Filomat 38:11 (2024), 3739–3747
https://doi.org/10.2298/FIL2411739H

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

On some sequence sets based on Lucas band matrix and modulus functions

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Abstract. In this study, the sequence sets $\ell_p(G^k, v, \widehat{E}(r,s))$ and $\ell_\infty(G^k, v, \widehat{E}(r,s))$ which are depended on the Lucas band matrix and a sequence of modulus functions, are presented. After that, a few inclusion relationships of these sequence sets are given. Furthermore, a geometrical property such as the modulus of convexity of the sequence set $\ell_p(G^k, v, \widehat{E}(r,s))$ is discussed.

1. Introduction

Assume that ω is the space containing all real and complex-valued sequences. A linear subspace of ω is called a sequence space. The symbols ℓ_∞, c_0, c and ℓ_p for $1 \leq p < \infty$ represent the sequence spaces of all bounded, null, convergent, and p-absolutely convergent series respectively, normed by $\|x\|_\infty = sup_n |x_n|$ and $\|x\|_p = (\sum_n |x_n|^p)^{\frac{1}{p}}$. For the first time, the difference sequence spaces were introduced by Kizmaz [1] in the form of $X(\Delta) = \{x \in \omega : (x_n - x_{n-1}) \in X\},\ X = \ell_\infty, c, c_0$. After that, these sequence spaces have been generalized by Et and Colak [2] such as $X(\Delta^r) = \{x \in \omega : \Delta^r x \in X\},\ X = \ell_\infty, c, c_0$. Kirisci and Basar [3] have recently defined and investigated the difference sequence spaces $\hat{X} = \{x \in \omega : B(r,s)x \in X\},\ 1 \leq p < \infty,\ X = \ell_\infty, c, c_0, \ell_p$ where $B(r,s) = (sx_{n-1} + rx_n);\ r, s \neq 0$. At the same time, Colak and Et [4], Mursaleen [5], Altin [6], and some other authors have investigated the difference sequence spaces in their studies.

A normed space X has a Schauder basis (a_n) (or, more simply basis), if there is indeed a unique scalar sequence (α_n) for every x in X with

$$\lim_{n\to\infty} \|x - (\alpha_0 a_0 + \alpha_1 a_1 + \cdots + \alpha_n a_n)\| = 0.$$

A BK-space is a Banach space X, if there is $Q_i(x) = x_i$ such that $Q_i : X \to \mathbb{C}$ ($\mathbb{C}$ provides the set of all complex numbers) is continuous for all $i \in \mathbb{N}$, refer to [7].

As a simple instance, the spaces ℓ_∞, c_0, c and ℓ_p for $1 \leq p < \infty$ are BK-spaces, respectively with the norms below

$$\|x\|_\infty = \sup_n |x_n| \ \ and \ \ \|x\|_p = \Big(\sum_{n=0}^{\infty} |x_n|^p\Big)^{\frac{1}{p}}.$$

2020 *Mathematics Subject Classification*. Primary 40A05; Secondary 47A08, 52A99
Keywords. Lucas numbers, Modulus function, Sequence space, BK-space, Modulus of convexity.
Received: 02 August 2023; Accepted: 02 October 2023
Communicated by Eberhard Malkowsky
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In 1953, Nakano [8] established the concept of the modulus function. We remember that $g : [0,\infty) \to [0,\infty)$ is a modulus function such that $g(x) = 0$ iff $x = 0$, g is increasing and continuous from 0^+, and $g(x+y) \leq g(x) + g(y)$ for every x, y in $[0,\infty)$. A modulus can be either bounded or unbounded. For illustration, $g(x) = 1/(1+x)$ is a bounded modulus, but $g(x) = x^p$, $(0 < p \leq 1)$ is unbounded. The notion of the modulus function has also been discussed in [6, 7, 9–16]

Consider $\Psi = (\psi_{\rm mn})$ as an infinite matrix and assume that X and Y are two sequence spaces. Then, Ψ provides a matrix mapping from X into Y if, for every $x = (x_n)$ in X, there exists the sequence $\Psi x = (\psi_m(x)) \in Y$, where

$$\Psi_m(x) = \sum_{n=0}^{\infty} \psi_{mn} x_n \quad (m \in \mathbb{N}). \tag{1.1}$$

The set of all matrices ψ that have the property $\psi : X \to Y$ is represented by (X, Y). For that, $\Psi \in (X : Y)$ if and only if the right hand of the above equality (1.1) is convergent for each $m \in \mathbb{N}$ and $x \in X$. The notion of matrix domain X_Ψ of $\Psi \in X$ is expressed by

$$X_\Psi = \{x \in \omega \ : \Psi x \in X\}, \tag{1.2}$$

that represents a sequence space, see [16]. In recent years, a few mathematicians have developed certain sequence spaces by use of the matrix domain for an infinite matrix, refer to [7, 13, 17, 18].

