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Viscosity approximation with generalized contractions for fixed point problems and split fixed point problems of nonlinear operators

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Abstract. In this paper, we construct the hybrid viscosity iterative algorithms with generalized contraction to approximate the fixed point of nonlinear operators such as demicontractive operators. Under appropriate conditions, we establish the corresponding strong convergence theorems. Moreover, we apply our results to approximating the common fixed points of nonlinear operators and solving the split common fixed point problems of nonlinear operators. Finally, we present numerical examples to demonstrate the convergence of our algorithm.

1. Introduction

Let H be an infinite dimensional real Hilbert space with inner product and norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H, P_C be the metric projection from H onto C (see, e.g., [16] for more details on the metric projection). Let $T: H \to H$ be a mapping, the set of fixed points of T is denoted by $F(T)$, that is, $F(T) = \{x \in H : Tx = x\}$.

In what follows, we recall some definitions of classes of operators often used in fixed point theory.

Definition 1.1. *Let $T : H \to H$ be a mapping, for $\forall x, y \in H$, then*

(i) T *is* firmly nonexpansive *if*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 - \|(I - T)x - (I - T)y\|^2;$$

(ii) T *is* nonexpansive *if*

$$\|Tx - Ty\| \le \|x - y\|;$$

(iii) T *is* κ-strictly pseudocontractive *with* $\kappa \in [0, 1)$ *if*

$$\|Tx - Ty\|^2 \le \|x - y\|^2 + \kappa\|(I - T)x - (I - T)y\|^2;$$

Definition 1.2. *Let $T : H \to H$ be a mapping with $F(T) \neq \emptyset$, for $\forall \omega \in F(T), x \in H$, then*

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(i) T *is* directed *if*

$$\|Tx-\omega\|^2 \le \|x-\omega\|^2 - \|x-Tx\|^2;$$

(ii) T *is* α-strongly quasi-nonexpansive *with* $\alpha > 0$ *if*

$$\|Tx-\omega\|^2 \le \|x-\omega\|^2 - \alpha\|x-Tx\|^2;$$

(iii) T *is* quasi-nonexpansive *if*

$$\|Tx-\omega\| \le \|x-\omega\|;$$

(iv) T *is* β-demicontractive *with* $\beta < 1$ *if*

$$\|Tx-\omega\|^2 \le \|x-\omega\|^2 + \beta\|x-Tx\|^2.$$

It is easily observed that

(i) a firmly nonexpansive operator with nonempty fixed points is a directed operator; a nonexpansive operator with nonempty fixed points is a quasi-nonexpansive operator; a directed operator or a α-strongly quasi-nonexpansive operator is a quasi-nonexpansive operator;

(ii) the class of demicontractive operators contains important classes of operators: directed operator for $\beta = -1$, quasi-nonexpansive operator for $\beta = 0$, and strictly pseudocontractive operator with nonempty fixed points for $\beta \in (0, 1)$.

(iii) firmly nonexpansive operators, nonexpansive operators and strictly pseudocontractive operators are continuous; directed operators, (strongly) quasi-nonexpansive operators and demicontractive operators are discontinuous.

In this paper, we consider the computation of fixed point of general operators T by means of the so-called viscosity approximation method, which formally consists of the sequence $\{x_n\}$ given by the iteration (see [4, 6, 13, 18, 19])

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Tx_n, \forall n \ge 0. \tag{1}$$

where f is a contraction on H. The above method was first considered with regard to the special case when $f = \mu$ (μ being any given element), in 1967 by Halpern [4] (for μ=0) and in 1977 by Lions [6]. It is worth noting that this procedure can be regarded as a regularization process for fixed point iterations which is supposed to induce the convergence in norm of the iterates. Moreover, it allows one to select a particular fixed point of T which satisfies some variational inequality.

There is an extensive literature regarding the convergence analysis of (1), with several types of operator T, in the setting of Hilbert spaces and Banach spaces. For instance, one of the main convergence results related to (1) goes back to Moudafi [13] in 2000 (see in 2004 [19])regarding the case when T is a nonexpansive operator with nonempty fixed points, under additional conditions on parameters α_n, the sequence $\{x_n\}$ generated by (1) converges strongly to a point $x^* \in F(T)$, which is also the unique solution of the following variational inequality

$$\langle (I-f)x^*, x-x^*\rangle \ge 0, \quad \forall x \in F(T). \tag{2}$$

In 2010, P. E. Maing [9] proposed the following viscosity-type approximation method:

$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)T_\omega x_n, \forall n \ge 0, \tag{3}$$

where f is a contraction on H, T is a quasi-nonexpansive operator(while this kind of operators appears naturally when using sub-gradient projection operator techniques in solving convexly constrained problems [8, 20, 21]), $T_\omega = (1-\omega)I + \omega T$, $\omega \in (0, 1]$, where T_ω is a $\frac{1-\omega}{\omega}$-strongly quasi-nonexpansive operator and $F(T) = F(T_\omega)$. Under appropriate conditions, the sequence $\{x_n\}$ generated by (3) converges strongly to a point $x^* \in F(T)$, which is also the unique solution of the variational inequality problem (2). In [9], the author also applied the result to demicontractive operators. Obviously, strongly quasi-nonexpansive operators play an the important role in researching fixed points of quasi-nonexpansive operators and demicontractive operators. Very recently, D. V. Thong [17] presented the convergence results of the method (1) with operator T which belongs to the class of strongly quasi-nonexpansive operators, and applied the main results to approximating the common fixed points of demicontractive operators.

Motivated by the above related results, we construct hybrid viscosity algorithms with generalized contraction (Meir-Keeler type mappings or (ψ, L)-contractions) to approximate the fixed point of strongly quasi-nonexpansive operators, quasi-nonexpansive operators and demicontractive operators. Moreover, we apply our results to approximating the common fixed points of nonlinear operators and solving the split fixed point problems of nonlinear operators.

This paper is organized as follows: In Section 2, we give some basic definitions, propositions and lemmas which will be used in proving our main results; In Section 3, we present hybrid viscosity iterative algorithms with generalized contraction for approximating the fixed pont of strongly quasi-nonexpansive operators, quasi-nonexpansive operators and demicontractive operators; In Section 4, we apply our main results to approximating the common fixed points of nonlinear operators and solving the split common fixed point problems of nonlinear operators. In Section 5, we present numerical examples to demonstrate the convergence of our algorithm.

