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Decomposability and structure of semigroups of nonnegative r-potent operators on $\mathcal{L}^2(\mathcal{X})$

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Abstract. An operator $\mathbf{A}$ on a separable Hilbert space $\mathcal{L}^2(\mathcal{X})$ (where $\mathcal{X}$ is a separable locally compact Hausdorff Space) is called an r-potent operator if $\mathbf{A}^r = \mathbf{A}$ for some natural number r [1]. For the special case of $r = 2$, an r-potent operator reduces to an idempotent operator. The decomposability of general r-potent operators is an unsolved problem of operator theory. The decomposability of the special case of *nonnegative* r-potent operators was established by the authors in [2]. In this paper, we extend the results of [2] to *semigroups* of nonnegative r-potent operators and derive precise conditions under which a semigroup of nonnegative r-potent operators is decomposable. Further, we provide key results related to the structure of single decomposable nonnegative r-potent operators and use these results to derive the structure of decomposable semigroups of r-potents.

1. Introduction

An operator $\mathbf{A}$ on a separable Hilbert space $\mathcal{L}^2(\mathcal{X})$ (where $\mathcal{X}$ denotes a separable, locally compact Hausdorff Space and μ is a Borel measure on $\mathcal{X}$ with $\mu(\mathcal{X}) < \infty$) is called an r-potent operator if for some natural number r, $\mathbf{A}^r = \mathbf{A}$ (see [1]). A semigroup of nonnegative r-potent operators is a collection of nonnegative r-potent operators in $B(\mathcal{L}^2(\mathcal{X}))$ which are closed under multiplication of r-potent operators. We study the decomposability of such semigroups of r-potents and derive conditions for their decomposability. We further analyze the structure of such semigroups.

2. Preliminaries

Definition 2.1 ([3]). *A function $f \in \mathcal{L}^2(\mathcal{X})$ is non-negative if $\mu\{x \in \mathcal{X} : f(x) < 0\} = 0$ and is written as $f \geq 0$.*

Definition 2.2 ([4]). *A standard subspace of $\mathcal{L}^2(\mathcal{X})$ is a norm-closed linear manifold in $\mathcal{L}^2(\mathcal{X})$ of the form $\mathcal{L}^2(\mathcal{U}) = \{f \in \mathcal{L}^2(\mathcal{X}) : f = 0$ a.e on $\mathcal{U}^c\}$ for some Borel subset $\mathcal{U}$ of $\mathcal{X}$. The space is nontrivial if $\mu(\mathcal{U})\mu(\mathcal{U}^c) > 0$.*

Definition 2.3. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be Borel Subsets of $\mathcal{X}$. An operator $\mathbf{A}$ from $\mathcal{L}^2(\mathcal{X}_1)$ to $L^2(\mathcal{X}_2)$ is called non-negative if $\mathbf{A}f \geq 0$ whenever $f \geq 0$ in $\mathcal{L}^2(\mathcal{X}_1)$.*

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Definition 2.4 ([4]). *An operator* $\mathbf{A}$ *on* $\mathcal{L}^2(\mathcal{X})$ *is said to be decomposable if there exists a nontrivial standard subspace of* $\mathcal{L}^2(\mathcal{X})$ *invariant under* $\mathbf{A}$.

The above is equivalent to the following definition of decomposability: [5, p. 39]

Definition 2.5. *A nonnegative operator* $\mathbf{A}$ *on* $\mathcal{L}^2(\mathcal{X})$ *is decomposable if and only if there exists a Borel subset* $\mathcal{U}$ *of* $\mathcal{X}$ *with* $\mu(\mathcal{U})\mu(\mathcal{U}^c) > 0$ *such that* $\langle \mathbf{A}\mathcal{X}_{\mathcal{U}}, \mathcal{X}_{\mathcal{U}^c}\rangle = 0$, *where* $\mathcal{X}_{\mathcal{U}} = \begin{cases} 1 & \text{on } \mathcal{U} \\ 0 & \text{on } \mathcal{U}^c. \end{cases}$

Definition 2.6 ([6]). *An operator* $\mathbf{A} : \mathcal{L}^2(\mathcal{X}_1) \to \mathcal{L}^2(\mathcal{X}_2)$ *is called a compact linear operator if* $\mathbf{A}$ *is linear and if for every bounded subset* M *of* $\mathcal{L}^2(\mathcal{X}_1)$*, the image* $\mathbf{A}(M)$ *is relatively compact, that is, the closure* $\overline{\mathbf{A}(M)}$ *is a compact subset of* $\mathcal{L}^2(\mathcal{X}_2)$.

