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Some Definite Integrals Involving Elliptic Integrals in Association with Hypergeometric Functions

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Abstract. In this paper, the authors make use of the Gamma function as well as the hypergeometric and the generalized hypergeometric functions in order to investigate and develop several definite integrals involving the elliptic integrals of the first and the second kind. The numerical approximation of these definite integrals and the corresponding hypergeometric functions are also presented. The results derived in this article are believed to be new and extend and unify those that are available in the scientific literature.

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1. Introduction, Definitions and Preliminaries

Following the usual notations and conventions, the complete elliptic integrals $K(k)$ and $E(k)$ of the first and the second kind, respectively,

$$\begin{aligned} K(k) &= \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\theta}{\sqrt{1-k^2\sin^2\theta}} \\ &= \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \qquad \left(k^2<1\right) \end{aligned} \tag{1}$$

and

$$\begin{aligned} E(k) &= \int_0^{\frac{\pi}{2}} \sqrt{1-k^2\sin^2\theta}\,\mathrm{d}\theta \\ &= \int_0^1 \sqrt{\frac{1-k^2t^2}{1-t^2}}\,\mathrm{d}t \qquad \left(k^2<1\right), \end{aligned} \tag{2}$$

where k denotes the *modulus* and $\kappa := \sqrt{1-k^2}$ is referred to as the *complementary modulus* of $K(k)$ and $E(k)$ (see, for details, [1] and [4]).

In the theory of hypergeometric functions, the celebrated Gauss hypergeometric function ${}_2F_1$ (see [9]) in honor of the German mathematician (Johann) Carl Friedrich Gauss (1777–1855) happens to be the special case $p-1=q=1$ of the generalized hypergeometric function ${}_pF_q$ with p numerator parameters $\alpha_j \in \mathbb{C}$ $(j=1,\cdots,p)$ and q denominator parameters $\beta_j \in \mathbb{C}\setminus\mathbb{Z}_0^-$ $(j=1,\cdots,q)$. Indeed, in terms of the general Pochhammer symbol or the *shifted factorial* $(\lambda)_\nu$, since

$$(1)_n = n! \qquad \left(n\in\mathbb{N}_0\right),$$

which is defined (for $\lambda,\nu\in\mathbb{C}$), in terms of the familiar (Euler's) Gamma function, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & \left(\nu=0;\ \lambda\in\mathbb{C}\setminus\{0\}\right) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & \left(\nu=n\in\mathbb{N};\ \lambda\in\mathbb{C}\right), \end{cases} \tag{3}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists, we have (see, for details, [14] and [16])

$$\begin{aligned} {}_pF_q\left[\begin{matrix} \alpha_1,\cdots,\alpha_p; \\ \beta_1,\cdots,\beta_q; \end{matrix}\ z\right] &:= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n\cdots(\alpha_p)_n}{(\beta_1)_n\cdots(\beta_q)_n}\,\frac{z^n}{n!} \\ &=: {}_pF_q\left(\alpha_1,\cdots,\alpha_p;\beta_1,\cdots,\beta_q;z\right), \end{aligned} \tag{4}$$

in which the infinite series
(i) converges absolutely for $|z|<\infty$ if $p \leqq q$,
(ii) converges absolutely for $|z|<1$ if $p=q+1$, and
(iii) diverges for all z $(z\neq 0)$ if $p>q+1$.
Furthermore, if we set

$$\omega := \sum_{j=1}^{q}\beta_j - \sum_{j=1}^{p}\alpha_j,$$

then it is known that the generalized hypergeometric ${}_pF_q$ series in (1) (with $p = q + 1$) is
I. absolutely convergent for $|z| = 1$ if $\Re(\omega) > 0$,
II. conditionally convergent for $|z| = 1$ $(z \neq 1)$ if $-1 < \Re(\omega) \leqq 0$, and
III. divergent for $|z| = 1$ if $\Re(\omega) \leqq -1$.

In particular, when $p - 1 = q = 2$, the function ${}_3F_2$ is known as the Clausen hypergeometric function (see [5]) in honor of the Danish mathematician and astronomer, Thomas Clausen (1801–1885).