In 1876, the sequence $\{L_n\}_{n=0}^{\infty}$ of Lucas numbers 1, 3, 4, 7, 11, 18, 29, ... was introduced by Edouard Lucas which is given by the Fibonacci recurrence relation in the form $L_n = L_{n-1} + L_{n-2}; n \geq 2$ with different initial conditions $L_0 = 2$ and $L_1 = 1$ where L_n is the nth term of the sequence $\{L_n\}_{n=0}^{\infty}$. Lucas numbers have various formulas and several properties, see [19, 20].

By using Lucas numbers with two real numbers r and s such that $r, s \neq 0$, the Lucas band matrix $\widehat{E}(r,s) = \left(\widehat{E}_{\rm nm}(r,s)\right)$ has been established in [21] as follows:

$$\widehat{E}(r,s) = \begin{cases} s\frac{L_n}{L_{n-1}} & (m = n-1) \\ r\frac{L_{n-1}}{L_n} & (m = n) \\ 0 & (m > n \ \ or \ \ 0 \leq m < n-1). \end{cases} \tag{1.3}$$

With the help of (1.3) the $\widehat{E}-$transform for a sequence $x = (x_n)$ is formed by

$$\widehat{E}_n(r,s)(x) = r\frac{L_{n-1}}{L_n}x_n + s\frac{L_n}{L_{n-1}}x_{n-1}, \ n \geq 1. \tag{1.4}$$

Recently, Karakas [21] and Mohiuddine [7] have used Lucas numbers and Lucas band matrix in constructing some sequence spaces in their studies.

Suppose that X is a normed linear space and S_X and B_X are the unit sphere and unit ball of X, respectively. Then the idea of modulus of convexity has been defined by Clarkson [22] and Gurarii [23] respectively, as follows:

$$\delta_X(\varepsilon) = inf\left\{1 - \frac{\left\|x+y\right\|}{2} \ : x, y \in S_X, \ \left\|x-y\right\| = \varepsilon\right\}, \quad \varepsilon \in [0,2]$$

and

$$\gamma_X(\varepsilon) = inf\left\{1 - \inf_{h\in[0,1]} \left\|hx + (1-h)y\right\| \ : x, y \in S_X, \ \left\|x-y\right\| = \varepsilon\right\}, \quad \varepsilon \in [0,2].$$

The case of $0 < \gamma_X(\varepsilon) < 1$ means that X is uniformly convex as well as the choice of $\gamma_X(\varepsilon) \leq 1$ means that X is strictly convex. The concept of modulus of convexity has also been studied in [24] and by some other authors.

Lemma 1.1. [25] If g is a modulus function, then for each $k \in \mathbb{N}$ the function $g^k = g \, og \, o \ldots o \, g$ (k times) is also a modulus.

2. Main results

In this part of our study, we introduce some difference sequence spaces based on the Lucas band matrix $\widehat{E}(r,s)$ and a sequence of modulus functions $G = (g_n)$, and study some interesting results through newly introduced sequence spaces. Let $\mathbb{G}$ be the set of sequence of modulus functions $G = (g_n)$ such that $\lim_{u \to 0^+} \sup_n g_n(u) = 0$. The sequence of modulus functions determined by G is indicated by $G = (g_n) \in \mathbb{G}$. We let $G^k = \left(g_n^k\right) = \left\{g_1^k, g_2^k, \ldots\right\}$ $(k \in \mathbb{N})$, and $v = (v_n)$ to be a sequence of strictly positive real numbers. We use these notations throughout this study. Given the information above, we provide these sequence spaces as follows:

$$\ell_p\left(G^k, v, \widehat{E}(r,s)\right) = \left\{x \in \omega : \sum_n \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p < \infty\right\}, \; 1 \le p < \infty$$

and

$$\ell_\infty\left(G^k, v, \widehat{E}(r,s)\right) = \left\{x \in \omega : \sup_n \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right] < \infty\right\}.$$

