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Conformal Ricci-Yamabe solitons on warped product manifolds

Jay Prakash Singh[a], **Robert Sumlalsanga**[b]

[a]*Department of Mathematics, Central University of South Bihar. Gaya-824236, India*
[b]*Department of Mathematics and Computer Science, Mizoram University. Aizawl-796004, India*

Abstract. Self-similar solutions of the conformal Ricci-Yamabe flow equation are called conformal Ricci-Yamabe solitons. This paper mainly concerned with the study of conformal Ricci-Yamabe solitons within the structure of warped product manifolds, which extend the notion of usual Riemannian product manifolds. First, the proof is provided that the base and the fiber sharing the same property implies the existence of a warped product manifold admitting a conformal Ricci-Yamabe soliton. In the next section, warped product manifolds are used to study the characterization of conformal Ricci-Yamabe solitons in terms of Killing and conformal vector fields. Finally, we prove that a conformal Ricci-Yamabe soliton with a concurrent potential vector field admitted on a warped product manifold is Ricci flat.

1. Introduction

The study of the theory of Ricci flow (1982) by Hamilton [2, 26, 27] reached its highest magnitude and popularity soon after Perelman [10, 11] successfully applied it to solve the *Poincaré* conjecture. Ricci solitons were also studied by Hamilton, who viewed them as fixed or stationary points of the Ricci flow in the space of parameterized metrics $g(t)$ on scaling and M modulo diffeomorphisms. Since then, both topics have been extensively explored by numerous mathematicians, including Brendle [30], Cao [13], Chen[6] and many others (see for instance [7, 9, 14, 18, 21]).

A smooth manifold M furnished with a Riemannian metric g is known to be a Ricci soliton if, for some constant λ, there exists a smooth vector field X on M satisfying the following equation:

$$Ric + \frac{1}{2}\mathcal{L}_X g = \lambda g,$$

where $\mathcal{L}_X$ denotes the Lie derivative and Ric is the Ricci tensor. The Ricci soliton is called shrinking, steady and expanding if $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$ respectively.

Recently, self-similar solutions, referred as soliton solutions of various geometric flow equations, have been introduced and explored due to their potential significance as models for singularities. Among these, Ricci soliton defined as fixed points of the Ricci flow stand as one of the most extensively studied and renowned classes. Substantial advancements have been achieved in this particular domain. Furthermore, Hamilton [20] established the Yamabe flow to address the Yamabe problem. This problem primarily

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Email addresses: jpsmaths@gmail.com (Jay Prakash Singh), robertsuma23@gmail.com (Robert Sumlalsanga)

involves the quest for a metric on a manifold with a dimension of $n \geq 3$, such that its scalar curvature remains constant. Hence, the Yamabe flow is precisely delineated by the metric $g(t)$ on a Riemannian manifold (M^n, g), which adheres to the condition $\frac{\partial g(t)}{\partial(t)} = -Rg(t)$, where R denotes the scalar curvature of M. Notably, the two-dimensional scenario had already been resolved through the application of the Uniformization Theorem. For an in-depth exploration of this concept, we direct the reader to references [22], [28], and [29]. It is pertinent to mention that this flow finds its solution through the lens of gradient Yamabe solitons, which are rigorously defined as follows:

A pseudo-Riemannian manifold (M^n, g) is said to be a gradient Yamabe soliton if there exists a smooth function ϕ on M and a constant λ satisfying

$$Hess(\phi) = (R - \lambda)g,$$

where $Hess(\phi)$ is the Hessian of ϕ, R is the scalar curvature and g is the metric.
A gradient Yamabe soliton on a pseudo-Riemannian manifold (M^n, g) is classified as shrinking if $\lambda > 0$, expanding if $\lambda < 0$ or steady if $\lambda = 0$. Notably, a seminal outcome pertaining to the resolution of the Yamabe problem is documented in [22], wherein it is demonstrated that the metric of any compact Yamabe gradient soliton is a metric characterized by constant scalar curvature.

In 2019, G*ü*ler and Crasmareanu [12] introduced the Ricci-Yamabe flow for a smooth n-dimensional manifold M^n. Later, the notion of the conformal Ricci-Yamabe soliton equation was introduced by Zhang et.al, [23] as

$$\mathcal{L}_X g + 2\alpha Ric = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] g, \tag{1}$$

where α, β, λ are constants and p is the conformal pressure. The equation is the generalization of the Ricci-Yamabe soliton equation and it satisfies the conformal Ricci-Yamabe flow equation (For details, see [1, 16, 17, 19]).

Bishop and O'Neill [24] pioneered the warped product notion, which provides an adaptable foundation for complete manifolds of negative curvature. The origins of this notion may be traced back to the surfaces of revolution. Warped products are important in differential geometry and have applications in mathematical physics and general relativity. This multidisciplinary appeal has piqued the interest of mathematicians and physicists alike, resulting in a thriving field of research [4, 5]. Consider B and F Riemannian manifolds, as well as a positive smooth function f defined on B. The product manifold metric $B \times F$ is defined as follows:

$$g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F, \tag{2}$$

where π and σ are the natural projections onto the base and fiber manifold respectively. The product manifold is designated by $M = B \times_f F$ and is said to be the warped product of B and F under this condition. In this context, manifolds B and F are referred to as the base and fiber respectively, where g_B and g_F are the induced metric on base and fiber respectively. The function f is called the warping function.