2. Preliminaries

Throughout the paper, let the symbol $\to$ and $\rightharpoonup$ denote the strong convergence and weak convergence, respectively. In addition, $\omega_w(x_n)$ denote the weak ω-limit set of $\{x_n\}$, that is, $\omega_w(x_n) = \{u : \exists x_{n_j} \rightharpoonup u\}$. To prove our main results, we recall some basic definitions and lemmas, which will be needed in the sequel.

It is well-known that in a real Hilbert space H, the following equality holds:

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \ \forall x, y \in H. \tag{4}$$

Recall that P_C is the *metric projection* from H into C, then for each point $x \in H$, the unique point $P_C x \in C$ satisfies the property:

$$\|x - P_C x\| = \inf_{y\in C} \|x - y\| =: d(x, C).$$

Lemma 2.1. ([16]) *For a given $x \in H$:*

(i) $z = P_C x$ if and only if $\langle x - z, z - y\rangle \geq 0, \forall y \in C$;

(ii) $z = P_C x$ if and only if $\|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$;

(iii) $\langle P_C x - P_C y, x - y\rangle \geq \|P_C x - P_C y\|^2, \forall x, y \in H$.

A linear bounded operator $A : H \to H$ is called *strongly positive* if and only if there exists $\overline{\gamma} > 0$ such that $\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2$ for all $x \in H$. and we call such A a strongly positive operator with coefficient $\overline{\gamma}$.

Lemma 2.2. ([10]) *Let H be a Hilbert space and let A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. If $0 < \delta \leq \|A\|^{-1}$, then $\|I - \delta A\| \leq 1 - \delta\overline{\gamma}$.*

A mapping $\psi : R^+ \to R^+$ is said to be a *L-function* if $\psi(0) = 0$, $\psi(t) > 0$ for each $t > 0$ and for every $s > 0$, there exists $u > s$ such that $\psi(t) \leq s$ for each $t \in [s, u]$. As a consequence, every L-function ψ satisfies $\psi(t) < t$ for each $t > 0$.

Definition 2.3. *Let (X, d) be a metric space. A mapping $f : X \to X$ is said to be*

*(i) a (ψ, L)-*contraction *if $\psi : R^+ \to R^+$ is said to be a L-function and $d(f(x), f(y)) < \psi(d(x, y))$, for all $x, y \in X$, $x \neq y$;*

(ii) *a* Meir-Keeler type mapping *if for each* $\epsilon > 0$ *there exists* $\delta = \delta(\epsilon) > 0$ *such that for each* $x, y \in X$, *with* $\epsilon \leq d(x, y) < \epsilon + \delta$, *we have* $d(f(x), f(y)) < \epsilon$.

Proposition 2.4. ([5]) *Let* (X, d) *be a metric space and* $f : X \to X$ *be a mapping. The following assertions are equivalent:*

(i) f *is a Meir-Keeler type mapping;*
(ii) *there exists a L-function* $\psi : R^+ \to R^+$ *such that* f *is a* (ψ, L)*-contraction.*

Proposition 2.5. ([15]) *Let* C *be a convex subset of a Banach space* X *and* $f : C \to C$ *be a Meir-Keeler type mapping. Then, for each* $\epsilon > 0$ *there exists* $k \in (0, 1)$ *such that*

$$\|x - y\| \geq \epsilon \text{ implies } \|f(x) - f(y)\| \leq k\|x - y\|.$$

Lemma 2.6. ([12]) *A Meir-Keeler contraction defined on a complete metric space has a unique fixed point.*

Lemma 2.7. ([15]) *Let* C *be a convex subset of a Banach space* E. *Let* T *be a nonexpansive mapping on* C, *and let* f *be a Meir-Keeler contraction on* C. *Then the following hold:*

(i) Tf *is a Meir-Keeler contraction on* C;
(ii) *for each* $\alpha \in (0, 1)$, $(1 - \alpha)T + \alpha f$ *is a Meir-Keeler contraction on* C.

Lemma 2.8. ([2]) *Suppose that* $T : H \to H$ *is* β*-demicontractive mapping. Then the fixed point set* $F(T)$ *of* T *is closed and convex.*

Definition 2.9. ([3]) *Assume that* $T : H \to H$ *is a nonlinear operator, then* $I - T$ *is said to be demiclosed at zero if for any sequence* $\{x_n\}$ *in* H, *the following implication holds:*

$$x_n \rightharpoonup x \text{ and } (I - T)x_n \to 0 \Rightarrow x \in F(T).$$

Lemma 2.10. ([11]) *Let* C *be a nonempty closed convex subset of a real Hilbert space* H, *and let* $T : C \to C$ *be a* β*-strict pseudo-contractive. Then* $I - T$ *is demiclosed at 0.*

Lemma 2.11. ([1]) *Assume* C *is a closed convex subset of a Hilbert space* H.

(i) *Given an integer* $N \geq 1$, *assume, for each* $1 \leq i \leq N$, $T_i : C \to C$ *is a* k_i*-strict pseudo-contraction for some* $0 \leq k_i < 1$. *Assume* $\{\lambda_i\}_{i=1}^N$ *is a positive sequence such that* $\sum_{i=1}^N \lambda_i = 1$. *Then* $\sum_{i=1}^N \lambda_i T_i$ *is a* k*-strict pseudo-contraction, with* $k = \max\{k_i : 1 \leq i \leq N\}$.
(ii) *Let* $\{T_i\}_{i=1}^N$ *and* $\{\lambda_i\}_{i=1}^N$ *be given as in (i) above. Suppose that* $\sum_{i=1}^N \lambda_i T_i$ *has a common fixed point. Then*

$$F(\sum_{i=1}^N \lambda_i T_i) = \bigcap_{i=1}^N F(T_i).$$

Lemma 2.12. ([17]) *Let* $U : H \to H$ *be a* β*-demicontractive operator and* $T : H \to H$ *be a* α_1*-strongly quasi-nonexpansive operator with* $\beta < \alpha_1$. *Then, the operator* UT *is* $\frac{\alpha_1 \beta}{\alpha_1 - \beta}$ *demicontractive and* $F(U) \bigcap F(T) = F(UT)$.

Lemma 2.13. ([17]) *Let* $U : H \to H$ *be a* β*-demicontractive operator with* $F(U) \neq \emptyset$ *and set* $U_\lambda = (1 - \lambda)I + \lambda U$, $\lambda \in (0, 1 - \beta)$ *then*

(i) $F(U) = F(U_\lambda)$;
(ii) $\|U_\lambda x - z\|^2 \leq \|x - z\|^2 - \frac{1}{\lambda}(1 - \beta - \lambda)\|(I - U_\lambda)x\|^2, \forall z \in F(U)$;
(iii) $F(U)$ *is a closed convex subset of* H.