In terms of the Gauss hypergeometric function ${}_2F_1$, the complete elliptic integrals $K(k)$ and $E(k)$ can be expressed as follows:

$$K(k) = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right) \tag{5}$$

and

$$E(k) = \frac{\pi}{2}\, {}_2F_1\left(\frac{1}{2}, -\frac{1}{2}; 1; k^2\right), \tag{6}$$

respectively. Equivalently, we have the following explicit representations:

$$K(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}\,(n!)^2}\right)^2 k^{2n} = \frac{\pi}{2} \sum_{n=0}^{\infty} [P_{2n}(0)]^2\, k^{2n} \tag{7}$$

and

$$E(k) = \frac{\pi}{2} \sum_{n=0}^{\infty} \left(\frac{(2n)!}{2^{2n}\,(n!)^2}\right)^2 \frac{k^{2n}}{1-2n}, \tag{8}$$

respectively, $P_n(x)$ being the Legendre polynomial of degree n in x, which is defined by (see, for details, [18])

$$P_n(x) := {}_2F_1\left(-n, n+1; 1; \frac{1-x}{2}\right) = (-1)^n\, P_n(-x). \tag{9}$$

In the year 1983, by evaluating the first term in a certain Born series in two different ways and comparing the resulting expression, Barton [2] found the following integral formula:

$$\int_0^1 k^{\mu}\, K(\kappa)\, \mathrm{d}k = \frac{\pi}{4} \left(\frac{\Gamma\left(\frac{1}{2}\,\mu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,\mu + 1\right)}\right)^2 \qquad \left(\mu > -1;\ \kappa := \sqrt{1-k^2}\right). \tag{10}$$

Subsequently, while addressing Barton's problem of finding a *direct* proof of his formula (10), Bushell [3] not only proved Barton's formula (10) *directly*, but also derived a number of additional results analogous to (10), thereby extending several known integral formulas which were recorded by, for example, Byrd and Friedman [4, p. 274] (see also Müller [12] and Kaplan [11]).

The following interesting generalization of Barton's integral formula (10) was given by Bushell [3, p. 2, Eq. (2.2)]:

$$\int_0^1 k^\mu \, H(\kappa, \gamma) \, \mathrm{d}k = \frac{\pi}{4} \, \frac{\Gamma\left(\frac{1}{2}\,\mu + \frac{1}{2}\right) \, \Gamma\left(\gamma + \frac{1}{2}\,\mu + \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\,\mu + 1\right) \, \Gamma\left(\gamma + \frac{1}{2}\,\mu + 1\right)} \tag{11}$$

$\left(\mu > -1; \; \kappa := \sqrt{1-k^2}\right)$,

where the function $H(k,\gamma)$ defined by

$$\begin{aligned} H(k,\gamma) &:= \int_0^1 \frac{\left(1 - k^2 \, t^2\right)^{\gamma - \frac{1}{2}}}{\sqrt{1-t^2}} \, \mathrm{d}t \\ &= \int_0^{\frac{\pi}{2}} \left(1 - k^2 \, \sin^2\theta\right)^{\gamma - \frac{1}{2}} \, \mathrm{d}\theta \qquad \left(\gamma \geqq 0; \; k^2 < 1\right) \end{aligned} \tag{12}$$

provides a unification of both $K(k)$ and $E(k)$, since

$$K(k) = H(k,0) \quad \text{and} \quad E(k) = H(k,1) \qquad \left(k^2 < 1\right).$$

Incidentally, under a slightly modified form, Bushell's elliptic-type integral $H(k,\gamma)$ was studied presumably independently by Das [7, p. 77, Eq. (7.1)]:

$$H_\nu(k) := \int_0^{\frac{\pi}{2}} \left(1 - k^2 \, \sin^2\phi\right)^\nu \, \mathrm{d}\phi \qquad \left(\nu \in \mathbb{C}; \; k^2 < 1\right), \tag{13}$$

so that, obviously,

$$H_{-\frac{1}{2}} = K(k) \quad \text{and} \quad H_{\frac{1}{2}} = E(k) \qquad \left(k^2 < 1\right),$$

the restriction on the parameter γ in Bushell's definition (12) being unnecessary. Henceforth, therefore, we assume that $\gamma \in \mathbb{C}$