With the help of (1.2) the sequence spaces above are redefined as follows:

$$\ell_p\left(G^k, v, \widehat{E}(r,s)\right) = \left(\ell_p\right)_{v_n g_n^k \widehat{E}(r,s)} \quad \text{and} \quad \ell_\infty\left(G^k, v, \widehat{E}(r,s)\right) = \left(\ell_\infty\right)_{v_n g_n^k \widehat{E}(r,s)} \tag{2.1}$$

Remark 2.1. If $k = 1$ then the spaces $\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ and $\ell_\infty\left(G^k, v, \widehat{E}(r,s)\right)$ will reduce to the spaces $\ell_p\left(G, v, \widehat{E}(r,s)\right)$ and $\ell_\infty\left(G, v, \widehat{E}(r,s)\right)$ of Mohiuddine [7], respectively, as well as if we put $g(x) = x$ for every $x \in [0, \infty)$ and every g in G with $v_n = 1$ for all $n \in \mathbb{N}$, then the spaces $\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ and $\ell_\infty\left(G^k, v, \widehat{E}(r,s)\right)$ will become the same as $\ell_p\left(\widehat{E}(r,s)\right)$ and $\ell_\infty\left(\widehat{E}(r,s)\right)$ of Karakas [21], respectively.

Theorem 2.1. Assume that $G = (g_n)$ is a sequence of modulus functions in $\mathbb{G}$. The sequence spaces $\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ and $\ell_\infty\left(G^k, v, \widehat{E}(r,s)\right)$ are Banach spaces for $1 \le p < \infty$, respectively, normed by

$$\|x\|_{\ell_p\left(G^k, v, \widehat{E}(r,s)\right)} = \left(\sum_{n=1}^{\infty} \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p\right)^{\frac{1}{p}} \tag{2.2}$$

and

$$\|x\|_{\ell_\infty\left(G^k, v, \widehat{E}(r,s)\right)} = \sup_n \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]. \tag{2.3}$$

Proof. We here only consider the proof for $\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$. It can be easily verified that $\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ is a normed linear space, normed by (2.2). Let $x^i = \left(x_n^i\right)_n$ be a Cauchy sequence such that $\left(x_n^i\right)_n = \left(x_1^i, x_2^i, \ldots\right) \in \ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ for each $i \in \mathbb{N}$. Then given $\varepsilon > 0$ there is a natural number $N \in \mathbb{N}$ such that for every $i, j \ge N$, we have

$$\left\|x^i - x^j\right\|_{\ell_p\left(G^k, v, \widehat{E}(r,s)\right)} = \left(\sum_n \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)\left(x^i - x^j\right)\right|\right)\right]^p\right)^{\frac{1}{p}} < \varepsilon$$

and so

$$\sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x^i - x^j \right) \right| \right) \right]^p < \varepsilon^p. \tag{2.4}$$

Then for every $i, j \geq N$ and for all $n \in \mathbb{N}$, we see that

$$v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x^i - x^j \right) \right| \right) < \varepsilon.$$

Hence for all $n \in \mathbb{N}$, we get

$$\left| \widehat{E}_n(r,s) \left(x^i - x^j \right) \right| \to 0$$

as $i, j \to \infty$. It follows that $\left(\widehat{E}_n(r,s) \left(x^i \right) \right) = \left(\widehat{E}_n(r,s) \left(x^1 \right), \widehat{E}_n(r,s) \left(x^2 \right), \ldots \right)$ is a Cauchy sequence of numbers. Since $\mathbb{R}$ and $\mathbb{C}$ are complete, then it is convergent, so there is $x = (x_n)$. Say $\left(\widehat{E}_n(r,s) \left(x^i \right) \right) \to \left(\widehat{E}_n(r,s)(x) \right)$ as $i \to \infty$ and for each $n \in \mathbb{N}$. Letting $j \to \infty$ in (2.4), we obtain for $i > N$

$$\sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x^i - x \right) \right| \right) \right]^p < \varepsilon^p. \tag{2.5}$$