Lemma 2.14. ([17]) *Let* $T : H_2 \to H_2$ *be a* μ*-demicontractive operator,* $A : H_1 \to H_2$ *be a linear bounded operator with* $L = \|A^*A\|$. *For a positive real number* γ, *define the operator* $V : H_1 \to H_1$ *by* $V = I + \gamma A^*(T - I)A$. *Then* *(i) for all* $x \in H_1$ *and* $z \in A^{-1}(Fix(T))$,

$$\|Vx - z\|^2 \le \|x - z\|^2 - \frac{1}{\gamma L}(1 - \mu - \gamma L)\|(I - V)x\|^2;$$

(ii) for all $x \in H_1$ *and* $z \in A^{-1}(Fix(T))$,

$$\|Vx - z\|^2 \le \|x - z\|^2 - \gamma(1 - \mu - \gamma L)\|(I - T)Ax\|^2;$$

(iii) $x \in F(V)$ *if* $Ax \in F(T)$ *provided that* $\gamma \in (0, \frac{1-\mu}{L})$.

Lemma 2.15. ([8]) *Let* $\{x_n\}$ *be a sequence of non-negative real numbers, such that there exists a subsequence* $\{x_{n_j}\}$ *of* $\{x_n\}$*, such that* $x_{n_j} < x_{n_j+1}$ *for all* $j \in N$*. Then, there exists a nondecreasing sequence* $\{m_k\}$ *of* N*, such that* $\lim_{k\to\infty} m_k = \infty$*, and the following properties are satisfied by all (sufficiently large) number* $k \in N$:

$$x_{m_k} \le x_{m_k+1} \text{ and } x_k \le x_{m_k+1}.$$

In fact, m_k *is the largest number* n *in the set* $\{1, 2, \cdots, k\}$*, such that* $x_n \le x_{n+1}$.

Lemma 2.16. ([19]) *Assume that* $\{a_n\}$ *is a sequence of nonnegative real numbers such that*

$$a_{n+1} \le (1 - b_n)a_n + c_n,$$

where b_n *is a sequence in* $(0, 1)$ *and* $\{c_n\}$ *is a sequence such that*

(i) $\sum_{n=1}^{\infty} b_n = \infty$;
(ii) $\limsup_{n\to\infty} \frac{c_n}{b_n} \le 0$ *or* $\sum_{n=1}^{\infty} |c_n| < \infty$.

Then $\lim_{n\to\infty} a_n = 0$.

3. Main results

In this section, we first give the properties of demicontractive operator (supplement with the content in [17]), and then present the main results of the paper.

The following Lemma supplements with the content of Lemma 2.12.

Lemma 3.1. *Let* $U : H \to H$ *be a* β*-demicontractive operator and* $T : H \to H$ *be a* α*-strongly quasi-nonexpansive operator with* $\beta < \alpha$*. Then, the operator* TU *is* $\frac{\alpha\beta}{\alpha-\beta}$ *demicontractive and* $F(U) \bigcap F(T) = F(TU)$.

Proof. First, we show that $F(U) \bigcap F(T) = F(TU)$.

It is suffices to show that $F(TU) \subseteq F(U) \bigcap F(T)$. Let $p \in F(TU)$, it is enough to show that $p \in F(U)$. Picking $q \in F(U) \bigcap F(T)$, from the definition of U and T, we have that

$$\begin{aligned}
\|p - q\|^2 &= \|TUp - q\|^2 \\
&\le \|Up - q\|^2 - \alpha\|Up - TUp\|^2 \\
&\le \|p - q\|^2 + \beta\|Up - p\|^2 - \alpha\|Up - TUp\|^2 \\
&= \|p - q\|^2 + \beta\|Up - p\|^2 - \alpha\|Up - p\|^2 \\
&= \|p - q\|^2 - (\alpha - \beta)\|Up - p\|^2.
\end{aligned}$$

It follows from the condition $\beta < \alpha$ that $Up = p$, that is, $p \in F(U)$. Then $Tp = TUp = p$, that is, $p \in F(T)$. Therefore, $F(U) \bigcap F(T) = F(TU)$.

Next, we show that the operator TU is $\frac{\alpha\beta}{\alpha-\beta}$ demicontractive.

Picking $x \in H$ and $q \in F(TU)$. Let $a = x - q$, $b = Ux - q$, $c = TUx - q$, then we have that $a - b = x - Ux$, $a - c = x - TUx$, $b - c = Ux - TUx$. From the definition of U and T, we obtain that

$$||b||^2 \leq ||a||^2 + \beta||a - b||^2$$

and

$$||c||^2 \leq ||b||^2 - \alpha||b - c||^2,$$

which imply that

$$2\beta\langle a, b\rangle \leq (1 + \beta)||a||^2 - (1 - \beta)||b||^2$$

and

$$-2\alpha\langle b, c\rangle \leq (1 - \alpha)||b||^2 - (1 + \alpha)||c||^2.$$

Moreover, we have that

$$\begin{aligned}
0 \leq ||\alpha c - (\alpha - \beta)b - \beta a||^2 &= \alpha^2||c||^2 + (\alpha - \beta)^2||b||^2 + \beta^2||a||^2 - 2\alpha(\alpha - \beta)\langle b, c\rangle \\
&\quad + 2\beta(\alpha - \beta)\langle b, a\rangle - 2\alpha\beta\langle a, c\rangle \\
&\leq \alpha^2||c||^2 + (\alpha - \beta)^2||b||^2 + \beta^2||a||^2 + (\alpha - \beta)[(1 - \alpha)||b||^2 - (1 + \alpha)||c||^2] \\
&\quad + (\alpha - \beta)[(1 + \beta)||a||^2 - (1 - \beta)||b||^2] - 2\alpha\beta\langle a, c\rangle \\
&= [\alpha^2 - (\alpha - \beta)(1 + \alpha)]||c||^2 + [\beta^2 + (\alpha - \beta)(1 + \beta)]||a||^2 - 2\alpha\beta\langle a, c\rangle \\
&= (\alpha\beta + \beta - \alpha)||c||^2 + (\alpha\beta + \alpha - \beta)||a||^2 - 2\alpha\beta\langle a, c\rangle \\
&= (\alpha\beta + \beta - \alpha)||c||^2 + (\alpha\beta + \alpha - \beta)||a||^2 + \alpha\beta(||a - c||^2 - ||a||^2 - ||c||^2) \\
&= (\beta - \alpha)||c||^2 + (\alpha - \beta)||a||^2 + \alpha\beta||a - c||^2,
\end{aligned}$$

then

$$(\alpha - \beta)||c||^2 \leq (\alpha - \beta)||a||^2 + \alpha\beta||a - c||^2,$$

hence

$$||c||^2 \leq ||a||^2 + \frac{\alpha\beta}{\alpha - \beta}||a - c||^2.$$

That is,

$$||TUx - q||^2 \leq ||x - q||^2 + \frac{\alpha\beta}{\alpha - \beta}||x - TUx||^2.$$

□

The following Lemma modifies Lemma 2.11 from strict pseudo-contraction to demicontractive operator.