Our investigation is centered around identifying the conditions that render the warped product a conformal Ricci-Yamabe soliton. To commence, let's begin by revisiting a pivotal result (for detailed information, refer to [25]) that will prove essential for our subsequent discussions.

Lemma 1.1. *[25] Let $(M, g) = (B \times_f B, g_B \oplus f^2 g_F)$ be a warped product of two Riemannian manifolds B and F with $dimB = m$ and $dimF = n$. Then, for all $X, Y \in \mathfrak{X}(B)$ and $U, V \in \mathfrak{X}(F)$*

(i) $D_X U = D_U X = \frac{X(f)}{f} U,$
(ii) $Ric(X, U) = 0,$
(iii) $Ric(X, Y) = Ric^B(X, Y) - \frac{n}{f} H^f(X, Y),$
(iv) $Ric(U, V) = Ric^F(U, V) - \left(\frac{\Delta f}{f} + (n-1)\frac{\|\nabla f\|^2}{f^2}\right) g(U, V),$

where $D_X Y$ is the lift of $\nabla_X Y$ on B and Ric^B, Ric^F are the lifts of the Ricci tensors on the base B and the fiber F respectively.

Motivated by their research [18], we extend our exploration to the realm of conformal Ricci-Yamabe solitons with $\alpha \neq 0$ on warped product manifolds. Our paper is structured as follows: In Section 2, we delve into the investigations surrounding conformal Ricci-Yamabe solitons on warped product manifolds, examining the impact on both the base and fiber manifolds when a warped product manifold admits a conformal Ricci-Yamabe soliton. Section 3 centers on the influence of specific types of smooth vector fields such as Killing vector fields, conformal vector fields, and concurrent vector fields on conformal Ricci-Yamabe solitons within warped product spaces. Section 4 is devoted to the study of the admittance of conformal Ricci-Yamabe soliton with a concurrent vector field in a warped product manifold.

2. Investigation of Conformal Ricci-Yamabe Soliton on Warped Product Manifolds

This section presents an exploration into the realm of conformal Ricci-Yamabe solitons on warped product manifolds. Our central objective here is to meticulously examine the repercussions of a warped product manifold adopting the characteristics of a conformal Ricci-Yamabe soliton. In particular, we are keen to elucidate how this alignment influences both the base manifold and the fiber manifold, effectively isolating the precise conditions that facilitate their transformation into conformal Ricci-Yamabe solitons. To facilitate our investigation, we consider the manifold $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, representing a warped product of two Riemannian manifolds B and F with $dimB = m$ and $dimF = n$ respectively. Proceeding further, let (M, g, μ, ξ) exemplify a conformal Ricci-Yamabe soliton, with $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$. Consequently, by invoking equation (1), we obtain:

$$\mathcal{L}_\xi g + 2\alpha Ric = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] g = \mu g, \tag{3}$$

where $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$. Recalling Lemma 1.1 from the preceding section, we readily deduce the following two widely recognized formulas tailored for warped product manifolds:

$$\mathcal{L}_\xi g = \mathcal{L}^B_{\xi_B} g_B + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f) g_F, \tag{4}$$

$$Ric = Ric^B - \frac{n}{f} H^f + Ric^F - \tilde{f} g_F, \tag{5}$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|^2_B$. Now, applying the definition of warped metric from equation (3) and using (4) and (5), we have

$$\begin{aligned} \mu(g_B + f^2 g_F) = \mu g =& \mathcal{L}_\xi g + 2\alpha Ric \\ =& \mathcal{L}^B_{\xi_B} + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f) g_F + 2\alpha Ric^B - 2\alpha \frac{n}{f} H^f + 2\alpha Ric^F - 2\alpha \tilde{f} g_F. \end{aligned} \tag{6}$$

Again, for all $U, V \in \mathfrak{X}(B)$, we can write

$$\left(\mathcal{L}^B_{\xi_B}\right)(U, V) = g_B(D^B_U \xi_B, V) + g_B(U, D^B_V \xi_B). \tag{7}$$

By definition of Hessian and (7), we get

$$\left(\mathcal{L}^B_{\xi_B} g_B - 2\alpha \frac{n}{f} H^f\right)(U, V) = g_B(D^B_U \xi_B, V) + g_B(U, D^B_V \xi_B) - 2\alpha \frac{n}{f} g_B(D^B_U \nabla^B f, V).$$