It is easily observed from Eq. (12) that [3, p. 2, Eq. (2.4)]

$$H(k,\gamma) = \frac{\pi}{2} \, {}_2F_1\left(\frac{1}{2}, \frac{1}{2} - \gamma; 1; k^2\right) \qquad \left(|k| < 1\right) \tag{14}$$

or, equivalently, that

$$H(k,\gamma) = \frac{\pi}{2} \, (1-k^2)^\gamma \, {}_2F_1\left(\frac{1}{2}, \gamma + \frac{1}{2}; 1; k^2\right) \qquad \left(|k| < 1\right), \tag{15}$$

where we have made use of Euler's transformation [8, p. 64]:

$$ {}_2F_1(a,b;c;z) = (1-z)^{c-a-b} \;\; {}_2F_1(c-a, c-b; c; z) \tag{16}$$

$$\left(|\arg(1-z)| \leqq \pi - \epsilon \;\; (0 < \epsilon < \pi); \; c \neq 0, -1, -2, \cdots\right).$$

Clearly, for the elliptic integral $E(k)$ of the second kind, we find from Eq. (15) that

$$H(k,-1)=\frac{E(k)}{\kappa^2} \qquad \Big(\kappa := \sqrt{1-k^2}\Big).$$

Many other unified and generalized studies of $K(k)$ and $E(k)$ can be found in the work by Srivastava and Siddiqi [17] in which the following unification (and generalization) of various known families of elliptic-type integrals was systematically investigated:

$$\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k) := \int_0^\pi \cos^{2\alpha-1}\left(\frac{\theta}{2}\right)\sin^{2\beta-1}\left(\frac{\theta}{2}\right)\frac{\left[1-\rho\,\sin^2\left(\frac{\theta}{2}\right)\right]^{-\lambda}}{\left(1-k^2\,\cos\theta\right)^{\mu+\frac{1}{2}}}\,\mathrm{d}\theta \tag{17}$$

$$\Big(0 \leqq k < 1;\ \min\{\Re(\alpha),\Re(\beta)\}>0;\ \lambda,\mu\in\mathbb{C};\ |\rho|<1\Big).$$

Indeed, as demonstrated by Srivastava and Siddiqi [17, p. 306, Eq. (1.42)], their elliptic-type integral $\Lambda_{\lambda,\mu}^{(\alpha,\beta)}(\rho;k)$ can also be specialized to yield the complete elliptic integral $\Pi\big(\alpha^2,k\big)$ of the *third* kind, which is defined by (see, for example, [4, p. 10, Entry 100.8]; see also [10, pp. xxx and 905])

$$\Pi\big(\chi^2,k\big) := \int_0^{\frac{\pi}{2}} \frac{\mathrm{d}\varphi}{\big(1-\chi^2\,\sin^2\varphi\big)\sqrt{1-k^2\,\sin^2\varphi}} \tag{18}$$

$$\Big(k^2<1;\ -\infty<\chi^2<\infty \quad (\chi^2\neq 1)\Big).$$

We now recall the following general result due to Srivastava [15], which provides a unification and generalization of a number of integrals involving elliptic integrals considered by, for example, Barton [2], Bushell [3], Cvijović and Klinowski [6], and others.

Theorem 1. (see Srivastava [15, p. 2307]) *If*

$$\Phi(z) := \sum_{n=0}^{\infty} a_n\, z^n \qquad \big(|z|<1\big) \tag{19}$$

and

$$\sum_{n=1}^{\infty}\left|\frac{a_n}{n^{1+\frac{1}{2}\sigma}}\right| < \infty \qquad \Big(\Re(\sigma)>-2\Big), \tag{20}$$

then the following integral formula holds true:

$$\begin{aligned}\int_0^1 & k^\rho\,\kappa^\sigma\,\Phi(zk)\,H(\zeta\kappa,\gamma)\,\mathrm{d}k \\ &= \frac{\pi}{4}\,\Gamma\left(\frac{1}{2}\sigma+1\right)\sum_{n=0}^{\infty}\frac{\Gamma\left(\frac{1}{2}(\rho+n+1)\right)}{\Gamma\left(\frac{1}{2}(\rho+\sigma+n+3)\right)} \\ &\quad\cdot {}_3F_2\left[\begin{array}{c}\frac{1}{2}-\gamma,\frac{1}{2}\sigma+1,\frac{1}{2};\\ \frac{1}{2}(\rho+\sigma+n+3),1;\end{array}\ \zeta^2\right] \qquad \Big(\kappa := \sqrt{1-k^2}\Big),\end{aligned} \tag{21}$$

provided further that $\Re(\rho)>-1$ *and*

$$\Big(|\zeta|<1 \ \ (\textit{or}\ \ |\zeta|=1 \quad \textit{and} \quad \Re(\rho+2\gamma)>-1)\Big).$$

This paper is motivated essentially by the above-mentioned developments involving the elliptic integrals $K(k)$, $E(k)$ and $\Pi(\chi^2, k)$ of the first, the second and the third kind, respectively. Our main objective here is to make use of the Gamma function as well as the hypergeometric and the generalized hypergeometric functions in order to investigate and develop several definite integrals involving an interesting an interesting unification of the elliptic integrals $K(k)$ and $E(k)$ of the first and the second kind. We also present the numerical approximations of these definite integrals and the corresponding hypergeometric functions. The results, which we have derived in this paper, are believed to be new and extend and unify those that are available in the scientific literature.

2. A Set of Integral Formulas

We begin by remarking that Srivastava's general result (21), which has been reproduced here as Theorem 1, can indeed be appropriately specialized to yield numerous integral formulas for the elliptic integrals $K(k)$ and $E(k)$ of the first and the second kind. Here, in this section, first prove the following theorem asserting a general definite integral involving Bushell's function $H(k, \gamma)$ defined by (12), which does indeed unify and generalize the elliptic integrals $K(k)$ and $E(k)$ of the first and the second kind.

Theorem 2. *Let the function $H(k, \gamma)$ be defined by* (12)*. Suppose also that p and q are integers such that $p \geqq 0$ and $q \geqq 1$. Then the following integral formula holds true for $\gamma \in \mathbb{C}$:*

$$\int_0^1 \frac{k^{2p+1}}{\sqrt{1-k^2}} H\Big(z^{\frac{1}{2}}\, k^q, \gamma\Big)\, \mathrm{d}k = \frac{\pi^{\frac{3}{2}}}{4} \frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)} \cdot {}_{q+2}F_{q+1}\left[\begin{array}{c} \frac{1}{2}, \frac{1}{2}-\gamma, \frac{p+1}{q}, \frac{p+2}{q}, \cdots, \frac{p+q}{q}; \\ 1, \frac{p+\frac{3}{2}}{q}, \frac{p+\frac{5}{2}}{q}, \cdots, \frac{p+q+\frac{1}{2}}{q}; \end{array} z\right] \qquad \Big(|z| < 1\Big), \tag{22}$$

provided that each member of the assertion (22) *exists.*

Proof. For convenience, we denote the left-hand side of (22) by $\Omega(\gamma)$. Then, upon applying the definition (12) of $H(k, \gamma)$, if we invert the order of summation and integration, we find that

$$\begin{aligned} \Omega(\gamma) &:= \int_0^1 \frac{k^{2p+1}}{\sqrt{1-k^2}} H\Big(z^{\frac{1}{2}}\, k^q, \gamma\Big)\, \mathrm{d}k \\ &= \frac{\pi}{2} \sum_{n=0}^{\infty} \frac{\left(\frac{1}{2}\right)_n \left(\frac{1}{2}-\gamma\right)_n}{(1)_n} \frac{z^n}{n!} \int_0^1 \frac{k^{2p+2qn+1}}{\sqrt{1-k^2}}\, \mathrm{d}k, \end{aligned} \tag{23}$$

wherein the inversion of the order of summation and integration is justifiable under the hypotheses of Theorem 2.