Lemma 3.2. *Let $U_i : H \to H$ be a β_i-demicontractive operator. Let $V = \sum_{i=1}^{N} \lambda_i U_i$, $\{\lambda_i\}_{i=1}^{N} \subset [0, 1]$ and $\sum_{i=1}^{N} \lambda_i = 1$. Let $\beta = \max_{1\leq i\leq N}\{\beta_i\}$. Then the operator V is β demicontractive and $\bigcap_{1\leq i\leq N} F(U_i) = F(V)$.*

Proof. Obviously, We only need to prove the case $\lambda_i \in (0, 1)$ for $1 \leq i \leq N$, and it is suffices to prove the Lemma for $N = 2$ and the method can easily be applied to the general case.

First, we show that $F(U) \bigcap F(T) = F(V)$.

It is suffices to show that $F(V) \subseteq F(U) \bigcap F(T)$. Let $p \in F(V)$, it is enough to show that $p \in F(U)$. Picking $q \in F(U) \bigcap F(T)$, from (4) and the definition of U and T, we have that

$$\begin{aligned}
\|p-q\|^2 = \|Vp-q\|^2 &= \|[\lambda U + (1-\lambda)T]p - q\|^2 \\
&= \|\lambda(Up-q) + (1-\lambda)(Tp-q)\|^2 \\
&= \lambda\|Up-q\|^2 + (1-\lambda)\|Tp-q\|^2 - \lambda(1-\lambda)\|Up-Tp\|^2 \\
&\leq \lambda[\|p-q\|^2 + \beta_1\|p-Up\|^2] + (1-\lambda)[\|p-q\|^2 + \beta_2\|p-Tp\|^2] - \lambda(1-\lambda)\|Up-Tp\|^2 \\
&= \|p-q\|^2 + \lambda\beta_1\|p-Up\|^2 + (1-\lambda)\beta_2\|p-Tp\|^2 - \lambda(1-\lambda)\|Up-Tp\|^2.
\end{aligned}$$

Since $p = Vp = \lambda Up + (1-\lambda)Tp$, let $Up = a$, then $Tp = \frac{p-\lambda a}{1-\lambda}$. We have

$$\begin{aligned}
\|p-q\|^2 &\leq \|p-q\|^2 + \lambda\beta_1\|p-Up\|^2 + (1-\lambda)\beta_2\|p-Tp\|^2 - \lambda(1-\lambda)\|Up-Tp\|^2 \\
&= \|p-q\|^2 + \lambda\beta_1\|p-a\|^2 + (1-\lambda)\beta_2\|p - \frac{p-\lambda a}{1-\lambda}\|^2 - \lambda(1-\lambda)\|a - \frac{p-\lambda a}{1-\lambda}\|^2 \\
&= \|p-q\|^2 + \frac{\lambda[\beta_1 - 1 + \lambda(\beta_2-\beta_1)]}{1-\lambda}\|p-a\|^2.
\end{aligned}$$

It is easy to vertify that $\beta_1 - 1 + \lambda(\beta_2 - \beta_1) < 0$ always holds, so we have $p = a$, that is, $p \in F(U)$. Therefore $F(V) \subseteq F(U) \bigcap F(T)$.

Next, we show that the operator V is β-demicontractive.

Picking $x \in H$ and $q \in F(V)$, from (4) and the definition of U and T, we have that

$$\begin{aligned}
\|Vx-q\|^2 &= \|[\lambda U + (1-\lambda)T]x - q\|^2 \\
&= \|\lambda(Ux-q) + (1-\lambda)(Tx-q)\|^2 \\
&= \lambda\|Ux-q\|^2 + (1-\lambda)\|Tx-q\|^2 - \lambda(1-\lambda)\|Ux-Tx\|^2 \\
&\leq \lambda[\|x-q\|^2 + \beta_1\|x-Ux\|^2] + (1-\lambda)[\|x-q\|^2 + \beta_2\|x-Tx\|^2] - \lambda(1-\lambda)\|Ux-Tx\|^2 \\
&= \|x-q\|^2 + \lambda\beta_1\|x-Ux\|^2 + (1-\lambda)\beta_2\|x-Tx\|^2 - \lambda(1-\lambda)\|Ux-Tx\|^2 \\
&\leq \|x-q\|^2 + \lambda\beta\|x-Ux\|^2 + (1-\lambda)\beta\|x-Tx\|^2 - \lambda(1-\lambda)\|Ux-Tx\|^2 \\
&= \|x-q\|^2 + \beta[\lambda\|x-Ux\|^2 + (1-\lambda)\|x-Tx\|^2 - \lambda(1-\lambda)\|Ux-Tx\|^2] \\
&\quad + \beta\lambda(1-\lambda)\|Ux-Tx\|^2 - \lambda(1-\lambda)\|Ux-Tx\|^2 \\
&= \|x-q\|^2 + \beta\|Vx-x\|^2 - (1-\beta)\lambda(1-\lambda)\|Ux-Tx\|^2 \\
&\leq \|x-q\|^2 + \beta\|Vx-x\|^2.
\end{aligned}$$

That is, V is β-demicontractive. □

Lemma 3.3. *Let H is an infinite dimensional real Hilbert space, $f : H \to H$ be a Meir-Keeler-type contraction and A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. For any nonempty closed convex subset D of H, if $\|A\| \leq 1$ and constant $\gamma \leq \bar{\gamma}$, then $P_D(I - A + \gamma f)$ has a unique fixed point in D. Or equivalently, the following variational inequality:*

$$\langle (A - \gamma f)x, z - x\rangle \geq 0, \ \ \forall z \in D.$$

has a unique solution in D.