This implies that $\left(x^i - x \right) \in \ell_p \left(G^k, v, \widehat{E}(r,s) \right)$, and since $x^i \in \ell_p \left(G^k, v, \widehat{E}(r,s) \right)$, then

$$\begin{aligned}
\sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s)(x) \right| \right) \right]^p &\leq \sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x - x^i \right) \right| + \left| \widehat{E}_n(r,s) \left(x^i \right) \right| \right) \right]^p \\
&\leq \sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x - x^i \right) \right| \right) + v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x^i \right) \right| \right) \right]^p \\
&\leq \sum_n 2^p \left(\left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x - x^i \right) \right| \right) \right]^p + \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x^i \right) \right| \right) \right]^p \right) \\
&= 2^p \sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x - x^i \right) \right| \right) \right]^p + 2^p \sum_n \left[v_n g_n^k \left(\left| \widehat{E}_n(r,s) \left(x^i \right) \right| \right) \right]^p \\
&< \infty.
\end{aligned}$$

Furthermore, by adding limits in (2.5) and letting $i \to \infty$, we have

$$\lim_{i \to \infty} \left\| x^i - x \right\|_{\ell_p \left(G^k, v, \widehat{E}(r,s) \right)} = 0.$$

So indeed, the sequence x^i converges to x and they are both in $\ell_p \left(G^k, v, \widehat{E}(r,s) \right)$ for $1 \leq p < \infty$. So, we conclude the completeness of $\ell_p \left(G^k, v, \widehat{E}(r,s) \right)$. Therefore it is a Banach space. Hence the proof.

Theorem 2.2. Assume that $G = (g_n)$ is a sequence of modulus functions in $\mathbb{G}$. Then the given sequence spaces $\ell_p \left(G^k, v, \widehat{E}(r,s) \right)$ and $\ell_\infty \left(G^k, v, \widehat{E}(r,s) \right)$ are BK-spaces for $1 \leq p < \infty$, respectively, with the norms (2.2) and (2.3).

Proof. The proof is simply obtained. Since the conditions of (2.1) hold, $\widehat{E}(r,s)$ is a triangle matrix and both ℓ_p and ℓ_∞ are BK-spaces by their typical norms. Then by Theorem 4.3.12 of Wilansky [26], the proof can be obtained straightforwardly. Therefore, our sequence spaces are BK-spaces. Hence the proof.

Remark 2.2. It's clear to see that $\|x\|_{\ell_p \left(G^k, v, \widehat{E}(r,s) \right)} \neq |||x|||_{\ell_p \left(G^k, v, \widehat{E}(r,s) \right)}$ and $\|x\|_{\ell_\infty \left(G^k, v, \widehat{E}(r,s) \right)} \neq |||x|||_{\ell_\infty \left(G^k, v, \widehat{E}(r,s) \right)}$, this means that the difference sequence spaces $\ell_p \left(G^k, v, \widehat{E}(r,s) \right)$ and $\ell_\infty \left(G^k, v, \widehat{E}(r,s) \right)$ are of non-absolute type. From the above nonequalities, it has come to notice that the absolute property may not hold for

$\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ and $\ell_\infty\left(G^k, v, \widehat{E}(r,s)\right)$ from at least one sequence in which $|x| = (|x_n|)$ and $1 \le p < \infty$.

Theorem 2.3. Assume that $G = (g_n)$ is a sequence of modulus functions in $\mathbb{G}$. Then for $1 \le p < q$, the following inclusion relationship is satisfied.

$$\ell_p\left(G^k, v, \widehat{E}(r,s)\right) \subseteq \ell_q\left(G^k, v, \widehat{E}(r,s)\right)$$

Proof. By the use of (1.4), we consider a transformation $M : \ell_p\left(G^k, v, \widehat{E}(r,s)\right) \to \ell_p$ defined as $M(x) = v_n g_n^k\left(\widehat{E}_n(r,s)(x)\right)$ ($n \in \mathbb{N}$). Now if $x \in \ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ then it gives us $M(x) \in \ell_p$. Since $\ell_p \subset \ell_q$ for $1 \le p < q$, so we have $M(x) \in \ell_q$. Thus $x \in \ell_q\left(G^k, v, \widehat{E}(r,s)\right)$. So that for $1 \le p < q$ the inclusion $\ell_p\left(G^k, v, \widehat{E}(r,s)\right) \subset \ell_q\left(G^k, v, \widehat{E}(r,s)\right)$ holds. Hence the proof.