The above equation can be reduced as

$$\begin{aligned} \left(\mathcal{L}^B_{\xi_B} g_B - 2\alpha \frac{n}{f} H^f\right)(U, V) =& g_B(D^B_U \xi_B, V) - \alpha \frac{n}{f} g_B(D^B_U \nabla^B f, V)) + (g_B(U, D^B_V \xi_B) - \alpha \frac{n}{f} g_B(D^B_U \nabla^B f, V) \\ =& g_B(D^B_U(\xi_B - \alpha n \nabla^B lnf), V) + g_B(U, D^B_V(\xi_B - \alpha n \nabla^B lnf)) \\ =& (\mathcal{L}^B_{\xi_B - \alpha n \nabla^B lnf} g_B)(U, V), \forall U, V \in \mathfrak{X}(B). \end{aligned} \tag{8}$$

Since it is true for all $U, V \in \mathfrak{X}(B)$, in operator notation we can write

$$\mathcal{L}^B_{\xi_B} g_B - 2\alpha\frac{n}{f}H^f = \mathcal{L}^B_{\xi_B - \alpha n \nabla^B lnf} g_B. \tag{9}$$

Putting the value of (9) in (6) gives

$$(\mathcal{L}_{\xi_B - \alpha n \nabla^B lnf} g_B + 2\alpha Ric^B) + (f^2 \mathcal{L}^F_{\xi_F} g_F + 2\alpha Ric^F) = \mu g_B + (\mu f^2 - 2f\xi_B(f) + 2\tilde{f})g_F. \tag{10}$$

Hence, we can state the following theorem.

Theorem 2.1. *Suppose a warped product of two Riemannian manifolds B and F is* $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, *with the warping function denoted as* f. *We have* $dimB = m$ *and* $dimF = n$. *Assuming that* (M, g, μ, ξ) *forms a conformal Ricci-Yamabe soliton, it follows that both the base* $(B, g_B, \mu, \xi_B - \alpha n \nabla^B lnf)$ *and the fiber* $(F, g_F, \mu f^2 - 2f\xi_B(f) + 2\tilde{f}, f^2\xi_F)$ *exhibit the characteristics of conformal Ricci-Yamabe solitons. Here,* $\tilde{f} = f\Delta f + (n-1)|\nabla f|^2_B$ *and* $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$, *where* λ, β *and* p *represent the soliton constants and the conformal pressure respectively.*

Now, let us consider the soliton vector field ξ of the conformal Ricci-Yamabe soliton (M, g, μ, ξ) to be a gradient of some smooth function ϕ, i.e., when $\xi = grad\phi = \nabla\phi$. The function ϕ is referred to as the potential function of the soliton, and the soliton is known as a conformal gradient Ricci-Yamabe soliton. To maintain clarity in notation, A conformal gradient Ricci-Yamabe soliton is designated as (M, g, μ, ϕ), where the final term identifies the soliton's potential function.

Let us consider a warped product of two Riemannian manifolds B and F be $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$. We have $dimB = m$ and $dimF = n$. Now, if (M, g, μ, ϕ) forms a conformal gradient Ricci-Yamabe soliton, for any vector fields $X, Y \in \mathfrak{X}(M)$, then (3) provides

$$2H^\phi(X, Y) + 2\alpha Ric(X, Y) = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] g(X, Y) = \mu g(X, Y). \tag{11}$$

Again, we consider $X = X_B$ and $Y = Y_B$, with X_B and Y_B representing the lifts of the vector fields X and Y in $\mathfrak{X}(B)$, then (11) reduces to

$$2H^\phi(X_B, Y_B) + 2\alpha Ric(X_B, Y_B) = \mu g(X_B, Y_B).$$

By substituting the value of the Ricci tensor for the base manifold as provided in Lemma 1.1, the equation above can be expressed as

$$2H^{\phi B}_B(X_B, Y_B) + 2\alpha Ric^B(X_B, Y_B) - 2\alpha\frac{n}{f}H^f_B(X_B, Y_B) = \mu g_B(X_B, Y_B),$$

where $\phi_B = \phi$ $\phi_B = \phi$ at a specific point of the fiber F. Finally, by employing the properties of the Hessian in the equation above, we obtain

$$2H^{\phi_B - \alpha n lnf}_B(X_B, Y_B) + 2\alpha Ric^B(X_B, Y_B) = \mu g_B(X_B, Y_B). \tag{12}$$

This demonstrates that $(B, g_B, \mu, \phi B - \alpha n lnf)$ qualifies as a conformal gradient Ricci-Yamabe soliton.

Furthermore, if we set $(X = X_F)$ and $(Y = Y_F)$, with X_F and Y_F representing the lifts of the vector fields X and Y in $\mathfrak{X}(F)$, then (11) yields

$$2H^\phi(X_F, Y_F) + 2\alpha Ric(X_F, Y_F) = \mu g(X_F, Y_F).$$

Using (5) and Lemma 1.1, the above equation becomes

$$2H^{\phi_F}_F(X_F, Y_F) + 2\alpha Ric^F(X_F, Y_F) - \tilde{f} g_F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F),$$

where $\phi_F = \phi$ at a fixed point of the base B and $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. From the above equation, we get

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha Ric^F(X_F, Y_F) = (\mu f^2 + \tilde{f})g_F(X_F, Y_F).$$