Proof. Since f is a Meir-Keeler-type contraction, then, for any $\|x-y\| \leq \epsilon + \delta$, we have that $\|f(x) - f(y)\| \leq \epsilon$. Observe that

$$\|(I - A + \gamma f)x - (I - A + \gamma f)y\| \leq \|(I-A)(x-y)\| + \gamma\|f(x) - f(y)\|.$$

Case 1. $\|x-y\| \le \epsilon$, from Lemma 2.2, we have

$$\begin{aligned}\|(I-A+\gamma f)x-(I-A+\gamma f)y\| &\le \|I-A\|\|x-y\|+\gamma\psi(\|x-y\|)\\ &\le (1-\bar{\gamma})\|x-y\|+\gamma\|x-y\|\\ &= (1-\bar{\gamma}+\gamma)\|x-y\|\\ &\le \|x-y\| \le \epsilon.\end{aligned}$$

Case 2. $\epsilon+\delta \ge \|x-y\| > \epsilon$, from Lemma 2.2 and Proposition 2.5, we have

$$\begin{aligned}\|(I-A+\gamma f)x-(I-A+\gamma f)y\| &\le \|I-A\|\|x-y\|+\gamma k_\epsilon\|x-y\|\\ &\le (1-\bar{\gamma})\|x-y\|+\gamma k_\epsilon\|x-y\|\\ &= (1-\bar{\gamma}+\gamma k_\epsilon)\|x-y\|\\ &\le (1-\bar{\gamma}+\gamma k_\epsilon)(\epsilon+\delta).\end{aligned}$$

Taking $\delta = \frac{(\bar{\gamma}-\gamma k_\epsilon)\epsilon}{1-\bar{\gamma}+\gamma k_\epsilon}$, we obtain that

$$\|(I-A+\gamma f)x-(I-A+\gamma f)y\| \le \epsilon.$$

Therefore, $I-A+\gamma f$ is a Meir-Keeler-type contraction on H. From Lemma 2.7, we have $P_D(I-A+\gamma f)$ is a Meir-Keeler-type contraction from H onto D. It follows from Lemma 2.6 that $P_D(I-A+\gamma f)$ has a unique fixed point in D. By Lemma 2.1, we have that the variational inequality:

$$\langle (A-\gamma f)x, z-x\rangle \ge 0, \ \ \forall z \in D.$$

has a unique solution in D. □

Lemma 3.4. *Let H be an infinite dimensional real Hilbert space, A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$, Asumme that $\{\beta_n\} \subset (0,1)$ and $\alpha_n \le (1-\beta_n)\|A\|^{-1}$, for $\forall n \ge 1$, then the following inequality holds:*

$$\|(1-\beta_n)I-\alpha_n A\| \le 1-\beta_n-\alpha_n\overline{\gamma}, \ \forall n \ge 1.$$

Proof. From condition $\alpha_n \le (1-\beta_n)\|A\|^{-1}$, for $\forall x \in H$, we have

$$\begin{aligned}\langle ((1-\beta_n)I-\alpha_n A)x, x\rangle &= (1-\beta_n)\|x\|^2-\alpha_n\langle Ax, x\rangle\\ &\ge (1-\beta_n-\alpha_n\|A\|)\|x\|^2\\ &\ge 0.\end{aligned}$$

This is, $(1-\beta_n)I-\alpha_n A$ is positive operator on H. Again since A be a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$, we have

$$\begin{aligned}\|(1-\beta_n)I-\alpha_n A\| &= \sup\{\langle ((1-\beta_n)I-\alpha_n A)x, x\rangle : x \in H, \|x\| = 1\}\\ &= \sup\{1-\beta_n-\alpha_n\langle Ax, x\rangle : x \in H, \|x\| = 1\}\\ &\le 1-\beta_n-\alpha_n\overline{\gamma}.\end{aligned}$$

□

In what follows, we state and prove the main results of this paper.

Theorem 3.5. *Let H be an infinite dimensional real Hilbert space. Let $T : H \to H$ be a α-strongly quasi-nonexpansive operator such that $I-T$ is demiclosed at zero. Suppose that $f : H \to H$ is a Meir-Keeler-type contraction and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $\|A\| \le 1$, constant $\gamma \le \bar{\gamma}$ and $F(T) \ne \emptyset$. For an arbitrary $x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n\gamma f(x_n)+\beta_n x_n+[(1-\beta_n)I-\alpha_n A]Tx_n. \tag{5}$$

Asumme that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_n \alpha_n = \infty$;
(ii) $\{\beta_n\} \subset [0,1)$, $\limsup_{n\to\infty} \beta_n < 1$

Then sequence $\{x_n\}$ converges strongly to a point $x^* \in F(T)$ which is also the unique solution of the variational inequality

$$\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0, \quad \forall x \in F(T). \tag{6}$$

or equivalently, $x^* = P_{F(T)}(I - A + \gamma f)x^*$.

Proof. First, we show that the variational inequality (6) has a unique solution. Indeed, from Lemma 2.8, we know that $F(T)$ is closed and convex set of H. Then it follows from 3.3 that the variational inequality (6) has a unique solution, denoted by x^*. That is, $x^* = P_{F(T)}(I - A + \gamma f)x^*$.

Next, we show that $\{x_n\}$ is bounded.

From the condition $\lim_{n\to\infty} \alpha_n = 0$, we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|A\|^{-1}$, by Lemma 3.4, we know that

$$\|(1 - \beta_n)I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \overline{\gamma}, \ \forall n \geq 1.$$

Suppose that $\forall p \in F(T)$, fixed ϵ_0, for $\forall n \geq 1$.

Case 1. $\|x_n - p\| < \epsilon_0$. It is obvious that $\{x_n\}$ is bounded.

Case 2. $\|x_n - p\| \geq \epsilon_0$. By Proposition 2.5, there exists $k_{\epsilon_0} \in (0,1)$ such that $\|f(x_n) - f(p)\| \leq k_{\epsilon_0}\|x_n - p\|$. From (5) and Lemma 3.4, we have

$$\begin{aligned}
\|x_{n+1} - p\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A)Tx_n - p\| \\
&= \|\alpha_n \gamma f(x_n) - \alpha_n \gamma f(p) + \beta_n x_n - \beta_n p + [(1 - \beta_n)I - \alpha_n A](Tx_n - p) \\
&\quad + \alpha_n \gamma f(p) + \beta_n p + [(1 - \beta_n)I - \alpha_n A]p - p\| \\
&\leq \alpha_n \gamma k_{\epsilon_0}\|x_n - p\| + \beta_n\|x_n - p\| + \|(1 - \beta_n)I - \alpha_n A\|\|Tx_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\leq \alpha_n \gamma k_{\epsilon_0}\|x_n - p\| + \beta_n\|x_n - p\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\leq (1 - \alpha_n\overline{\gamma} + \alpha_n \gamma k_{\epsilon_0})\|x_n - p\| + \alpha_n\|\gamma f(p) - Ap\| \\
&\leq (1 - \alpha_n(\overline{\gamma} - \gamma k_{\epsilon_0}))\|x_n - p\| + \alpha_n(\overline{\gamma} - \gamma k_{\epsilon_0})\frac{1}{\overline{\gamma} - \gamma k_{\epsilon_0}}\|\gamma f(p) - Ap\|.
\end{aligned}$$