Assume that $G = (g_n)$ is a sequence of modulus functions in $\mathbb{G}$. And given two non-zero real numbers r and s as given in (1.3). Then we define the following equalities as follows:

$$D_5 = max\left\{5, \sup_n\left\{g_n^k(5)\right\}\right\}, \quad D_6 = max\left\{6, \sup_n\left\{g_n^k(6)\right\}\right\},$$

$$D_r = max\left\{|r|, \sup_n\left\{g_n^k(|r|)\right\}\right\} \quad \text{and} \quad D_s = max\left\{|s|, \sup_n\left\{g_n^k(|s|)\right\}\right\}.$$

The above equalities may be used in some steps of our study.

Theorem 2.4. Assume that $G = (g_n)$ is a sequence of modulus functions in $\mathbb{G}$. Then the indicated inclusion relationship below is valid.

$$\ell_p \subset \ell_p\left(G^k, v, \widehat{E}(r,s)\right) \text{ for } 1 \le p < \infty.$$

Proof. To verify the validity of the inclusion we need to find a number $D > 0$ such that $\|x\|_{\ell_p\left(G^k,v,\widehat{E}(r,s)\right)} \le D\|x\|_{\ell_p}$ for $x \in \ell_p$. From the Lucas sequence, we write $\frac{L_{n-1}}{L_n} \le 2$ and $\frac{L_n}{L_{n-1}} \le 3$ ($n \in \mathbb{N}$). Now we assume that $x \in \ell_p$, $1 \le p < \infty$. Then by using (1.4) and the above inequalities, we have

$$\begin{aligned}
\sum_n \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p &= \sum_n \left[v_n g_n^k\left(\left|r\frac{L_{n-1}}{L_n}x_n + s\frac{L_n}{L_{n-1}}x_{n-1}\right|\right)\right]^p \\
&\le \sum_n D_6^{p-1}\left[v_n g_n^k\left(|2rx_n| + |3sx_{n-1}|\right)\right]^p \\
&\le D_6^{2p-1}\max\{D_r, D_s\}\left(\sum_n\left[v_n g_n^k(|x_n|)\right]^p + \sum_n\left[v_n g_n^k(|x_{n-1}|)\right]^p\right)
\end{aligned}$$

and

$$\begin{aligned}
\sup_n\left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right] &= \sup_n\left[v_n g_n^k\left(\left|r\frac{L_{n-1}}{L_n}x_n + s\frac{L_n}{L_{n-1}}x_{n-1}\right|\right)\right] \\
&\le D_5\max\{D_r, D_s\}\sup_n\left[v_n g_n^k(|x_n|)\right].
\end{aligned}$$

Then for $1 < p \le \infty$, we have

$$\|x\|_{\ell_p\left(G^k,v,\widehat{E}(r,s)\right)} \le D_6^2\max\{D_r, D_s\}\|x\|_{\ell_p} \tag{2.6}$$

and

$$\|x\|_{\ell_\infty\left(G^k,v,\widehat{E}(r,s)\right)} \le D_5\max\{D_r, D_s\}\|x\|_{\ell_\infty}. \tag{2.7}$$

For $p = 1$, the inequality (2.6) is easily obtained. Hence the proof.

Example 2.1. The sequence $x = (x_n) = \left(\frac{1}{r}\left(-\frac{s}{r}\right)^n L_n^2\right)$ assures the strictness of the above inclusion relationship since $x \in \ell_p\left(G^k, v, \widehat{E}(r,s)\right) - \ell_p$.

