{\alpha_2(18)} S^{18},$$

respectively. So we get the following exact sequences

$$\pi_{11}(Sp(3)) \xrightarrow{\alpha_2(11)^*} \pi_{18}(Sp(3)) \to [A_1, Sp(3)] \to \pi_{10}(Sp(3)),$$
$$\pi_{18}(Sp(3)) \xrightarrow{\alpha_2(18)^*} \pi_{25}(Sp(3)) \to [A_2, Sp(3)] \to \pi_{17}(Sp(3)),$$

respectively. By Lemmas 2.4 and 2.5, we get the following exact sequence

$$\mathbb{Z} \xrightarrow{\alpha_2(11)^*} \mathbb{Z}_5 \to [A_1, Sp(3)] \to 0.$$

We know that there is a 5-local homotopy equivalence $Sp(3) \simeq B_1^2 \times S^7$, Toda in [21] showed that the map $S^{11} \to B_1^2$ representing the generator of $\pi_{11}(B_1^2)$ is a 5-local homotopy equivalence in dimensions ≤ 18. Therefore localized at 5, as $\pi_{18}(S^7) \cong 0$, we have $\pi_{18}(Sp(3)) \cong \pi_{18}(B_1^2)$, so the map $S^{18} \xrightarrow{\alpha_2(11)} S^{11} \to Sp(3)$ is nontrivial because $\alpha_2(11)$ is nontrivial. Now the composition $\varepsilon_{11}^3 \circ \alpha_2(11)$ is nontrivial and generate $\pi_{18}(Sp(3)) \cong \mathbb{Z}_5\{\varepsilon_{18}^3\}$. Since $\alpha_2(11)^*$ sends ε_{11}^3 to ε_{18}^3, so $\alpha_2(11)^*$ is surjective. Therefore we can conclude that the group $[A_1, Sp(3)]$ is isomorphic to zero. Also, since $\pi_{25}(Sp(3))$ and $\pi_{17}(Sp(3))$ are isomorphic to zero, we get the group $[A_2, Sp(3)]$ is zero. Also, we have $\pi_{21}(Sp(3)) \cong \pi_{28}(Sp(3)) \cong 0$. Thus we get $[\Sigma^7 B_1^2, Sp(3)] \cong [\Sigma^{14} B_1^2, Sp(3)] \cong 0$. □

On the other hand, by Lemma 2.4 we have $\pi_{14}(Sp(3)) \cong \mathbb{Z}_5$. Therefore we get the following proposition.

Proposition 4.5. *The 5-local order of Samelson product $S^7 \wedge Sp(3) \to Sp(3)$ is 5.* □

7-primary
According to the [10], we have the following proposition.

Proposition 4.6. *Localized at 7, if $(7, k) = (7, k')$ then $\mathcal{G}_k(Sp(3)) \simeq \mathcal{G}_{k'}(Sp(3))$.* □

Therefore according to the Propositions 4.3, 4.5 and 4.6, we get the following theorem.

Theorem 4.7. *Localized at an odd prime p, the order of Samelson product $S^7 \wedge Sp(3) \to Sp(3)$ is at most $2835 = 3^4 \cdot 5 \cdot 7$.* □

Note that for primes $p > 7$, by [16] there is a homotopy equivalence

$$Sp(3) \simeq S^3 \times S^7 \times S^{11}.$$

Therefore we get the following isomorphism

$$\begin{aligned}[\Sigma^7 Sp(3), Sp(3)] &\cong [\Sigma^7(S^3 \times (S^7 \times S^{11})), Sp(3)] \\ &\cong \pi_{10}(Sp(3)) \oplus \pi_{14}(Sp(3)) \oplus \pi_{17}(Sp(3)) \oplus \pi_{18}(Sp(3)) \\ &\oplus \pi_{21}(Sp(3)) \oplus \pi_{25}(Sp(3)) \oplus \pi_{28}(Sp(3)).\end{aligned}$$

By [8] and Lemma 2.4 the homotopy groups in the displayed equation are all zero. Therefore we conclude the order of Samelson product $S^7 \wedge Sp(3) \to Sp(3)$ is trivial. According to the exact sequence ($\star\star$), for primes $p > 7$, we can obtain this result, also.