Set $M = \max\{\|x_1 - p\|, \frac{1}{\overline{\gamma} - \gamma k_{\epsilon_0}}\|\gamma f(p) - Ap\|\}$. Assume that $\|x_n - p\| \leq M$, By induction, we have $\|x_{n+1} - p\| \leq M$. Hence $\{x_n\}$ is bounded and $\{f(x_n)\}$, $\{Tx_n\}$, $\{Ax_n\}$ and $\{A(Tx_n)\}$ are also bounded.

Observe that

$$\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]Tx_n - x_n\|^2 \\
&= \|\alpha_n(\gamma f(x_n) - Ax_n) + [(1 - \beta_n)I - \alpha_n A](Tx_n - x_n)\|^2 \\
&= \alpha_n^2\|\gamma f(x_n) - Ax_n\|^2 + \|[(1 - \beta_n)I - \alpha_n A](Tx_n - x_n)\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ax_n, [(1 - \beta_n)I - \alpha_n A](Tx_n - x_n)\rangle \\
&\leq \alpha_n^2\|\gamma f(x_n) - Ax_n\|^2 + (1 - \beta_n - \alpha_n\overline{\gamma})^2\|Tx_n - x_n\|^2 \\
&\quad + 2\alpha_n\langle \gamma f(x_n) - Ax_n, [(1 - \beta_n)I - \alpha_n A](Tx_n - x_n)\rangle.
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1} - p\|^2 - \|x_n - p\|^2 - \|x_{n+1} - x_n\|^2 &= 2\langle x_{n+1} - x_n, x_n - p\rangle \\
&= 2\langle \alpha_n(\gamma f(x_n) - Ax_n) + [(1 - \beta_n)I - \alpha_n A](Tx_n - x_n), x_n - p\rangle \\
&= 2\alpha_n\langle \gamma f(x_n) - Ax_n, x_n - p\rangle + 2\langle [(1 - \beta_n)I - \alpha_n A](Tx_n - x_n), x_n - p\rangle \\
&= 2\alpha_n\langle \gamma f(x_n) - Ax_n, x_n - p\rangle + 2(1 - \beta_n)\langle Tx_n - x_n, x_n - p\rangle \\
&\quad - 2\alpha_n\langle A(Tx_n - x_n), x_n - p\rangle \\
&\leq 2\alpha_n\langle \gamma f(x_n) - Ax_n, x_n - p\rangle - (1 - \beta_n)(\alpha + 1)\|Tx_n - x_n\|^2 \\
&\quad - 2\alpha_n\langle A(Tx_n - x_n), x_n - p\rangle.
\end{aligned}$$

Therefore

$$\begin{aligned}\|x_{n+1}-p\|^2-\|x_n-p\|^2 &\le 2\alpha_n\langle \gamma f(x_n)-Ax_n, x_n-p\rangle-(1-\beta_n)(\alpha+1)\|Tx_n-x_n\|^2\\ &\quad -2\alpha_n\langle A(Tx_n-x_n), x_n-p\rangle+\alpha_n^2\|\gamma f(x_n)-Ax_n\|^2+(1-\beta_n-\alpha_n\overline{\gamma})^2\|Tx_n-x_n\|^2\\ &\quad +2\alpha_n\langle \gamma f(x_n)-Ax_n, [(1-\beta_n)I-\alpha_nA](Tx_n-x_n)\rangle\\ &= 2\alpha_n\langle \gamma f(x_n)-Ax_n, x_n-p\rangle-[(1-\beta_n)(\alpha+1)-(1-\beta_n-\alpha_n\overline{\gamma})^2]\|Tx_n-x_n\|^2\\ &\quad -2\alpha_n\langle A(Tx_n-x_n), x_n-p\rangle+\alpha_n^2\|\gamma f(x_n)-Ax_n\|^2\\ &\quad +2\alpha_n\langle \gamma f(x_n)-Ax_n, [(1-\beta_n)I-\alpha_nA](Tx_n-x_n)\rangle.\end{aligned}$$

Let $\Lambda=(1-\beta_n)(\alpha+1)-(1-\beta_n-\alpha_n\overline{\gamma})^2$, it is easy to vertify that $\liminf_{n\to\infty}\Lambda>0$. Then we can obtain that

$$\begin{aligned}\Lambda\|Tx_n-x_n\|^2 &\le 2\alpha_n\langle \gamma f(x_n)-Ax_n, x_n-p\rangle-2\alpha_n\langle A(Tx_n-x_n), x_n-p\rangle+\alpha_n^2\|\gamma f(x_n)-Ax_n\|^2\\ &\quad +2\alpha_n\langle \gamma f(x_n)-Ax_n, [(1-\beta_n)I-\alpha_nA](Tx_n-x_n)\rangle+\|x_n-p\|^2-\|x_{n+1}-p\|^2\\ &\le 2\alpha_n\|\gamma f(x_n)-Ax_n\|\|x_n-p\|+2\alpha_n\|A(Tx_n-x_n)\|\|x_n-p\|+\alpha_n^2\|\gamma f(x_n)-Ax_n\|^2\\ &\quad +\|x_n-p\|^2-\|x_{n+1}-p\|^2+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\|\gamma f(x_n)-Ax_n\|\|Tx_n-x_n\|.\end{aligned}\tag{7}$$

Next, We analyze the inequality (7) by considering the following two cases.

Case 1. Assume that there exists n_0 large enough such that $\|x_{n+1}-p\|^2\le\|x_n-p\|^2$ for all $n\ge n_0$. Since $\|x_n-p\|^2$ is bounded, we have that $\lim_{n\to\infty}\|x_n-p\|^2$ exists. Since $\liminf_{n\to\infty}\Lambda>0$, $\lim_{n\to\infty}\alpha_n=0$ and $\{x_n\}$, $\{f(x_n)\}$, $\{T(x_n)\}$, $\{A(x_n)\}$ and $\{A(Tx_n)\}$ are bounded, we can obtain

$$\|Tx_n-x_n\|\to 0\ (n\to\infty).$$

For any $q\in\omega_\omega(x_n)$, there exists some subsequence $\{x_{n_l}\}$ of $\{x_n\}$ such that $x_{n_l}\rightharpoonup q$ as $l\to\infty$. Since $\|Tx_n-x_n\|\to 0$ and $I-T$ is demiclosed at zero, it follows from Definition 2.9 that $q\in F(T)$, that is, $\omega_\omega(x_n)\subseteq F(T)$.