Theorem 2.5. Assume that $G = (g_n)$ is a sequence of modulus functions in $\mathbb{G}$, and let $\beta_m = v_n g_n^k\left(\widehat{E}_m(r,s)(x)\right)$. Then for $1 \le p < \infty$, the sequence $\left(h^{(m)}\right)_{n=1}^{\infty}$ provides a basis for $\ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ which is formed as

$$\left(h^{(m)}\right)_n = \begin{cases} \frac{1}{r}\left(-\frac{s}{r}\right)^{m-n} \frac{{L_n}^2}{L_{m-1}L_m}, & n \ge m \\ 0, & m > n. \end{cases}$$

For that, every $x \in \ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ can be uniquely represented in the form

$$x = \sum_m \beta_m h^{(m)}. \tag{2.8}$$

Proof. By using the sequence $h^{(m)}$ we get $v_n g_n^k\left(\widehat{E}(r,s)\left(h^{(m)}\right)\right) = e^{(m)} \in \ell_p$ where $e^{(m)} = (0, 0, \ldots, 1, 0, \ldots)$ (i.e. 1 at the m^{th} place and zero elsewhere) for each $m \in \mathbb{N}$. Hence $h^{(m)} \in \ell_p\left(G^k, v, \widehat{E}(r,s)\right)$. In addition, let $x \in \ell_p\left(G^k, v, \widehat{E}(r,s)\right)$ and for every $i \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, take

$$x^{(i)} = \sum_{m=1}^{i} \beta_m h^{(m)}.$$

Thus

$$\begin{aligned} v_n g_n^k\left(\widehat{E}\,(r,s)\left(x^{(i)}\right)\right) &= \sum_{m=1}^{i} v_n g_n^k\left(\widehat{E}_m(r,s)(x)\right) v_n g_n^k\left(\widehat{E}(r,s)\left(h^{(m)}\right)\right) \\ &= \sum_{m=1}^{i} \beta_m e^{(m)}. \end{aligned}$$

Also

$$v_n g_n^k\left(\widehat{E}_{\mathrm{n}}\,(r,s)\left(x - x^{(i)}\right)\right) = \begin{cases} v_n g_n^k\left(\widehat{E}_{\mathrm{n}}\,(r,s)(x)\right), & n > i \\ 0, & 0 \le n \le i. \end{cases}$$

Then, there is $i_0 \in \mathbb{N}_0$ such that

$$\sum_{n=i_0+1}^{\infty} \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p \le \left(\frac{\varepsilon}{2}\right)^p$$

for any $\varepsilon > 0$. Therefore, for every $i > i_0$, we have

$$\begin{aligned} \left\|x - x^{(i)}\right\|_{\ell_p\left(G^k, v, \widehat{E}(r,s)\right)} &= \left(\sum_{n=i+1}^{\infty} \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p\right)^{\frac{1}{p}} \\ &\le \left(\sum_{n=i_0+1}^{\infty} \left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p\right)^{\frac{1}{p}} \le \frac{\varepsilon}{2} < \varepsilon. \end{aligned}$$

This concludes that

$$\lim_{i \to \infty} \left\|x - x^{(i)}\right\|_{\ell_p\left(G^k, v, \widehat{E}(r,s)\right)} = 0.$$

Moreover, to show that (2.8) is unique, let us consider

$$x=\sum_m \xi_m h^{(m)}\ ,$$

for $x\in\ell_p\left(G^k,v,\widehat{E}(r,s)\right)$. Then, we have

$$\begin{aligned} v_n g_n^k\left(\widehat{E}_n(r,s)(x)\right) &= \sum_m \xi_m\left(v_n g_n^k\left(\widehat{E}_n(r,s)\left(h^{(m)}\right)\right)\right) \\ &= \sum_m \xi_m e_n^{(m)}=\xi_n. \end{aligned}$$

Hence the proof.

Theorem 2.6. Assume that $G=(g_n)$ is a sequence of modulus functions in $\mathbb{G}$. Then the Gurarii's modulus of convexity for $\ell_p\left(G^k,v,\ \widehat{E}(r,s)\right)$, $1\leq p<\infty$ can be represented in the form of the following inequality

$$\gamma_{\ell_p\left(G^k,v,\ \widehat{E}(r,s)\right)}(\varepsilon)\leq 1-\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}},\quad \varepsilon\in[0,2].$$

Proof. Take $x\in\ell_p\left(G^k,v,\ \widehat{E}(r,s)\right)$. Then,

$$\begin{aligned} \|x\|_{\ell_p\left(G^k,v,\ \widehat{E}(r,s)\right)} &= \left\|v_n g_n^k\left(\widehat{E}_n(r,s)(x)\right)\right\|_{\ell_p} \\ &= \left(\sum_n\left[v_n g_n^k\left(\left|\widehat{E}_n(r,s)(x)\right|\right)\right]^p\right)^{\frac{1}{p}}. \end{aligned}$$