Proof of part (b) of Theorem 1.1 for G=Sp(3)

Localized at an odd prime p, by Theorem 4.7 and Lemma 2.3 we can conclude that if $(2835, k) = (2835, k')$ then $\mathcal{G}_k(Sp(3)) \simeq \mathcal{G}_{k'}(Sp(3))$. □

Proof of Theorem 1.1 for G=Spin(7)

All of the material for the $Spin(7)$ case follows immediately from Friedlander's odd primary homotopy equivalence $BSpin(7) \simeq BSp(3)$. Therefore the proof of Theorem 1.1 for the $Spin(7)$ case immediately follows from the $Sp(3)$ case. □

5. *Spin*(8)-gauge group

In this section, we study the homotopy types of $Spin(8)$-gauge group over S^8 by giving a lower bound and an upper bound for the number of homotopy types of $\mathcal{G}_k(Spin(8))$. The following lemma is an important role in determining the number of homotopy types of $\mathcal{G}_k(Spin(8))$ that was proved in [18].

Lemma 5.1. *Let $F \to X \to Y$ be a homotopy fibration, where F is an H-space, and let $\lambda\colon \Omega Y \to F$ be the homotopy fibration connecting map. Let $\lambda'\colon A \to \Omega Y$ and $\lambda''\colon B \to \Omega Y$ be maps such that*
(i) the composition $\mu \circ (\lambda' \times \lambda'')\colon A \times B \to \Omega Y$ is a homotopy equivalence, where μ is the loop multiplication on ΩY,
(ii) the composition $\lambda \circ \lambda''\colon B \to F$ is null-homotopic.
Then the orders of maps λ and $\lambda \circ \lambda'$ are equal. □

We have the following lemma.

Lemma 5.2. *Localized at an odd prime p, the map $Spin(8) \to \Omega^7_0 Spin(8)$ has order at most $2835 = 3^4 \cdot 5 \cdot 7$.*

Proof. Consider the homotopy fibration sequence

$$Spin(8) \xrightarrow{\alpha''_k} \Omega^7_0 Spin(8) \longrightarrow B\mathcal{G}_k(Spin(8)) \xrightarrow{ev} B(Spin(8)), \tag{5.1}$$

where α''_k is the fibration connecting map. Localized at prime p, there is a fibration

$$Spin(7) \xrightarrow{\lambda'} Spin(8) \to S^7,$$

that is split. Thus there is a homotopy equivalence

$$Spin(8) \simeq Spin(7) \times S^7.$$

Note that the following composition

$$Spin(7) \times S^7 \xrightarrow{\lambda' \times \lambda''} Spin(8) \times Spin(8) \xrightarrow{\mu} Spin(8),$$

is a homotopy equivalence, where λ'' is a homotopy inverse for the map $Spin(8) \to S^7$. On the other hand, $\pi_{14}(Spin(8)) \cong \pi_{14}(Spin(7)) \times \pi_{14}(S^7)$. By [13] we know that $\pi_{14}(Spin(7)) \cong \mathbb{Z}_{2520} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2$. Also by [22], we have $\pi_{14}(S^7) \cong \mathbb{Z}_{120}$. Therefore $\pi_{14}(Spin(8)) \cong \mathbb{Z}_{2520} \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_{120}$.

First, we consider the 7-local order of map $Spin(8) \to \Omega_0^7 Spin(8)$. Since $\alpha''_7 \simeq 7 \circ \alpha''_1$, the composition $S^7 \xrightarrow{\lambda''} Spin(8) \xrightarrow{\alpha''_7} \Omega_0^7(Spin(8))$ is null-homotopic. Therefore by Lemma 5.1 we obtain the orders of maps α''_7 and $\alpha''_7 \circ \lambda'$ are equal. Thus we will need to calculate the order of map $\alpha''_7 \circ \lambda'$. Consider the following diagram

$$\begin{array}{ccc} Spin(7) & \xrightarrow{\alpha'_7} & \Omega_0^7 Spin(7) \\ {\scriptstyle \lambda'}\downarrow & & \downarrow{\scriptstyle \Omega^7\lambda'} \\ Spin(8) & \xrightarrow{\alpha''_7} & \Omega_0^7 Spin(8). \end{array}$$

By Theorem 4.7, we have that $\alpha'_7 \simeq 7 \circ \alpha'_1$ is null-homotopic. Therefore we can conclude the map $\alpha''_7 \circ \lambda'$ is null-homotopic, also. Thus the 7-local order of map $Spin(8) \to \Omega_0^7 Spin(8)$ is at most 7. Similarly, we can obtain the 5-local and 3-local orders of the map $Spin(8) \to \Omega_0^7 Spin(8)$ are at most 5 and 81, respectively. □