We first state the results for decomposability of a single operator on $\mathcal{L}^2(\mathcal{X})$ which are given in [2].

Definition 2.7 ([2]). *Let* $\varepsilon = \{e_1, e_2 \ldots, e_N\}$ *be a set of non-negative basis functions in the range space* $R(\mathbf{A})$ *of a nonnegative compact r-potent operator* $\mathbf{A}$*. Then, there must exist an alternate set of basis functions* $\varepsilon' = \{e'_1, e'_2, \ldots, e'_N\}$ *such that* e'_j *for all* j*, are nonnegative and have nonoverlapping supports.*

Theorem 2.8 ([2]). *If there exists a basis* $\{e_1, e_2, \ldots, e_N\}$ *in the range space* $R(\mathbf{A})$ *of a nonnegative compact operator* $\mathbf{A}$ *with* $\mathbf{A} \leq N$ *such that* e_j*, for all* j*, are nonnegative and have nonoverlapping supports, then* $\mathbf{A}$ *must be decomposable over some support set* $\mathcal{U}$.

The aforestated theorem involved a novel constructive algorithm to show that the basis function of the range space of a nonnegative r-potent operator can be chosen to be nonnegative and mutually orthogonal. This construction helped in proving the decomposability of a nonnegative compact r-potent operator of rank $> r - 1$.

In the next section, we analyze the structure of a decomposable compact nonnegative r-potent operator.

3. Structure of nonnegative decomposable compact *r*-potent operator

Our key results on the structure of nonnegative decomposable compact r-potent operators can be stated as follows:

Theorem 3.1. *Let* $\boldsymbol{A}$ *be a nonnegative compact r-potent operator with range space of dimension greater than* $r - 1$*. Then, the following hold:*

(1) $\boldsymbol{A}$ *is decomposable and for any maximal block triangular decomposition of* $\boldsymbol{A}$ *via standard subspaces, the diagonal blocks must be indecomposable nonnegative r-potents of rank* $\leq r - 1$.
(2) *If application of* $\boldsymbol{A}$ *on any function in the nonnegative orthogonal basis forms a cycle of length* s*, then* s *must be a factor of* $r - 1$.
(3) *The least common multiple (LCM) of the lengths of all such cycles is* $r - 1$.

Proof. We prove the three parts of the above theorem in order:

(1) As $\boldsymbol{A}$ is a compact nonnegative r-potent of rank $> r - 1$, it must be decomposable. Therefore, $\boldsymbol{A}$ must have a block-triangularization form as

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ 0 & \boldsymbol{D} \end{pmatrix}$$

with respect to

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{U}) \oplus \mathcal{L}^2(\mathcal{U}^c)$$

for some Borel Subset $\mathcal{U}$ of $\mathcal{X}$ with $\mu(\mathcal{U}) \cdot \mu(\mathcal{U}^c) > 0$. Further, using Zorn's lemma, we have a maximal triangularizing chain in $\mathcal{L}at'(\mathbf{A})$. Therefore, we have the following:

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{X}_1) \oplus \mathcal{L}^2(\mathcal{X}_2) \oplus \cdots \oplus \mathcal{L}^2(\mathcal{X}_k)$$

and

$$\boldsymbol{A} = \begin{pmatrix} \boldsymbol{A}_{11} & & & * \\ & \boldsymbol{A}_{22} & & \\ & & \ddots & \\ 0 & & & \boldsymbol{A}_{kk} \end{pmatrix}$$

where

$$\boldsymbol{A}_{ii} : \mathcal{L}^2(\mathcal{X}_i) \to \mathcal{L}^2(\mathcal{X}_i)$$

and

$$\mu(\mathcal{X}_i) > 0, \quad \forall\, i = 1, 2, \ldots, k.$$

As $\boldsymbol{A}^r = \boldsymbol{A}$, we have

$$\boldsymbol{A}_{ii}^r = \boldsymbol{A}_{ii}, \quad \forall\, i = 1, 2, \ldots, k.$$

Therefore, each $\boldsymbol{A}_{ii}$ is a nonnegative r-potent. Further, as the chain is maximal, the compression of $\boldsymbol{A}$ to each gap is indecomposable. Hence, each $\boldsymbol{A}_{ii}$ is a non-negative indecomposable r-potent. Further, $\text{rank}(\boldsymbol{A}_{ii}) \leq r-1, \forall\, i$, because if rank $(\boldsymbol{A}_{ii}) > r-1$ for some i, then $\boldsymbol{A}_{ii}$ would be decomposable, contradicting the maximality of the chain.