In the integral in (23), we now set

$$k = \sin\theta \quad \text{and} \quad \mathrm{d}k = \cos\theta\, \mathrm{d}\theta \qquad \left(0 \leqq \theta \leqq \frac{\pi}{2}\right)$$

and make use of the following familiar integral formula:

$$\int_0^{\frac{\pi}{2}} \sin^\alpha\theta\ \cos^\beta\theta\ \mathrm{d}\theta = \frac{\Gamma\left(\dfrac{\alpha+1}{2}\right)\Gamma\left(\dfrac{\beta+1}{2}\right)}{2\,\Gamma\left(\dfrac{\alpha+\beta+2}{2}\right)} \tag{24}$$

$\Big(\min\{\Re(\alpha),\Re(\beta)\} > -1\Big).$

We thus find from (23) that

$$\begin{aligned}\Omega(\gamma) &= \frac{\pi}{2}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-\gamma\right)_n}{(1)_n}\,\frac{z^n}{n!}\,\frac{\Gamma(p+qn+1)\,\Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(p+qn+\frac{3}{2}\right)}\\ &= \frac{\pi^{\frac{3}{2}}}{4}\sum_{n=0}^{\infty}\frac{\left(\frac{1}{2}\right)_n\left(\frac{1}{2}-\gamma\right)_n}{(1)_n}\,\frac{z^n}{n!}\,\frac{(p+1)_{qn}\,\Gamma(p+1)}{\left(p+\frac{3}{2}\right)_n\,\Gamma\left(p+\frac{3}{2}\right)},\end{aligned} \tag{25}$$

which, in view of the Gauss-Legendre multiplication formula for the Pochhammer symbol defined by (3) given by

$$(\lambda)_{mn} = m^{mn}\prod_{j=1}^{m}\left(\frac{\lambda+j-1}{m}\right)_n \qquad \Big(m\in\mathbb{N};\ n\in\mathbb{N}_0\Big), \tag{26}$$

leads us to the integral formula (22) just as asserted by Theorem 1. □

Corollary 1 and Corollary 2 below would follow readily from Theorem 2 when we set $\gamma = 0$ and $\gamma = 1$, respectively.

Corollary 1. *Let p and q be integers such that* $p \geqq 0$ *and* $q \geqq 1$. *Then the following integral formula holds true for the elliptic integral* $K(k)$ *of the first kind*:

$$\begin{aligned}\int_0^1 \frac{k^{2p+1}}{\sqrt{1-k^2}}\,K\Big(z^{\frac{1}{2}}\,k^q\Big)\,\mathrm{d}k &= \frac{\pi^{\frac{3}{2}}}{4}\,\frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}\\ &\cdot\ {}_{q+2}F_{q+1}\left[\begin{matrix}\frac{1}{2},\frac{1}{2},\frac{p+1}{q},\frac{p+2}{q},\cdots,\frac{p+q}{q};\\ 1,\frac{p+\frac{3}{2}}{q},\frac{p+\frac{5}{2}}{q},\cdots,\frac{p+q+\frac{1}{2}}{q};\end{matrix}\ z\right] \qquad \Big(|z|<1\Big),\end{aligned} \tag{27}$$

provided that each member of the assertion (27) *exists.*

Corollary 2. *Let p and q be integers such that* $p \geqq 0$ *and* $q \geqq 1$. *Then the following integral formula holds true for the elliptic integral* $E(k)$ *of the second kind*:

$$\int_0^1 \frac{k^{2p+1}}{\sqrt{1-k^2}}\, E\left(z^{\frac{1}{2}}\, k^q\right) \mathrm{d}k = \frac{\pi^{\frac{3}{2}}}{4}\, \frac{\Gamma(p+1)}{\Gamma\left(p+\frac{3}{2}\right)}$$
$$\cdot\ {}_{q+2}F_{q+1}\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{p+1}{q}, \frac{p+2}{q}, \cdots, \frac{p+q}{q}; \\ \\ 1, \frac{p+\frac{3}{2}}{q}, \frac{p+\frac{5}{2}}{q}, \cdots, \frac{p+q+\frac{1}{2}}{q}; \end{array} z\right] \qquad \left(|z|<1\right), \tag{28}$$

provided that each member of the assertion (28) *exists.*

In their *further* special case when $p = 0$, Corollary 1 and Corollary 2 reduce to the following relatively simpler results.