Note that any map $Q_2 \to BSpin(8)$ lifts through $BSpin(7) \to BSpin(8)$ induces an epimorphism $[Q_2, BSpin(7)] \to [Q_2, BSpin(8)]$. Also, note that the induced epimorphism on homotopy sets comes from the fact that Q_2 is 7-dimensional while $BSpin(7) \to BSpin(8)$ induces an isomorphism on π_m for $1 \leq m \leq 6$ and an epimorphism on π_7. We recall that $[Q_2, BSpin(7)] \cong [Q_2, BSp(3)] \cong 0$, so we get the following lemma.

Lemma 5.3. *There is an isomorphism* $[Q_2, BSpin(8)] \cong 0$. □

Consider the homotopy cofibration sequence

$$S^9 \xrightarrow{g} Q_2 \to Sp(2) \xrightarrow{\pi'} S^{10}, \quad (*)$$

where the maps g and π' are the attaching map for the top cell and the pinch map to the top cell, respectively. Applying the functor $[-, B\mathcal{G}_k(Spin(8))]$ to cofibration $(*)$, there is an exact sequence of pointed sets

$$[S^{10}, B\mathcal{G}_k(Spin(8))] \xrightarrow{(\pi')^*} [Sp(2), B\mathcal{G}_k(Spin(8))] \to [Q_2, B\mathcal{G}_k(Spin(8))] \xrightarrow{g^*} [S^9, B\mathcal{G}_k(Spin(8))].$$

Localized at an odd prime p, apply π_9 to homotopy fibration (5.1). By [13] and [22], we know that the groups $\pi_8(Spin(7)), \pi_{16}(Spin(7))$ and $\pi_8(S^7), \pi_{16}(S^7)$ are zero, respectively. Thus we have

$$\pi_9(\Omega_0^7 Spin(8)) \cong \pi_{16}(Spin(8)) \cong \pi_{16}(Spin(7)) \times \pi_{16}(S^7) \cong 0,$$
$$\pi_9(BSpin(8)) \cong \pi_8(Spin(8)) \cong \pi_8(Spin(7)) \times \pi_8(S^7) \cong 0.$$

Therefore we obtain $\pi_9(B\mathcal{G}_k(Spin(8)))$ is zero. Therefore we get the following lemma.

Lemma 5.4. *Localized at an odd prime p, there is a bijection between the set $[Q_2, B\mathcal{G}_k(Spin(8)]$ and the group $Coker(\pi')^*$.* □

We have the following lemma.

Lemma 5.5. *Localized at an odd prime p, if $\mathcal{G}_k(Spin(8)$ is homotopy equivalent to $\mathcal{G}_{k'}(Spin(8)$ then we have $(105, k) = (105, k')$.*

Proof. Apply the functor $[Q_2, -]$ to fibration (5.1) to obtain the following exact sequence

$$[Q_2, Spin(8)] \xrightarrow{(\alpha''_k)_*} [Q_2, \Omega_0^7 Spin(8)] \to [Q_2, B\mathcal{G}_k(Spin(8))] \to [Q_2, BSpin(8)], \quad (5.2)$$

where by Lemma 5.3, we have $[Q_2, BSpin(8)] \cong 0$. Thus the set $[Q_2, B\mathcal{G}_k(Spin(8))]$ bijects with $Coker(\alpha''_k)_*$. By adjunction, $[Q_2, \Omega_0^7 Spin(8)] \cong [\Sigma^7 Q_2, Spin(8)]$. On the other hand, we have the isomorphism

$$[\Sigma^7 Q_2, Spin(8)] \cong [\Sigma^7 Q_2, Spin(7) \times S^7] \cong [\Sigma^7 Q_2, Spin(7)] \oplus [\Sigma^7 Q_2, S^7].$$

Therefore we get the decomposition $(\alpha''_k)_* = (\alpha'_k)_* \oplus (\beta_k)_*$, where

$$(\alpha'_k)_* \colon [Q_2, Spin(7)] \longrightarrow [\Sigma^7 Q_2, Spin(7)], \qquad (\beta_k)_* \colon [Q_2, S^7] \longrightarrow [\Sigma^7 Q_2, S^7].$$