Hence, given the constancy of the warping function f, the term $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$ disappears from the right-hand side of the above equation, we get

$$2H_F^{\phi_F}(X_F, Y_F) + 2\alpha Ric^F(X_F, Y_F) = \mu f^2 g_F(X_F, Y_F). \tag{13}$$

Thus, $(F, g_F, \mu f^2, \phi_F)$ is a conformal gradient Ricci-Yamabe soliton. Hence, from the above observations and (12) and (13), we can state the following:

Theorem 2.2. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f. We have $dimB = m$ and $dimF = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, then*

(i) *the base $(B, g_B, \mu, \phi_B - nlnf)$ forms a conformal gradient Ricci-Yamabe soliton with $\phi_B = \phi$ at a point within the fiber F.*
(ii) *the fiber $(F, g_F, \mu f^2, \phi_f)$ forms a conformal gradient Ricci-Yamabe soliton with $\phi_F = \phi$ at a point within the base B, given that the warping function f is constant.*

Remark 2.3. *For particular values of $\alpha = 1, \beta = 0$, the above theorem reduces to Theorem 2.2 of [18].*

3. Influence of Distinctive Vector Field Types on Conformal Ricci-Yamabe Solitons in Warped Product Manifolds

This section's major goal is to investigate how particular forms of smooth vector fields affect conformal Ricci-Yamabe solitons in warped product spaces. We shall pay special attention to the conformal and killing vector fields, which are defined as follows:

Definition 3.1. *A smooth vector field X on a Riemannian manifold (M, g) is called*

(i) *Killing vector field or an infinitesimal isometry, if the local 1-parameter group of transformations generated by X in a neighbourhood of each point of M consists of local isometries, or in other words, if X satisfies $\mathcal{L}_X g = 0$ and*
(ii) *conformal vector field if X satisfies $\mathcal{L}_X g = \rho g$,*

Here, ρ represents a smooth function defined on the manifold M and $\mathcal{L}_X g$ refers the Lie derivative of the Riemannian metric g along the vector field X.

As per the definition outlined above, our initial result in this section is presented below:

Proposition 3.2. *Let $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$ represent a warped product of two Riemannian manifolds B and F denoted as a warping function f. Here, $dimB = m$ and $dimF = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, and under the condition that any one of the following conditions holds*

(i) *$\xi = \xi_B$ and ξ_B is a Killing vector field on the base B.*
(ii) *$\xi = \xi_F$ and ξ_F is a Killing vector field on the fiber F.*

Then, under any one of the above conditions, the manifold (M, g) transforms into an Einstein manifold.

Proof. Based on our assumption that (M, g, μ, ξ) constitutes a conformal Ricci-Yamabe soliton, it complies with equation (3) gives

$$\mathcal{L}_\xi g + 2\alpha Ric = \mu g. \tag{14}$$

Now, let $\xi = \xi_B$, and ξ_B is a Killing on B, we get $\mathcal{L}^B_{\xi_B} g_B = 0$. Then, (4) gives $\mathcal{L}_\xi g = 0$. Therefore, (14) yields $Ric = \frac{\mu}{2\alpha} g$ and this implies (M, g) is an Einstein manifold.
Furthermore, if we consider $\xi = \xi_F$ and note that ξ_F is a Killing vector field on F, we find that $\mathcal{L}^F_{\xi_F} g_F = 0$. Proceeding similarly to the previous part of the proof, we get the same result. □

Theorem 3.3. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f. We have $dimB = m$ and $dimF = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton, it follows that ξ_B is a Killing vector field on the base B, then the base $(B, g_B, \mu, -n\alpha lnf)$ forms a conformal Ricci-Yamabe soliton, where ξ_B is the lift of the vector field ξ to $\mathfrak{X}(B)$.*

Proof. Considering the given information that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton as per Theorem 2.1, we can conclude that the base $(B, g_B, \mu, \xi_B - n\alpha\nabla^B lnf)$ also qualifies as a conformal Ricci-Yamabe soliton and hence it satisfies (3). Thus, we can write

$$\mathcal{L}^B_{\xi_B - n\alpha\nabla^B lnf} g_B + 2\alpha Ric^B = \mu g_B. \tag{15}$$

Again, using (9) in the above equation, (15) becomes

$$\mathcal{L}^B_{\xi_B} g_B - 2\alpha\frac{n}{f}H^f + 2\alpha Ric^B = \mu g_B.$$

Given that ξ_B is a Killing vector field on the base B, we can assert that $\mathcal{L}^B_{\xi_B} g_B = 0$. Then, the above equation gives us

$$-2\alpha\frac{n}{f}H^f + 2\alpha Ric^B = \mu g_B.$$

Thus, using the properties of Hessian, the above equation results in

$$2H^{-\alpha n lnf} + 2\alpha Ric^B = \mu g_B. \tag{16}$$

Hence, comparing (16) with (11) completes the proof. □

The following conclusion is reached from the study of Killing vector fields on conformal Ricci-Yamabe soliton warped product manifolds:

Theorem 3.4. *Suppose a warped product of two Riemannian manifolds B and F is $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, with the warping function denoted as f. We have $dimB = m$ and $dimF = n$. Assuming that (M, g, μ, ξ) forms a conformal Ricci-Yamabe soliton and the lifts ξ_B and ξ_F are both Killing on the base B and the fiber F respectively. Then, the manifold (M, g) is Einstein provided $\xi_B(f) = 0$.*

Proof. As provided, both ξ_B and ξ_F are Killing vector fields, implying $\mathcal{L}^B_{\xi_B} g_B = 0$ and $\mathcal{L}^F_{\xi_F} g_F = 0$. Utilizing these values in equation (4), we get

$$\mathcal{L}_\xi g = 2f\xi_B(f)g_F. \tag{17}$$

Again, assuming that (3) is true and that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton, we obtain

$$\mathcal{L}_\xi g + 2\alpha Ric = \mu g.$$

Now, using (17) in the above equation gives us

$$2f\xi_B(f)g_F + 2\alpha Ric = \mu g. \tag{18}$$

The proof is complete if $\xi_B(f) = 0$, in which case the above equation (18) produces $Ric = \frac{\mu g}{2\alpha}$, indicating that the manifold (M, g) is Einstein. □

Now, our attention turns towards examining the impact of conformal vector fields on warped product manifolds that admit conformal Ricci-Yamabe solitons. An immediate result is given below:

Proposition 3.5. *Suppose a warped product of two Riemannian manifolds B and F is* $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, *with the warping function denoted as* f. *We have* $dimB = m$ *and* $dimF = n$. *Assuming that* (M, g, μ, ξ) *forms a conformal Ricci-Yamabe soliton. Then, the manifold* (M, g) *transforms into an Einstein manifold with the factor* $\left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha}\right)$ *if and only if the vector field* ξ *is conformal with a factor of* 2ρ.

Proof. Since (M, g, μ, ξ) being a conformal Ricci-Yamabe soliton, equation (3) gives

$$\mathcal{L}_\xi g + 2\alpha Ric = \mu g. \tag{19}$$

Assuming that the vector field ξ is conformal with a factor of 2ρ, according to Definition 3.1, we have $\mathcal{L}_X g = 2\rho g$, where ρ is a smooth function. Therefore, by substituting this value into equation (19), we obtain

$$Ric = \left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha}\right) g. \tag{20}$$

As a result, (M, g) is an Einstein manifold from this. The reverse calculation method can also be used to demonstrate that (M, g) is an Einstein manifold with factor $\left(\frac{\mu}{2\alpha} - \frac{\rho}{\alpha}\right)$. Afterwards, ξ conforms to factor 2ρ. The evidence is now complete. □

It should be noted that the vector field ξ was assumed to be conformal when discussing conformal Ricci-Yamabe solitons in the aforementioned result. It follows that the question of whether it is necessary to consider ξ to be conformal as a whole or if a weaker condition exists is a natural one. The following theorem may help to clarify the situation.

Theorem 3.6. *Suppose a warped product of two Riemannian manifolds B and F is* $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, *with the warping function denoted as* f. *We have* $dimB = m$ *and* $dimF = n$. *Assuming that* (M, g, μ, ξ) *forms a conformal Ricci-Yamabe soliton, it follows that the lifts* ξ_B *and* ξ_F *are both conformal on the base B and the fiber F with the factors* $2\rho_B$ *and* $2\rho_F$ *respectively. Here* ρ_B *and* ρ_F *are two smooth functions. Under this conditions, if* $\rho_B = \rho_F + \xi_B(lnf)$, *then the manifold* (M, g) *turns out to be Einstein.*

Proof. Given that ξ_B is conformal on the base B with a factor of $2\rho_B$, it follows that $\mathcal{L}^B_{\xi_B} g_B = 2\rho_B g_B$. Similarly, for ξ_F being conformal with a factor of $2\rho_F$, we obtain $\mathcal{L}^F_{\xi_F} g_F = 2\rho_F g_F$. Then, using these values in (4), we get

$$\mathcal{L}_\xi g = 2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F). \tag{21}$$

Again, (M, g, μ, ξ) being a conformal Ricci-Yamabe soliton, from (3) and the above equation (21), we have

$$2(\rho_B g_B + f^2 \rho_F g_F + f \xi_B(f) g_F) = \mu g.$$

Now, the above equation gives us

$$Ric = \frac{1}{\alpha}\left(\frac{\mu g}{2} - \rho_B g_B - f^2(\rho_F + \xi_B(lnf)\right) g_F. \tag{22}$$

Setting $\rho_B = \rho_F + \xi_B(lnf)$ and using (2), the above equation (22) yields $Ric = \left(\frac{\mu}{2\alpha} - \frac{\rho_B}{\alpha}\right) g$. This implies that the manifold (M, g) is Einstein and this concludes the proof. □

Our final theorem, which provides the converse of the prior theorem, brings this section to a close. The conformal Ricci-Yamabe soliton (M, g, μ, ξ) was described in the previous result, and the next result outlines the circumstances in which a warped product manifold (M, g) admits a conformal Ricci-Yamabe soliton.