Next, we show that

$$\limsup_{n\to\infty}\langle (A-rf)x^*, x_n-x^*\rangle\ge 0,\tag{8}$$

Indeed, taking a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle (A-rf)x^*, x_n-x^*\rangle=\lim_{i\to\infty}\langle (A-rf)x^*, x_{n_i}-x^*\rangle.$$

Since $\{x_n\}$ is bounded, without loss of generality, we may assume that $x_{n_i}\rightharpoonup q\in F(T)$. From Lemma 3.3, we have

$$\limsup_{n\to\infty}\langle (A-rf)x^*, x_n-x^*\rangle=\langle (A-rf)x^*, q-x^*\rangle\ge 0.$$

Again from $\|Tx_n-x_n\|\to 0, (n\to\infty)$, we have that

$$\limsup_{n\to\infty}\langle (A-rf)x^*, Tx_n-x^*\rangle=\langle (A-rf)x^*, q-x^*\rangle\ge 0.\tag{9}$$

Finally, we show that $x_n\to x^*$ as $n\to\infty$.

Assume that the sequence $\{x_n\}$ does not converge strongly to $x^*\in F(T)$, then there exists $\epsilon>0$ and a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $\|x_{n_i}-x^*\|\ge\epsilon$, for all $i\ge 0$. From Proposition 2.5, for this ϵ there exists $k_\epsilon\in(0,1)$ such that

$$\|f(x_{n_i})-f(x^*)\|\le k_\epsilon\|x_{n_i}-x^*\|.$$

Then we have

$$\begin{aligned}
\|x_{n_i+1}-x^*\|^2 &= \|\alpha_{n_i}\gamma f(x_{n_i})+\beta_{n_i}x_{n_i}+((1-\beta_{n_i})I-\alpha_{n_i}A)Tx_{n_i}-x^*\|^2\\
&= \|\alpha_{n_i}(\gamma f(x_{n_i})-Ax^*)+\beta_{n_i}(x_{n_i}-x^*)+((1-\beta_{n_i})I-\alpha_{n_i}A)(Tx_{n_i}-x^*)\|^2\\
&= \alpha_{n_i}^2\|\gamma f(x_{n_i})-Ax^*\|^2+\|\beta_{n_i}(x_{n_i}-x^*)+((1-\beta_{n_i})I-\alpha_{n_i}A)(Tx_{n_i}-x^*)\|^2\\
&\quad+2\langle\alpha_{n_i}(\gamma f(x_{n_i})-Ax^*),\beta_{n_i}(x_{n_i}-x^*)+((1-\beta_{n_i})I-\alpha_{n_i}A)(Tx_{n_i}-x^*)\rangle\\
&\le \alpha_{n_i}^2\|\gamma f(x_{n_i})-Ax^*\|^2+[\beta_{n_i}\|x_{n_i}-x^*\|+(1-\beta_{n_i}-\alpha_{n_i}\overline{\gamma})\|Tx_{n_i}-x^*\|]^2\\
&\quad+2\alpha_{n_i}\beta_{n_i}\langle\gamma f(x_{n_i})-Ax^*,x_{n_i}-x^*\rangle+2\alpha_{n_i}\langle\gamma f(x_{n_i})-Ax^*,((1-\beta_{n_i})I-\alpha_{n_i}A)(Tx_{n_i}-x^*)\rangle\\
&\le \alpha_{n_i}^2\|\gamma f(x_{n_i})-Ax^*\|^2+(1-\alpha_{n_i}\overline{\gamma})^2\|x_{n_i}-x^*\|^2+2\alpha_{n_i}\beta_{n_i}\langle\gamma f(x_{n_i})-\gamma f(x^*),x_{n_i}-x^*\rangle\\
&\quad+2\alpha_{n_i}\beta_{n_i}\langle\gamma f(x^*)-Ax^*,x_{n_i}-x^*\rangle+2\alpha_{n_i}\langle\gamma f(x_{n_i})-\gamma f(x^*),((1-\beta_{n_i})I-\alpha_{n_i}A)(Tx_{n_i}-x^*)\rangle\\
&\quad+2\alpha_{n_i}\langle\gamma f(x^*)-Ax^*,((1-\beta_{n_i})I-\alpha_{n_i}A)(Tx_{n_i}-x^*)\rangle\\
&\le [(1-\alpha_{n_i}\overline{\gamma})^2+2\alpha_{n_i}\beta_{n_i}\gamma k_\epsilon+2\alpha_{n_i}\gamma k_\epsilon(1-\beta_{n_i}-\alpha_{n_i}\overline{\gamma})]\|x_{n_i}-x^*\|^2\\
&\quad+\alpha_{n_i}^2\|\gamma f(x_{n_i})-Ax^*\|^2+2\alpha_{n_i}\beta_{n_i}\langle\gamma f(x^*)-Ax^*,x_{n_i}-x^*\rangle\\
&\quad+2\alpha_{n_i}(1-\beta_{n_i})\langle\gamma f(x^*)-Ax^*,Tx_{n_i}-x^*\rangle-2\alpha_{n_i}^2\langle\gamma f(x^*)-Ax^*,A(Tx_{n_i}-x^*)\rangle\\
&\le [1-2\alpha_{n_i}(\overline{\gamma}-\gamma k_\epsilon)+\alpha_{n_i}^2\overline{\gamma}^2]\|x_{n_i}-x^*\|^2+\alpha_{n_i}^2\|\gamma f(x_{n_i})-Ax^*\|^2+2\alpha_{n_i}\beta_{n_i}\langle\gamma f(x^*)-Ax^*,x_{n_i}-x^*\rangle\\
&\quad+2\alpha_{n_i}(1-\beta_{n_i})\langle\gamma f(x^*)-Ax^*,Tx_{n_i}-x^*\rangle+2\alpha_{n_i}^2\|\gamma f(x^*)-Ax^*\|\|A(Tx_{n_i}-x^*)\|\\
&\le [1-2\alpha_{n_i}(\overline{\gamma}-\gamma k_\epsilon)]\|x_{n_i}-x^*\|^2+\alpha_{n_i}\xi_{n_i}.
\end{aligned}\tag{10}$$

where

$$\begin{aligned}
\xi_{n_i} &= \alpha_{n_i}\overline{\gamma}^2\|x_{n_i}-x^*\|^2+\alpha_{n_i}\|\gamma f(x_{n_i})-Ax^*\|^2+2\beta_{n_i}\langle\gamma f(x^*)-Ax^*,x_{n_i}-x^*\rangle\\
&\quad+2\alpha_{n_i}\|\gamma f(x^*)-Ax^*\|\|A(Tx_{n_i}-x^*)\|+2(1-\beta_{n_i})\langle\gamma f(x^*)-Ax^*,Tx_{n_i}-x^*\rangle.
\end{aligned}$$