Therefore we can conclude $Coker(\alpha''_k)_* \cong Coker(\alpha'_k)_* \oplus Coker(\beta_k)_*$. By Lemmas 3.2 and 3.3, we have $Coker(\alpha'_k)_* \cong \mathbb{Z}_{(105,k)}$. We need to calculate the $Coker(\beta_k)_*$, for this, we calculate the cohomotopy group $[\Sigma^7 Q_2, S^7]$. First, localized at 3, by using the following method, we show that $[\Sigma^7 Q_2, S^7] \cong \mathbb{Z}_9$. Consider the homotopy cofibration diagram that rows are cofibrations

$$\begin{array}{ccccccc} S^{13} & \xrightarrow{\Sigma^7\nu'} & S^{10} & \longrightarrow & \Sigma^7 Q_2 & \longrightarrow & S^{14} \\ \downarrow{\scriptstyle 2} & & \downarrow{\scriptstyle \|} & & \downarrow & & \downarrow{\scriptstyle \|} \\ S^{13} & \xrightarrow{\nu_{10}} & S^{10} & \longrightarrow & \Sigma^6 \mathbb{H}P^2 & \longrightarrow & S^{14} \end{array} \tag{5.3}$$

where by relation (5.5) in [22] we have $2\nu_{10} = \Sigma^7\nu'$. By [22], we know that the groups $\pi_{11}(S^7), \pi_{13}(S^7)$ are zero and $\pi_{10}(S^7) \cong \mathbb{Z}_3$. By applying $[-, S^7]$ to diagram (5.3), we get the following diagram that rows are exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \pi_{14}(S^7) \cong \mathbb{Z}_3 & \longrightarrow & [\Sigma^7 Q_2, S^7] & \longrightarrow & \pi_{10}(S^7) \cong \mathbb{Z}_3 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \cong} & & \downarrow & & \downarrow{\scriptstyle \cong} & & \\ 0 & \longrightarrow & \pi_{14}(S^7) \cong \mathbb{Z}_3 & \longrightarrow & [\Sigma^6 \mathbb{H}P^2, S^7] & \longrightarrow & \pi_{10}(S^7) \cong \mathbb{Z}_3 & \longrightarrow & 0. \end{array}$$

The Five Lemma therefore implies that there is an isomorphism $[\Sigma^7 Q_2, S^7] \cong [\Sigma^6 \mathbb{H}P^2, S^7]$. Now, by Theorem 1.2 in [11] we have that the cohomotopy group $[\Sigma^6 \mathbb{H}P^2, S^7]$ is isomorphic to $\mathbb{Z}_9$. Therefore the cohomotopy group $[\Sigma^7 Q_2, S^7]$ is also isomorphic to $\mathbb{Z}_9$. Also, it is obvious that localized at 5, the group $[\Sigma^7 Q_2, S^7]$ is isomorphic to $\pi_{14}(S^7) \cong \mathbb{Z}_5\{\alpha_2(7)\}$. Therefore, localized at an odd prime p, we obtain the group $[\Sigma^7 Q_2, S^7]$ is isomorphic to $\mathbb{Z}_9 \oplus \mathbb{Z}_5$. Since the map $(\beta_k)_* \colon \mathbb{Z} \to \mathbb{Z}_9 \oplus \mathbb{Z}_5$ is surjective, so we can conclude $Coker(\beta_k)_*$ is isomorphic to zero. Therefore we get $Coker(\alpha''_k)_*$ is isomorphic to $\mathbb{Z}_{(105,k)}$.

Now suppose that $\mathcal{G}_k(Spin(8)) \simeq \mathcal{G}_{k'}(Spin(8))$. By Lemma 5.4, localized at an odd prime p, the set $[Q_2, B\mathcal{G}_k(Spin(8))]$ bijects with $Coker(\pi')^*$. Similar to the discussion in the proof of part (a) of Theorem 1.1 for $G = Sp(3)$, localized at an odd prime p, there is an isomorphism of sets $[Q_2, B\mathcal{G}_k(Spin(8))] \cong [Q_2, B\mathcal{G}_{k'}(Spin(8))]$. We have that the set $[Q_2, B\mathcal{G}_k(Spin(8))]$ is isomorphic to $\mathbb{Z}_{(105,k)}$. Similarly, $[Q_2, B\mathcal{G}_{k'}(Spin(8))]$ is isomorphic to $\mathbb{Z}_{(105,k')}$. Thus the bijection between $[Q_2, B\mathcal{G}_k(Spin(8))]$ and $[Q_2, B\mathcal{G}_{k'}(Spin(8))]$ implies that $(105, k) = (105, k')$. □

Here, we prove Theorem 1.1 for $G = Spin(8)$.