(2) Let $\{e_1, e_2, \ldots, e_n\}$ be a nonnegative orthogonal basis of $R(\mathbf{A})$. Then, since

$$\boldsymbol{A}^r = \boldsymbol{A}$$

we have

$$\begin{aligned} & \boldsymbol{A}^{r-1} = \boldsymbol{I} \ \text{ on } R(\boldsymbol{A}) \\ \Rightarrow \quad & \boldsymbol{A}^{r-1} e_i = e_i \quad \forall\, i = 1, 2, \ldots, n. \end{aligned}$$

Proceeding exactly as in the proof of Theorem 2.8, given as [2, Theorem 23], we get

$$\begin{aligned} & \boldsymbol{A} e_{l_1} = \alpha_1 e_{l_2} \\ \Rightarrow \quad & \boldsymbol{A}^2 e_{l_1} = \alpha_1 \boldsymbol{A} e_{l_2} = \alpha_1 \alpha_2 e_{l_3} \\ & \quad \vdots \\ \Rightarrow \quad & \boldsymbol{A}^{r-1} e_{l_1} = \alpha_1 \alpha_2 \ldots \alpha_{r-2} \boldsymbol{A} e_{l_{r-1}} = \alpha_1 \alpha_2 \ldots \alpha_{r-1} e_{l_1} = e_{l_1} \quad (\text{since } \boldsymbol{A}^{r-1} = \boldsymbol{I} \text{ on } R(\boldsymbol{A})) \\ \Rightarrow \quad & \boldsymbol{A}^r e_{l_1} = \boldsymbol{A} e_{l_1} = \alpha_1 e_{l_2} \end{aligned}$$

Therefore, $\boldsymbol{A}$ permutes the function of the nonnegative orthogonal basis of $R(\boldsymbol{A})$ in disjoint cycles of length $\leq r-1$. Consequently, if application of $\boldsymbol{A}$ on any basis function e_{i_1} leads to a cycle of length s, then

$$\begin{aligned} \boldsymbol{A} e_{i_1} &= \beta_i^1 e_{i_2} \\ \boldsymbol{A} e_{i_2} &= \beta_i^2 e_{i_3} \\ &\ \ \vdots \\ \boldsymbol{A} e_{i_s} &= \beta_i^s e_{i_1} \end{aligned}$$

so that

$$A^s e_{i_j} = e_{i_j}, \quad \forall\ j = 1, 2, \dots, s.$$

Moreover, since $s, r-1 \in Z^+$, by division algorithm, there must exist some p, q such that

$$r - 1 = ps + q, \quad \text{where } 0 \le q < s.$$

Then,

$$\begin{aligned}
& A^{ps+q} = A^{r-1} = I = A^0 \\
\Rightarrow\ & A^{ps} \cdot A^q = A^0 \\
\Rightarrow\ & I \cdot A^q = A^0 \quad (\text{since } A^s = I \text{ for this cycle}) \\
\Rightarrow\ & q = 0
\end{aligned}$$

and hence s must be a factor of $r - 1$.