Corollary 3. *Let $q \geqq 1$ be and integer. Then the following integral formula holds true for the elliptic integral $K(k)$ of the first kind*:

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, K\left(z^{\frac{1}{2}}\, k^q\right) \mathrm{d}k$$
$$= \frac{\pi}{2}\ {}_{q+1}F_q\left[\begin{array}{c} \frac{1}{2}, \frac{1}{2}, \frac{1}{q}, \frac{2}{q}, \cdots, \frac{q-1}{q}; \\ \\ \frac{3}{2q}, \frac{5}{2q}, \cdots, \frac{2q+1}{2q}; \end{array} z\right] \qquad \left(|z|<1\right), \tag{29}$$

provided that each member of the assertion (29) *exists.*

Corollary 4. *Let $q \geqq 1$ be an integer. Then the following integral formula holds true for the elliptic integral $E(k)$ of the second kind*:

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(z^{\frac{1}{2}}\, k^q\right) \mathrm{d}k$$
$$= \frac{\pi}{2}\ {}_{q+1}F_q\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{q}, \frac{2}{q}, \cdots, \frac{q-1}{q}; \\ \\ \frac{3}{2q}, \frac{5}{2q}, \cdots, \frac{2q+1}{2q}; \end{array} z\right] \qquad \left(|z|<1\right), \tag{30}$$

provided that each member of the assertion (30) *exists.*

By assigning various special values to the integer parameter $q \geqq 1$ in Corollary 3 and Corollary 4, we can deduce a number of integral formulas involving the elliptic integrals $K(k)$ and $E(k)$, respectively. For example, if we choose to set $q = 6, 7, 8, 9, 10$, we obtain the following immediate consequences of the assertion (30) of Corollary 4:

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(z^{\frac{1}{2}}\, k^6\right) \mathrm{d}k$$
$$= \frac{\pi}{2}\ {}_7F_6\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}; \\ \\ \frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12}; \end{array} z\right] \qquad \left(|z|<1\right), \tag{31}$$

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\Big(z^{\frac{1}{2}}\, k^7\Big)\, \mathrm{d}k$$
$$= \frac{\pi}{2}\; {}_7F_6\left[\begin{array}{c} -\frac{1}{2}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}; \\ \frac{3}{14}, \frac{5}{14}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}, \frac{15}{14}; \end{array} z\right] \qquad \Big(|z| < 1\Big), \tag{32}$$

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\Big(z^{\frac{1}{2}}\, k^8\Big)\, \mathrm{d}k$$
$$= \frac{\pi}{2}\; {}_9F_8\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}; \\ \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}, \frac{17}{16}; \end{array} z\right] \qquad \Big(|z| < 1\Big), \tag{33}$$

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\Big(z^{\frac{1}{2}}\, k^9\Big)\, \mathrm{d}k$$
$$= \frac{\pi}{2}\; {}_9F_8\left[\begin{array}{c} -\frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}; \\ \frac{1}{6}, \frac{5}{18}, \frac{7}{18}, \frac{11}{18}, \frac{13}{18}, \frac{5}{6}, \frac{17}{18}, \frac{19}{18}; \end{array} z\right] \qquad \Big(|z| < 1\Big) \tag{34}$$

and

$$\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\Big(z^{\frac{1}{2}}\, k^{10}\Big)\, \mathrm{d}k$$
$$= \frac{\pi}{2}\; {}_{11}F_{10}\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}; \\ \frac{3}{20}, \frac{1}{4}, \frac{7}{20}, \frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{3}{4}, \frac{17}{20}, \frac{19}{20}, \frac{21}{20}; \end{array} z\right] \qquad \Big(|z| < 1\Big), \tag{35}$$

Analogous integral formulas involving the elliptic integral $K(k)$ of the first kind can be deduced by similarly specializing the integer parameter $q \geqq 1$ in Corollary 4. The details involved are being omitted here.