Theorem 3.7. *Assume* (B, g_B, μ, ξ_B) *be a conformal Ricci-Yamabe soliton, F be an Einstein manifold with factor* β. *We have* $dimB = m$ *and* $dimF = n$. *Assume that* $(M, g) = ((B \times_f F, g_B \oplus f^2 g_F)$ *is a warped product of two Riemannian manifolds B and F with a warping function* f *and* ξ_F *is conformal vector field with factor* 2ρ, *if* $H^f = 0$ *and the warping function* f *satisfies the quadratic equation*

$$(2\rho - \mu) f^2 + 2f\xi_B(f) + 2\beta + 2(1 - n)k^2 = 0,$$

where $k^2 = \|\nabla f\|^2_B = g_B(\nabla f, \nabla f))$ *for some real number* k, *then* (M, g, μ, ξ) *is a conformal Ricci-Yamabe soliton.*

Proof. Let (B, g_B, μ, ξ_B) be a conformal Ricci-Yamabe soliton, from (3) we get

$$\mathcal{L}^B_{\xi_B} g_B + 2\alpha Ric^B = \mu g_B. \tag{23}$$

Since F is an Einstein manifold with a factor of β, we can express the Ricci tensor as $Ric^F = \beta g_F$. By substituting this value into equation (5), we get

$$Ric = Ric^B - \alpha \frac{n}{f} H^f + \beta g_F - \tilde{f} g_F, \tag{24}$$

where $\tilde{f} = f\Delta f + (n-1)||\nabla f||^2_B$. Using (23) in (4), we have

$$\mathcal{L}_\xi g = \mu g_B - 2\alpha Ric^B + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f) g_F. \tag{25}$$

Multiplying both sides of the equation (24) by 2 and then adding it with (25) gives

$$\mathcal{L}_\xi g + 2\alpha Ric = \mu g_B + f^2 \mathcal{L}^F_{\xi_F} g_F + 2f\xi_B(f) g_F + 2\left(-\frac{n}{f} H^f + \beta g_F - \tilde{f} g_F\right).$$

Again, considering the vector field ξ_F is conformal with a factor of 2ρ, i.e., $\mathcal{L}^F_{\xi_F} g_F = 2\rho g_F$, we get

$$\mathcal{L}_\xi g + 2\alpha Ric = \mu g_B + 2\alpha f^2 \rho g_F + 2\alpha f \xi_B(f) g_F + 2\alpha\left(-\frac{n}{f} H^f + \beta g_F - \tilde{f} g_F\right). \tag{26}$$

Given that $H^f = 0$, it follows that $\Delta f = 0$, and consequently, $\tilde{f} = f\Delta f + (n-1)|\nabla f|^2_B$ simplifies to $\tilde{f} = (n-1)|\nabla f|^2_B = (n-1)k^2$, where $k^2 = |\nabla f|^2_B = g_B(\nabla f, \nabla f)$ for some real number k. Thus, utilizing these results in equation (26), we obtain

$$\begin{aligned}\mathcal{L}_\xi g + 2\alpha Ric =& \mu g_B + 2\alpha f^2 \rho g_F + 2\alpha f \xi_B(f) g_F + 2\alpha(\beta g_F - (n-1)k^2 g_F) \\ =& \mu(g_B + 2f^2 g_F) + \{2\alpha f^2 \rho - \mu f^2 + 2\alpha f \xi_B(f) + 2\alpha(\beta - (n-1)k^2)\} g_F.\end{aligned}$$

Thus, if $2\alpha f^2\rho - \mu f^2 + 2\alpha f\xi_B(f) + 2\alpha(\beta - (n-1)k^2) = 0$ i.e., if f satisfies the quadratic equation $(2\alpha\rho - \mu)f^2 + 2\alpha f \xi_B(f) + 2\alpha\beta + 2\alpha(1-n)k^2 = 0$, the foregoing equation reduces to

$$\mathcal{L}_\xi g + 2\alpha Ric = \mu(g_B + f^2 f_F) = \mu g. \tag{27}$$

Hence, we can establish that (M, g, μ, ξ) indeed forms a conformal Ricci-Yamabe soliton, thereby concluding the proof. □

4. Conformal Ricci-Yamabe Soliton on Warped Product Manifolds with Concurrent Vector Field

The concept of concircular vector fields is introduced by Yano [15] as a means to investigate concircular mappings, which essentially correspond to conformal mappings that uphold the integrity of geodesic circles. These vector fields find a multitude of applications in mathematical physics and general relativity. Chen [3] established the theorem that a Lorentzian manifold can be classified as a generalized Robertson-Walker spacetime if and only if it accommodates a timelike concircular vector field. A vector field ξ on a Riemannian manifold M satisfying

$$\nabla_X \xi = \alpha X, \tag{28}$$

for all vector fields $X \in \mathfrak{X}(M)$, is called a concircular vector field [6], where α is a non-trivial function on M. In particular, if the function α is constant, then the vector field ξ is called a concurrent vector field. Thus, we have the following definition [6]:

Definition 4.1. *A vector field ξ defined on a Riemannian manifold M is termed a concurrent vector field if it satisfies the following equation for all vector fields $X \in \mathfrak{X}(M)$:*

$$\nabla_X \xi = X. \tag{29}$$

The soliton vector field *xi* is a circular (and concurrent) vector field, and we study conformal Ricci-Yamabe solitons based on the definition given above. Our initial result can be stated as follows:

Theorem 4.2. *Consider (M, g, μ, ξ) as a conformal Ricci-Yamabe soliton situated on an n-dimensional Riemannian manifold (M, g), and the soliton vector field ξ possesses a concircular factor of α, then*

(i) *the manifold (M, g) is an Einstein manifold with factor $\left(\frac{\mu}{2\alpha} - 1\right)$ and*

(ii) *the soliton is expanding if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) < 0$, steady if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) = 0$ or shrinking if $2\left(1 - \frac{1}{n}\right)\alpha + \beta R + \left(p + \frac{2}{n}\right) > 0$.*

Proof. Based on our assumption, the soliton vector field ξ is considered to be concircular with a factor of α. Therefore, from equation (28), we obtain

$$(\mathcal{L}_\xi g)(X, Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi) = g(\alpha X, Y) + g(X, \alpha Y) = 2\alpha g(X, Y), \tag{30}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Again, considering (M, g, μ, ξ) as a conformal Ricci-Yamabe soliton, utilize (30) in equation (3), we get

$$Ric(X, Y) = \left(\frac{\mu}{2\alpha} - 1\right) g(X, Y), \tag{31}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$, and with $\mu = \left[\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$, equation (31) demonstrates that (M, g) is an Einstein manifold with a factor of $\left(\frac{\mu}{2\alpha} - 1\right)$. This concludes the initial portion of the theorem.

Furthermore, it is worth noting that for conformal Ricci flow, the scalar curvature $r(g) = -1$. Therefore, taking an orthonormal basis $\{e_i : 1 \le i \le n\}$ of the manifold M and summing over $1 \le i \le n$ on both sides of equation (31) yields

$$-1 = r(g) = n\left(\frac{\mu}{2\alpha} - 1\right).$$

Lastly, by substituting the value $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right]$ into the equation above and simplifying, we obtain

$$\lambda = \left(1 - \frac{1}{n}\right)\alpha + \frac{\beta R}{2} + \frac{p}{2} + \frac{1}{n}. \tag{32}$$

We know that the soliton is expanding if $\lambda < 0$, steady if $\lambda = 0$ or shrinking if $\lambda > 0$. Therefore, applying this to equation (32) concludes the proof. □

We then get a concurrent vector field result that directly follows the aforementioned theorem.

Corollary 4.3. *Assume (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton with a concurrent soliton vector field ξ. Under these circumstances*

(i) *the manifold (M, g) becomes an Einstein with factor $\left(\frac{\mu}{2} - 1\right)$ and*

(ii) *the soliton is undergoing expansion as $\left(1 + \frac{p}{n} + \frac{\beta R}{2}\right) < 0$, steady as $\left(1 + \frac{p}{n} + \frac{\beta R}{2}\right) = 0$ or shrinking as $\left(1 + \frac{p}{n} + \frac{\beta R}{2}\right) > 0$.*

Proof. The proof is finished by proceeding as in Theorem 4.1 and then entering $\alpha = 1$ into equations (31) and (32). □

We conclude this section with the concurrent vector field theorem shown below:

Theorem 4.4. *Suppose a warped product of two Riemannian manifolds B and F is* $(M, g) = (B \times_f F, g_B \oplus f^2 g_F)$, *with the warping function denoted as* f. *We have* $dimB = m$ *and* $dimF = n$. *Assuming that* (M, g, μ, ξ) *forms a conformal Ricci-Yamabe soliton with concurrent vector field* ξ. *If a warping function* f *is constant and both lifts* ξ_B *and* ξ_F *on the base B and the fiber F are concurrent, then*

(i) *the soliton* (M, g, μ, ξ) *is expanding as* $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) < 0$, *steady as* $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) = 0$ *or shrinking as* $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) = 0$ *or* $\left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right) > 0$,

(ii) *all the three manifolds M, B and F are Ricci flat manifolds and*

(iii) *all the three manifolds M, B and F admit conformal gradient Ricci-Yamabe solitons.*

Proof. Given that (M, g, μ, ξ) is a conformal Ricci-Yamabe soliton on M with a concurrent vector field ξ, the first part of the corollary 4.3 can be used to prove this.