(3) We intend to show that LCM of lengths of all cycles be $r - 1$. Equivalently, we need to show that (1) if there is a cycle of length s, then s must be a factor of $r - 1$ and (2) if cycles have lengths $s_1, s_2, \dots$ such that $s_i, \forall i$ are factors of ℓ, then $r - 1$ should also a factor of ℓ. While (1) is already proved above, (2) can be shown as follows:
Since $r - 1, \ell \in Z^+$, by Division algorithm, there must exist a $p, q \in Z^+$ such that

$$\begin{aligned}
& l = p(r-1) + q \\
\Rightarrow\ & A^l = A^{p(r-1)} \cdot A^q \\
\Rightarrow\ & A^{\lambda_i s_i} = (A^{(r-1)})^p \cdot A^q \quad (\text{for some } \lambda_i) \\
\Rightarrow\ & (A^{s_i})^{\lambda_i} = I^p \cdot A^q \\
\Rightarrow\ & (I^{\lambda_i}) = A^q \\
\Rightarrow\ & q = 0 \\
\Rightarrow\ & l = p(r-1)
\end{aligned}$$

that is, $r - 1$ is a factor of ℓ. □

4. Decomposability of Semigroups of *r*-Potent Operators

We first present our results on decomposability of special class of semi-groups and then generalize our results.

Theorem 4.1. *Let A be a non-negative compact r-potent of rank $> r - 1$. Then, the semigroup $S = \{A, A^2, \cdots, A^{r-1}\}$ is decomposable.*

Proof. We start by noticing that A, being a nonnegative compact r-potent of rank $> r - 1$, is decomposable. Therefore, there exists a Borel subset $\mathcal{U}$ of $\mathcal{X}$ with

$$\mu(\mathcal{U}) \cdot \mu(\mathcal{U}^c) > 0$$

such that

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{U}) \oplus \mathcal{L}^2(\mathcal{U}^c)$$

and

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix}$$

with respect to the above decomposition of $L^2(\mathcal{X})$. Therefore,

$$\begin{aligned} \boldsymbol{A}^2 &= \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ 0 & \boldsymbol{D} \end{pmatrix} \cdot \begin{pmatrix} \boldsymbol{B} & \boldsymbol{C} \\ 0 & \boldsymbol{D} \end{pmatrix} \\ &= \begin{pmatrix} \boldsymbol{B}^2 & * \\ 0 & \boldsymbol{D}^2 \end{pmatrix} \\ &\vdots \\ \boldsymbol{A}^{r-1} &= \begin{pmatrix} \boldsymbol{B}^{r-1} & * \\ 0 & \boldsymbol{D}^{r-1} \end{pmatrix}. \end{aligned}$$

Therefore, each member of $\mathcal{S}$ has Block-Triangular decomposition via the same decomposing space $\mathcal{L}^2(\mathcal{U})$, and hence, the semigroup $\mathcal{S}$ is decomposable. □

The above result naturally motivates us to study the decomposability of general semigroups of nonnegative r-potent operators on $\mathcal{L}^2(\mathcal{X})$. Before stating our key result for such semigroups, we shall need the following two propositions and a corollary:

Proposition 4.2 ([5, p. 42]). *If $\mathcal{S}$ is an indecomposable semigroup of nonnegative operators on $\mathcal{L}^2(\mathcal{X})$, then so is every nonzero ideal of $\mathcal{S}$.*

Proposition 4.3 ([5, p. 43]). *Let $\mathcal{S}$ be a collection of nonnegative operators from $\mathcal{L}^2(\mathcal{X}) \to \mathcal{L}^2(\mathcal{Y})$. Let $\mathbf{A}$ and $\mathbf{B}$ be nonzero nonnegative operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{Y}))$ and $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$, respectively, satisfying $\boldsymbol{A}\mathcal{S}\boldsymbol{B} = \{0\}$. Then, there exists Borel subsets $E \subseteq \mathcal{X}$ and $F \subseteq \mathcal{Y}$ with positive measures such that $\langle \mathbf{S}\mathcal{X}_E, \mathcal{X}_F \rangle = 0$, for all $\mathbf{S} \in \mathcal{S}$.*

Corollary 4.4. *A nonnegative semigroup of operators in $\mathcal{B}(\mathcal{L}^2(\mathcal{X}))$ is decomposable if and only if there exist nonzero nonnegative operators $\mathbf{A}$ and $\mathbf{B}$ on $\mathcal{L}^2(\mathcal{X})$, not necessarily in $\mathcal{S}$, such that $\boldsymbol{A}\mathcal{S}\boldsymbol{B} = \{0\}$.*

We next state the main result of this section:

Theorem 4.5. *Let $\mathcal{S}$ be a semigroup of nonnegative r-potent operators of rank $> r-1$ where at least one operator is compact, then, $\mathcal{S}$ is decomposable.*

Proof. We start by noting that every compact r-potent operator is finite-dimensional. Let m be the minimum of rank($\mathcal{S}$), $\mathbf{S} \in \mathcal{S}$. Then, $m > r-1$.