3. Numerical Approximations of the Definite Integrals and Hypergeometric Functions

The generalized hypergeometric series, which occur on the right-hand sides of the integral formulas (31) to (35), are seen to satisfy the ω-condition of (4) for their convergence when the argument $z = 1$. In this section, we present numerical approximations of the case $z = 1$ of the definite integrals (31) to (35) and the corresponding hypergeometric functions. We choose to state our results as follows:

$$\begin{aligned}
&\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(k^6\right) \mathrm{d}k \\
&\quad = \frac{\pi}{2}\, {}_7F_6\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{5}{6}; \\ \frac{1}{4}, \frac{5}{12}, \frac{7}{12}, \frac{3}{4}, \frac{11}{12}, \frac{13}{12}; \end{array} 1\right] \\
&\quad \approx 1.40291,
\end{aligned} \tag{36}$$

$$\begin{aligned}
&\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(k^7\right) \mathrm{d}k \\
&\quad = \frac{\pi}{2}\, {}_7F_6\left[\begin{array}{c} -\frac{1}{2}, \frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7}, \frac{6}{7}; \\ \frac{3}{14}, \frac{5}{14}, \frac{9}{14}, \frac{11}{14}, \frac{13}{14}, \frac{15}{14}; \end{array} 1\right] \\
&\quad \approx 1.41426,
\end{aligned} \tag{37}$$

$$\begin{aligned}
&\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(k^8\right) \mathrm{d}k \\
&\quad = \frac{\pi}{2}\, {}_9F_8\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \frac{1}{2}, \frac{5}{8}, \frac{3}{4}, \frac{7}{8}; \\ \frac{3}{16}, \frac{5}{16}, \frac{7}{16}, \frac{9}{16}, \frac{11}{16}, \frac{13}{16}, \frac{15}{16}, \frac{17}{16}; \end{array} 1\right] \\
&\quad \approx 1.42357,
\end{aligned} \tag{38}$$

$$\begin{aligned}
&\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(k^9\right) \mathrm{d}k \\
&\quad = \frac{\pi}{2}\, {}_9F_8\left[\begin{array}{c} -\frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{1}{3}, \frac{4}{9}, \frac{5}{9}, \frac{2}{3}, \frac{7}{9}, \frac{8}{9}; \\ \frac{1}{6}, \frac{5}{18}, \frac{7}{18}, \frac{11}{18}, \frac{13}{18}, \frac{5}{6}, \frac{17}{18}, \frac{19}{18}; \end{array} 1\right] \\
&\quad \approx 1.4314
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
&\int_0^1 \frac{k}{\sqrt{1-k^2}}\, E\left(k^{10}\right) \mathrm{d}k \\
&\quad = \frac{\pi}{2}\, {}_{11}F_{10}\left[\begin{array}{c} \frac{1}{2}, -\frac{1}{2}, \frac{1}{10}, \frac{1}{5}, \frac{3}{10}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{7}{10}, \frac{4}{5}, \frac{9}{10}; \\ \frac{3}{20}, \frac{1}{4}, \frac{7}{20}, \frac{9}{20}, \frac{11}{20}, \frac{13}{20}, \frac{3}{4}, \frac{17}{20}, \frac{19}{20}, \frac{21}{20}; \end{array} 1\right] \\
&\quad \approx 1.4381.
\end{aligned} \tag{40}$$

The corresponding numerical approximations for the case $z = 1$ of the integral formulas involving the elliptic integral $K(k)$ of the first kind can be deduced in a similar manner.

4. Concluding Remarks and Observations

In our present investigation, we have made use of the Gamma function as well as the hypergeometric and the generalized hypergeometric functions with a view developing several definite integrals involving the elliptic integrals $K(k)$ and $E(k)$ of the first and the second kind, respectively. The numerical approximation of these definite integrals and the corresponding hypergeometric functions are also presented. The results derived in this article are believed to be new and would extend and unify those that are available in the scientific literature.

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References

[1] M. Abramowitz and I. A. Stegun (Editors), *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, Applied Mathematics Series **55**, National Bureau of Standards, Washington, D.C., 1964; Reprinted by Dover Publications, New York, 1965 (see also [13]).

[2] G. Barton, Do attractive scattering potentials concentrate particles at the origin in one, two and three dimensions? III: High energies in quantum mechanics, *Proc. Roy. Soc. London Ser. A Math. Phys. Engrg. Sci.* **388** (1983), 445–456.