We also take the sequences $a=(a_m)$ and $b=(b_m)$, where

$$a_m=\left(\left(\left(v_n g_n^k\right)^{-1}\right)\widehat{E}^{-1}(r,s)\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}},\ \left(v_n g_n^k\right)^{-1}\widehat{E}^{-1}(r,s)\left(\frac{\varepsilon}{2}\right),0,\ 0,\ \ldots\right)$$

and

$$b_m=\left(\left(\left(v_n g_n^k\right)^{-1}\right)\widehat{E}^{-1}(r,s)\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}},\ \left(v_n g_n^k\right)^{-1}\widehat{E}^{-1}(r,s)\left(-\frac{\varepsilon}{2}\right),0,\ 0,\ \ldots\right).$$

Where $\widehat{E}^{-1}(r,s)$ represents the inverse of the matrix $\widehat{E}(r,s)$ and $\varepsilon\in[0,2]$. The $\widehat{E}$-transforms of the sequences a and b are given by

$$v_n g_n^k\widehat{E}(r,s)(a)=\left(\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}},\ \left(\frac{\varepsilon}{2}\right),0,\ 0,\ \ldots\right)$$

and

$$v_n g_n^k\widehat{E}(r,s)(b)=\left(\left(1-\left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}},\ \left(-\frac{\varepsilon}{2}\right),0,\ 0,\ \ldots\right).$$

Then, we have

$$\left\|v_n g_n^k\widehat{E}(r,s)(a)\right\|_{\ell_p}=\|a\|_{\ell_p\left(G^k,v,\ \widehat{E}(r,s)\right)}=1,$$

and

$$\left\|v_n g_n^k\widehat{E}(r,s)(b)\right\|_{\ell_p}=\|b\|_{\ell_p\left(G^k,v,\ \widehat{E}(r,s)\right)}=1.$$

Hence, $a,\ b \in S_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)}$, and

$$\left\| v_n g_n^k \widehat{E}(r,s)(a) - v_n g_n^k \widehat{E}(r,s)(b) \right\|_{\ell_p} = \|a-b\|_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)} = \varepsilon.$$

Now, for $\alpha \in [0,1]$,

$$\begin{aligned} \|\alpha a + (1-\alpha)b\|^p_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)} &= \left\| \alpha v_n g_n^k \widehat{E}(r,s)(a) + (1-\alpha) v_n g_n^k \widehat{E}(r,s)(b) \right\|^p_{\ell_p} \\ &= 1 - \left(\frac{\varepsilon}{2}\right)^p + |2\alpha - 1|^p \left(\frac{\varepsilon}{2}\right)^p. \end{aligned}$$

From here,

$$\inf_{\alpha \in [0,1]} \|\alpha a + (1-\alpha)b\|_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)} = \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Therefore, for $1 \le p < \infty$,

$$\gamma_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)}(\varepsilon) \le 1 - \left(1 - \left(\frac{\varepsilon}{2}\right)^p\right)^{\frac{1}{p}}.$$

Hence the proof.

Corollary 2.7

(i) If $\varepsilon = 2$, then $\gamma_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)}(\varepsilon) \le 1$ and so that $\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)$ is strictly convex.

(ii) If $0 < \varepsilon < 2$, then $0 < \gamma_{\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)}(\varepsilon) < 1$ and so that $\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)$ is uniformly convex.

3. Conclusions

In this study, by using the Lucas band matrix $\widehat{E}(r,s)$, a sequence of strictly positive real numbers $v = (v_n)$ and a sequence of modulus functions $G = (g_n)$ with $1 \le p < \infty$, the sequence spaces $\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)$ and $\ell_\infty\left(G^k, v,\ \widehat{E}(r,s)\right)$ have been generalized, then they are established as BK-spaces with some given norms. After that, the connection between ℓ_p and $\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)$ for $1 \le p < \infty$ was founded. Finally the basis and the Gurarii's modulus of convexity for the space $\ell_p\left(G^k, v,\ \widehat{E}(r,s)\right)$ $(1 \le p < \infty)$ have been determined independently.

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