Let $\mathcal{J}$ be the collection of all those members of $\mathcal{S}$ which have rank m. Then, for any $\mathbf{S} \in \mathcal{S}$ and $\mathbf{J} \in \mathcal{J}$,

$$\text{rank}(\mathbf{SJ}) = \text{rank}(\mathbf{JS}) = m$$

and hence, both $\mathbf{SJ}$ and $\mathbf{JS}$ belong to $\mathcal{J}$. Therefore, $\mathcal{J}$ is a nonzero ideal of $\mathcal{S}$. By Proposition 4.2, $\mathcal{S}$ is decomposable if and only if $\mathcal{J}$ is decomposable. In the remaining part of this proof, we therefore only need to show that $\mathcal{J}$ is decomposable.

To this end, we start by noting that for any $\mathbf{A} \in \mathcal{J}$, we have $\mathbf{A}^{r-1} \in \mathcal{J}$ and $\mathbf{A}^{r-1}$ is idempotent. Therefore, $\mathcal{J}$ always contains an idempotent. Let us call this idempotent $\mathbf{P}$. As rank($\mathbf{P}$) is greater than $r-1$, which is greater than 1, $\mathbf{P}$ must be decomposable. Consider any $\mathbf{S} \in \mathcal{J}$, then $\mathbf{P}\mathcal{S}\mathbf{P}$ is also a member of $\mathcal{J}$ and is hence r-potent. This however implies that $(\mathbf{PSP})^{r-1}$ is an idempotent. Further,

$$R((\mathbf{PSP})^{r-1}) \subseteq R(\mathbf{P})$$

so that

$$\text{rank}((\mathbf{PSP})^{r-1}) \leq \text{rank}(\mathbf{P}),$$

which, due to minimality of rank of $\mathbf{P}$, further implies

$$\text{rank}((\mathbf{PSP})^{r-1}) = \text{rank}(\mathbf{P}).$$

In addition,

$$N(\mathbf{P}) \subseteq N((\mathbf{PSP})^{r-1})$$

which, along with

$$\begin{aligned}\mathcal{X} &= R(\mathbf{P}) \oplus N(\mathbf{P})\\ &= R((\mathbf{PSP})^{r-1}) \oplus N((\mathbf{PSP})^{r-1})\end{aligned}$$

implies that

$$N(\mathbf{P}) = N((\mathbf{PSP})^{r-1})$$

and therefore

$$\begin{aligned}&\mathbf{P} = (\mathbf{PSP})^{r-1}\\ \Rightarrow\quad &\mathbf{PSP} = \mathbf{P}^{\frac{1}{r-1}}.\end{aligned}$$

Then, since

$$\mathbf{P}^2 = \mathbf{P} \Rightarrow \mathbf{P}^{r-1} = \mathbf{P},$$

we can conclude that $\mathbf{P}$ is definitely an $(r-1)$-th root of $\mathbf{P}$. However, there might be other nonnegative $(r-1)$-th roots of $\mathbf{P}$ in $\mathcal{S}$, that is,

$$\mathbf{PSP} = \mathbf{P}^{\frac{1}{r-1}} = \mathbf{P} \text{ or } \mathbf{K}^*\text{(say)}.$$

We can therefore have two possibilities:

Case 1: $PSP = \mathbf{K}^*$

As $\mathbf{K}^*$ belongs to $\mathcal{J}$, we have

$$\operatorname{rank}(\mathbf{K}^*) > r-1,$$

which, along with the fact that $\mathbf{K}^*$ is r-potent, implies that $\mathbf{K}^*$ must be decomposable. We can therefore write

$$\mathbf{K}^* = \begin{pmatrix}\mathbf{K}_1 & \mathbf{Y}\\ 0 & \mathbf{K}_2\end{pmatrix}$$

with respect to some decomposition $\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{U}) \oplus \mathcal{L}^2(\mathcal{U}^c)$ for some Borel subset $\mathcal{U}$ of $\mathcal{X}$. Therefore,

$$\begin{aligned}\mathbf{P} &= (\mathbf{K}^*)^{r-1}\\ &= \begin{pmatrix}\mathbf{K}_1^{r-1} & *\\ 0 & \mathbf{K}_2^{r-1}\end{pmatrix}\\ &= \begin{pmatrix}\mathbf{L}_1 & *\\ 0 & \mathbf{L}_2\end{pmatrix}\quad\text{(say)}.\end{aligned}$$