[3] P. J. Bushell, On a generalization of Barton's integral and related integrals of complete elliptic integrals, *Math. Proc. Cambridge Philos. Soc.* **101** (1987), 1–5.

[4] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists*, Second edition (Revised), Die Grundlehren der Mathematischen Wissenschaften in Einzeldarstellungen, Band **67**, Springer-Verlag, New York, Heidelberg and Berlin, 1971.

[5] T. Clausen, Über die Fälle, wenn die Reihe von der Form

$$y = 1 + \frac{\alpha}{1} \cdot \frac{\beta}{\gamma}\, x + \frac{\alpha(\alpha+1)}{1 \cdot 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)}\, x^2 + \cdots$$

ein Quadrat von der Form

$$z = 1 + \frac{\alpha'}{1} \cdot \frac{\beta'}{\gamma'} \cdot \frac{\delta'}{\epsilon'}\, x + \frac{\alpha'}{1 \cdot 2} \cdot \frac{\beta'(\beta'+1)}{\gamma'(\gamma'+1)} \cdot \frac{\delta'(\delta'+1)}{\epsilon'(\epsilon'+1)}\, x^2 + \cdots$$

hat, *J. Reine Angew. Math.* **3** (1828), 89–91.

[6] D. Cvijović and J. Klinowski, Integrals involving complete elliptic integrals, *J. Comput. Appl. Math.* **106** (1999) 169–175.

[7] J. Das, A generalisation of elliptic integrals, *Bull. Calcutta Math. Soc.* (Festschrift for Professor Mahadev Datta) (1987), 73–82.

[8] A. Erdélyi, W. Magnus, F. Oberhettinger and F. G. Tricomi, *Higher Transcendental Functions*, Vol. **I**, McGraw-Hill Book Company, New York, Toronto and London, 1953.

[9] C. F. Gauss, Disquisitiones generales circa seriem infinitam

$$1 + \frac{\alpha}{1} \cdot \frac{\beta}{\gamma}\, x + \frac{\alpha(\alpha+1)}{1 \cdot 2} \cdot \frac{\beta(\beta+1)}{\gamma(\gamma+1)}\, x^2 + \cdots,$$

Comment. Soc. Reg. Sci. Göttingen. Recent. Thesis, Göttingen, 1811; *Ges. Werke Göttingen* **3** (1866), 123–163.

[10] I. S. Gradshteyn and I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Corrected and enlarged edition prepared by A. Jeffrey (incorporating the fourth edition prepared by Yu. V. Geronimus and M. Yu. Tseytlin), Academic Press, New York, 1980.

[11] E. L. Kaplan, Multiple elliptic integrals, *J. Math. Phys.* **29** (1950), 69–75.

[12] K. F. Müller, Berechnung der Induktivität Spulen, *Arch. Electrotech.* **17** (1926), 336–353.

[13] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark (Editors), *NIST Handbook of Mathematical Functions* [With 1 CD-ROM (Windows, Macintosh and UNIX)], U. S. Department of Commerce, National Institute of Standards and Technology, Washington, D. C., 2010; Cambridge University Press, Cambridge, London and New York, 2010 (see also [1]).

[14] E. D. Rainville, *Special Functions*, Macmillan Company, New York, 1960; Reprinted by Chelsea publishing Company, Bronx, New York, 1971.

[15] H. M. Srivastava, Some elliptic integrals of Barton and Bushell, *J. Phys. A*: *Math. Gen.* **28** (1995), 2305–2312.

[16] H. M. Srivastava and P. W. Karlsson, *Multiple Gaussian Hypergeometric Series*, Halsted Press (Ellis Horwood Limited, Chichester), John Wiley and Sons, New York, Chichester, Brisbane and Toronto, 1985.

[17] H. M. Srivastava and R. N. Siddiqi, A unified presentation of certain families of elliptic-type integrals related to radiation field problems, *Radiat. Phys. Chem.* **46** (1995), 303–315.

[18] G. Szegö, *Orthogonal Polynomials*, Fourth edition, American Mathematical Society Colloquium Publications, Vol. **23**, American Mathematical Society, Providence, Rhode Island, 1975.