Proof of Theorem 1.1 for G = Spin(8)

By Lemma 5.5, we have part (a). For part (b), by lemmas 2.3 and 5.2 we can conclude that if $(2835, k)$ is equal to $(2835, k')$ then $\mathcal{G}_k(Spin(8)) \simeq \mathcal{G}_{k'}(Spin(8))$. □

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References

[1] M. F. Atiyah, R. Bott, *The Yang-Mills equations over Riemann Surfaces*, Philos. Trans. Roy. Soc. London Ser. A **308** (1983), 523–615.
[2] T. Cutler, *The homotopy type of Sp(3)-gauge groups*, Topology and its Applications. **236** (2018), 44–58.
[3] G. Cooke, L. Smith, *Mod p decompositions of co-H-spaces and applications*, Math. Z. **157** (1977), 155–177.
[4] M. C. Crabb, W. A. Sutherland, *Counting homotopy types of gauge groups*, Proc. London Math. Soc. **83** (2000), 747–768.

[5] E. M. Friedlander, *Exceptional isogenies and the classifying spaces of simple Lie groups*, Ann. Math. **101** (1975), 510–520.
[6] B. Harris, *On the homotopy groups of the classical groups*, Ann. Math. **74** (1961), 407–413.
[7] H. Hamanaka, A. Kono, *Homotopy type of gauge groups of* $SU(3)$*-bundles over* S^6, Topology and its Applications. **154** (2007), 1377–1380.
[8] H. Imanishi, *Unstable Homotopy Groups of Classical Groups (odd primary components)*, J. Math. Kyoto Univ. **7-3** (1967), 221–243.
[9] I. James, J. Whitehead, *The homotopy theory of sphere-bundles over spheres I*, Proc. Lond. Math. Soc. **4** (1954), 196–218.
[10] D. Kishimoto, A. Kono, M. Tsutaya, *On p-local homotopy types of gauge groups*, Proc. Roy. Soc. Edinburgh, Sect. A **144** (2014), 149–160.
[11] H. Kachi, J. Mukai, T. Nozaki, Y. Sumita, D. Tamaki, *Some cohomotopy groups of suspended projective planes*, Math. J. Okayama Univ. **43** (2001), 105–121.
[12] G. E. Lang, *The evaluation map and EHP sequences*, Pacific J. Math. **44** (1973), 201–210.
[13] M. Mimura, *The homotopy groups of Lie groups of low rank*, J. Math. Kyoto Univ. **6** (1967), no. 2, 131–176.
[14] S. Mohammadi, *The homotopy types of* $SU(n)$*-gauge groups over* S^{2m}, Homology, Homotopy and Applications. **24(1)** (2022), 55–70.
[15] S. Mohammadi, *The homotopy type of* $Sp(n)$*-gauge groups over* S^{4m}, arXiv:2204.08867.
[16] M. Mimura, G. Nishida, H. Toda, *Mod p decomposition of compact Lie groups*, Publ. Res. Inst. Math. Sci. **13** (1977), 627–680.
[17] M. Mimura, H. Toda, *Homotopy groups of symplectic groups*, J. Math. Kyoto Univ. **3** (1963), 251–273.
[18] S. Rea, *Homotopy type of* $Spin^c(n)$*-gauge groups over* S^4, arXiv:2107.06000.
[19] W. Sutherland, *Function spaces related to gauge groups*, Proc. Roy. Soc. Edinburgh Sect. A **121** (1992), 185–190.
[20] T. So, S. D. Theriault, *The homotopy types of* $Sp(2)$*-gauge groups over closed, simply-connected four-manifolds*, Proceedings of the Steklov Institute of Mathematics. **305** (2019), 309–329.
[21] H. Toda, *On homotopy groups of* S^3*-bundles over Spheres*, Kyoto J. Math. **2-2** (1963), 193–207.
[22] H. Toda, *Composition Methods in Homotopy Groups of Spheres*, Annals of Mathematics Studies 49, Princeton University Press, Princeton N.J., 1962.
[23] S. D. Theriault, *The homotopy types of* $Sp(2)$*-gauge groups*, Kyoto J. Math. **50** (2010), 591–605.
[24] S. D. Theriault, *The homotopy types of* $SU(3)$*-gauge groups over simply connected* 4*-manifolds*, Publ. Res. Inst. Math. Sci. **48** (2012), 543–563.