Therefore, $\mathbf{PSP} = \mathbf{K}^*$ implies

$$\begin{aligned}\begin{pmatrix}\mathbf{L}_1 & *\\ 0 & \mathbf{L}_2\end{pmatrix}\begin{pmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{pmatrix}\begin{pmatrix}\mathbf{L}_1 & *\\ 0 & \mathbf{L}_2\end{pmatrix} &= \begin{pmatrix}\mathbf{K}_1 & \mathbf{Y}\\ 0 & \mathbf{K}_2\end{pmatrix}\\ \Rightarrow \mathbf{S} &= \begin{pmatrix}\mathbf{S}_{11} & \mathbf{S}_{12}\\ \mathbf{S}_{21} & \mathbf{S}_{22}\end{pmatrix}\end{aligned}$$

is the decomposition of $\mathbf{S}$ with respect to $\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{U}) \oplus \mathcal{L}^2(\mathcal{U}^c)$. This, however, gives

$$\mathbf{L}_2\mathbf{S}_{21}\mathbf{L}_1 = 0\,.$$

By Proposition 4.3, there exist Borel subsets $\mathbf{E}$ and $\mathbf{F}$ in $\mathcal{U}$ and $\mathcal{U}^c$, respectively, with positive measures such that

$$\langle \mathbf{S}_{21}\mathcal{X}_{\mathbf{E}}, \mathcal{X}_{\mathbf{F}} \rangle = 0.$$

Therefore, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathbf{E}) \oplus \mathcal{L}^2(\mathbf{F}) \oplus \mathcal{L}^2(G)$$

where $\mathbf{G} = (\mathcal{U} \sim \mathbf{E}) \cup (\mathcal{U}^c \sim \mathbf{F})$, each $\mathbf{S} \in \mathcal{J}$ has the form

$$\mathbf{S} = \begin{pmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} & \mathbf{R}_{13} \\ 0 & \mathbf{R}_{22} & \mathbf{R}_{23} \\ \mathbf{R}_{31} & \mathbf{R}_{32} & \mathbf{R}_{33} \end{pmatrix}$$

so that

$$\langle \mathbf{S}\mathcal{X}_{\mathbf{E}}, \mathcal{X}_{\mathbf{F}} \rangle = 0, \quad \forall\, \mathbf{S} \in \mathcal{J}.$$

Case 2: $\boldsymbol{PSP = P}$

Since $\mathbf{P}$ is decomposable,

$$\mathbf{P} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{Z} \\ 0 & \mathbf{P}_2 \end{pmatrix}$$

with respect to some decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathcal{W}) \oplus \mathcal{L}^2(\mathcal{W}^c)$$

for some Borel subset $\mathcal{W}$ of $\mathcal{X}$. Therefore, $\boldsymbol{PSP = P}$ implies

$$\begin{pmatrix} \mathbf{P}_1 & \mathbf{Z} \\ 0 & \mathbf{P}_2 \end{pmatrix} \begin{pmatrix} \mathbf{S}'_{11} & \mathbf{S}'_{12} \\ \mathbf{S}'_{21} & \mathbf{S}'_{22} \end{pmatrix} \begin{pmatrix} \mathbf{P}_1 & \mathbf{Z} \\ 0 & \mathbf{P}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{P}_1 & \mathbf{Z} \\ 0 & \mathbf{P}_2 \end{pmatrix}$$

where

$$\mathbf{S} = \begin{pmatrix} \mathbf{S}'_{11} & \mathbf{S}'_{12} \\ \mathbf{S}'_{21} & \mathbf{S}'_{22} \end{pmatrix}$$

is the decomposition of $\mathbf{S}$ with respect to $\mathcal{L}^2(\mathcal{W}) \oplus \mathcal{L}^2(\mathcal{W}^c)$. This gives

$$\mathbf{P}_2\mathbf{S}'_{21}\mathbf{P}_1 = 0\,.$$

Again, by Proposition 4.3, there exist Borel subsets $\mathbf{M}$ and $\mathbf{N}$ in $\mathcal{W}$ and $\mathcal{W}^c$, respectively, with positive measures such that

$$\langle \mathbf{S}'_{21}\mathcal{X}_{\mathbf{M}}, \mathcal{X}_{\mathbf{N}} \rangle = 0\,.$$