$$Ric(X, Y) = \left(\frac{\mu}{2\alpha} - 1\right) g(X, Y), \tag{33}$$

for all vector fields $X, Y \in \mathfrak{X}(M)$. Suppose we take $X = X_F$ and $Y = Y_F$. Then, by utilizing Lemma 1.1 and equation (5), we obtain

$$Ric(X_F, Y_F) = Ric^F(X_F, Y_F) - \tilde{f} g_F(X_F, Y_F), \tag{34}$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. Now, using equations (33) and (2) in the above equation (34) yields

$$Ric^F(X_F, Y_F) = \tilde{f} g_F(X_F, Y_F) + \left(\frac{\mu}{2\alpha} - 1\right) f^2 g_F(X_F, Y_F),$$

where $\tilde{f} = f\Delta f + (n-1)\|\nabla f\|_B^2$. According to the given condition that f is constant, denoted as $f = c$ for some constant c, it follows that $\tilde{f} = 0$. Consequently, the equation above simplifies to

$$Ric^F(X_F, Y_F) = c^2 \left(\frac{\mu}{2\alpha} - 1\right) g_F(X_F, Y_F), \tag{35}$$

for all vector fields $X_F, Y_F \in \mathfrak{X}(F)$. Therefore, based on the above equation (35), we can conclude that F is Einstein. Given that equation (35) holds true for any vector field in $\mathfrak{X}(F)$, by substituting $X_F = Y_F = \xi_F$ into the equation (35), we obtain

$$Ric^F(\xi_F, \xi_F) = c^2 \left(\frac{\mu}{2\alpha} - 1\right) g_F(\xi_F, \xi_F) = c^2 \left(\frac{\mu}{2\alpha} - 1\right) \|\xi_F\|_F^2. \tag{36}$$

Consider $\xi_F, e_1, e_2, e_3, ..., e_{n-1}$ as an orthonormal basis of $\mathfrak{X}(F)$. In this basis, the curvature tensor of the manifold F can be expressed as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(R^F(\xi_F, e_i)\xi_F, e_i).$$

Utilizing the formula for the curvature tensor, we can rewrite the above equation as

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla^F_{\xi_F} \nabla^R_{e_i} \xi_F - \nabla^R_{e_i} \nabla^F_{\xi_F} \xi_F - \nabla^F_{[\xi_F, e_i]} \xi_F, e_i). \tag{37}$$

Furthermore, given that ξ_F is a concurrent vector field, equation (29) yields $\nabla_X \xi_F = X$ for all $X \in \mathfrak{X}(F)$. Utilizing this result in equation (37), we obtain

$$R^F(\xi_F, e_i, \xi_F, e_i) = g_F(\nabla^F_{\xi_F} e_i - \nabla^F_{e_i} \xi_F - [\xi_F, e_i], e_i) = 0.$$

This implication leads to $Ric^F(\xi_F, \xi_F) = 0$, and consequently, from equation (36), we obtain $\mu = 2\alpha$, i.e., $\mu = \left[2\lambda - \beta R - \left(p + \frac{2}{n}\right)\right] = 2\alpha$. After simplification, this gives $\lambda = \left(\frac{p}{2} + \frac{1}{n} + \frac{\beta R}{2} + \alpha\right)$ and the soliton is shrinking if $\lambda > 0$, steady $\lambda = 0$ or expanding $\lambda < 0$ .This demonstrates the theorem's initial proposition.

We now have $Ric = Ric^F = 0$ using the value $\mu = 2\alpha$ in equations (35) and (33). This establishes the Ricci flatness of M and F.

Now, assuming that $X = X_B$ and $Y = Y_B$, we can write from Lemma 1.1

$$Ric(X_B, Y_B) = Ric^B(X_B, Y_B) - \frac{n}{f} H^f(X_B, Y_B), \tag{38}$$

for all $X_B, Y_B \in \mathfrak{X}(B)$, the equation above, given that $Ric = 0$, simplifies to

$$Ric^B(X_B, Y_B) = \frac{n}{f} H^f(X_B, Y_B). \tag{39}$$

As a result of our presumption that f is constant, $H^f = 0$ implies that the above equation (39) yields $Ric^B(X_B, Y_B) = 0$, for all $X_B, Y_B \in \mathfrak{X}(B)$. As a result, we arrive at $Ric^B = 0$, which establishes the Ricci flatness of the manifold B. This concludes the second part of the theorem's proof.

Let's suppose that $\phi = \frac{1}{2} g(\xi, \xi)$ in order to prove the final portion of the theorem. Then

$$g(X, grad\phi) = X(\phi) = g(\nabla_X \xi, \xi), \tag{40}$$

for all $X \in \mathfrak{X}(M)$. Again, as ξ being concurrent, using equation (29) in equation (40), we get

$$g(X, grad\phi) = g(X, \xi),$$

for all $X \in \mathfrak{X}(M)$. We can conclude that $\xi = grad\phi$ because the above equation holds true for any vector field $X \in \mathfrak{X}(M)$. As a result, (M, g) admits a Ricci-Yamabe soliton with a conformal gradient.

Taking $\phi_B = \frac{1}{2} g(\xi_B, \xi_B)$ and $\phi_F = \frac{1}{2} g(\xi_F, \xi_F)$ and proceeding in the same manner, we can show that $\xi_B = grad\phi_B$ and $\phi_F = grad\phi_F$. Also, from Theorem 2.1, we know that since (M, g) is a conformal Ricci-Yamabe soliton, B and F are both conformal Ricci-Yamabe soliton. Therefore, we can say that the conformal gradient Ricci-Yamabe soliton is admissible on both the manifolds B and F. □

5. References

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