Therefore, with respect to the decomposition

$$\mathcal{L}^2(\mathcal{X}) = \mathcal{L}^2(\mathbf{M}) \oplus \mathcal{L}^2(\mathbf{N}) \oplus \mathcal{L}^2(\mathbf{O})$$

when $\mathbf{O} = (\mathcal{W} \sim \mathbf{M}) \cup (\mathcal{W}^c \sim \mathbf{N})$, each $\mathbf{S} \in \mathcal{J}$ has the form

$$\mathbf{S} = \begin{pmatrix} \mathbf{T}_{11} & T_{12} & \mathbf{T}_{13} \\ 0 & T_{22} & \mathbf{T}_{23} \\ \mathbf{T}_{31} & T_{32} & \mathbf{T}_{33} \end{pmatrix}$$

so that

$$\langle \mathbf{S}\mathcal{X}_{\mathbf{M}}, \mathcal{X}_{\mathbf{N}} \rangle = \quad \forall \ \mathbf{S} \in \mathcal{J}.$$

Therefore, $\mathcal{J}$ is decomposable, and hence, S is decomposable.

This concludes our proof that the semigroups of nonnegative r-potent operators on $\mathcal{L}^2(\mathcal{X})$ are decomposable. □

We now discuss the structure of this decomposable semigroup.

Theorem 4.6. *Let* $\mathbf{S}$ *be a semigroup of nonnegative r-potent operators of rank* $> r - 1$ *with atleast one operator compact. Then, any maximal standard block triangularization of* $\mathbf{S}$ *has the property that each non-zero diagonal block is a semigroup of nonnegative r-potent operators with atleast one element of rank* $\leq r - 1$.

Proof. By Theorem 4.5, $\mathbf{S}$ is decomposable. Consider any maximal chain in $\mathcal{L}at'\mathbf{S}$, where $\mathcal{L}at'\mathbf{S}$ is the lattice of all standard subspaces which are invariant under every member of $\mathbf{S}$, resulting in a standard block triangularization of $\mathbf{S}$. Consider any two subspaces $\mathcal{N}_1$ and $\mathcal{N}_2$ in the chain $\mathcal{N}_1 \subseteq \mathcal{N}_2$ such that $\mathcal{N}_2 \ominus \mathcal{N}_1$ is a gap. If the compression of $\mathbf{S}$ to $\mathcal{N}_2 \ominus \mathcal{N}_1$ is non-zero, it forms a semigroup of nonnegative r-potents. Further, it must be indecomposable, for otherwise, if it has a standard invariant subspace $\mathcal{K}$, then $\mathcal{N}_1 \oplus \mathcal{K}$ is in $\mathcal{L}at'\mathbf{S}$ and lies strictly between $\mathcal{N}_1$ and $\mathcal{N}_2$, contradicting the maximality of this chain. Thus, non-zero compression (or diagonal block) constitutes an indecomposable semigroup of r-potents. By Theorem 4.5, it must contain atleast one element of rank $\leq r - 1$. □

5. Concluding Remarks

We had shown in [2] that a compact nonnegative r-potent operator on $\mathcal{L}^2(\mathcal{X})$ of rank $> r - 1$ is always decomposable. In this paper, we established decomposability of semigroups of nonnegative r-potent operators of rank $> r - 1$ and having at least one compact operator. We further provided concrete insights into the structure of both single decomposable nonnegative r-potent operators as well as their semigroups. A potential area of future research relates to doing away with the compactness condition. In particular, it should be noted that in our work so far, the assumption of compactness, coupled with r-potence, gives a finite-dimensional range space which leads to decomposability. On dropping the compactness assumption, the treatment of the problem would change completely. In addition, the decomposability of semigroups of nonnegative noncompact r-potent operators is not known. Further, the decomposability of r-potent operators without nonnegativity assumption is an open problem.

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