Set $b_{n_i}=2\alpha_{n_i}(\overline{\gamma}-\gamma k_\epsilon)$, $c_{n_i}=\alpha_{n_i}\xi_{n_i}$. then (10) reduces to formula

$$\|x_{n_i+1}-x^*\|^2\le(1-b_{n_i})\|x_{n_i}-x^*\|^2+c_{n_i}.$$

From condition $\sum_{n=0}^{\infty}\alpha_n=\infty$, we know that $\sum_{i=0}^{\infty}b_{n_i}=\infty$. Also from condition $\lim_{n\to}\alpha_n=0$, (8) and (9), we have that $\limsup_{i\to\infty}\frac{c_{n_i}}{b_{n_i}}=\limsup_{i\to\infty}\frac{\xi_{n_i}}{2(\overline{\gamma}-\gamma k_\epsilon)}\le 0$, then it follows form Lemma 2.16 that $x_{n_i}\to x^*$ as $i\to\infty$. The contradiction permits us to conclude that sequence $\{x_n\}$ converges strongly to $x^*\in F(T)$.

Case 2. Assume that there exists a subsequence $\{\|x_{n_j}-p\|^2\}$ of $\{\|x_n-p\|^2\}$ such that $\|x_{n_j}-p\|^2<\|x_{n_j+1}-p\|^2$ for all $j\in N$. Then it follows from Lemma 2.15 that there exists a nondecreasing sequence $\{m_k\}$ of N, such that $\lim_{k\to\infty}m_k=\infty$, and the following inequalities hold for all $k\in N$:

$$\|x_{m_k}-p\|^2\le\|x_{m_k+1}-p\|^2 \ and\ \|x_k-p\|^2\le\|x_{m_k+1}-p\|^2.\tag{11}$$

Similarly, we get

$$\|Tx_{m_k}-x_{m_k}\|\to 0\ (n\to\infty).$$

Following an argument similar to that in Case 1 we have $\omega_\omega(x_{m_k})\subseteq F(T)$. Also, we can obtain that

$$\limsup_{n\to\infty}\langle(A-rf)x^*,x_{m_k}-x^*\rangle\ge 0,\tag{12}$$

and

$$\limsup_{n\to\infty}\langle(A-rf)x^*,Tx_{m_k}-x^*\rangle=\langle(A-rf)x^*,q-x^*\rangle\ge 0.\tag{13}$$

Assume that the sequence $\{x_{m_k}\}$ does not converge strongly to $x^* \in F(T)$, then there exists $\epsilon > 0$ and a subsequence $\{x_{m_{k_i}}\}$ of $\{x_{m_k}\}$ such that $\|x_{m_{k_i}} - x^*\| \geq \epsilon$, for all $i \geq 0$. From Proposition 2.5, for this ϵ there exists $k_\epsilon \in (0,1)$ such that

$$\|f(x_{m_{k_i}}) - f(x^*)\| \leq k_\epsilon \|x_{m_{k_i}} - x^*\|.$$

Similarly, we get

$$\|x_{m_{k_i}+1} - x^*\|^2 \leq (1 - b_{m_{k_i}})\|x_{m_{k_i}} - x^*\|^2 + c_{m_{k_i}}.$$

where $b_{m_{k_i}} = 2\alpha_{m_{k_i}}(\overline{\gamma} - \gamma k_\epsilon)$, $c_{m_{k_i}} = \alpha_{m_{k_i}}\xi_{m_{k_i}}$.

$$\begin{aligned}\xi_{m_{k_i}} &= \alpha_{m_{k_i}}\overline{\gamma}^2\|x_{m_{k_i}} - x^*\|^2 + \alpha_{m_{k_i}}\|\gamma f(x_{m_{k_i}}) - Ax^*\|^2 + 2\beta_{m_{k_i}}\langle \gamma f(x^*) - Ax^*, x_{m_{k_i}} - x^*\rangle \\ &\quad + 2\alpha_{m_{k_i}}\|\gamma f(x^*) - Ax^*\|\|A(Tx_{m_{k_i}} - x^*)\| + 2(1-\beta_{m_{k_i}})\langle \gamma f(x^*) - Ax^*, Tx_{m_{k_i}} - x^*\rangle.\end{aligned}$$

From condition $\sum_{n=0}^{\infty}\alpha_n = \infty$, we know that $\sum_{i=0}^{\infty} b_{m_{k_i}} = \infty$. Also from condition $\lim_{n\to}\alpha_n = 0$, (12) and (13), we have that $\limsup_{i\to\infty}\frac{c_{m_{k_i}}}{b_{m_{k_i}}} = \limsup_{i\to\infty}\frac{\xi_{m_{k_i}}}{2(\overline{\gamma}-\gamma k_\epsilon)} \leq 0$, then it follows form Lemma 2.16 that $x_{m_{k_i}} \to x^*$ as $i \to \infty$. The contradiction permits us to conclude that sequence $\{x_{m_k}\}$ converges strongly to $x^* \in F(T)$. By (11), we get $\|x_k - x^*\| \leq \|x_{m_k} - x^*\|$, $\forall k \in N$. Therefore, $x_k \to x^*$ as $k \to \infty$. □

4. Applications

4.1. Applications to approximating the common fixed points of nonlinear operators

Theorem 4.1. *Let H be an infinite dimensional real Hilbert space. Let $T : H \to H$ be a β-demicontractive such that $I - T$ is demiclosed at zero. Suppose that $f : H \to H$ is a Meir-Keeler-type contraction and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $\|A\| \leq 1$, constant $\gamma \leq \bar{\gamma}$ and $F(T) \neq \emptyset$. For an arbitrary $x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n A]T_\lambda x_n. \tag{14}$$

where $T_\lambda = (1-\lambda)I + \lambda T$. Assume that the parameter λ and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\lambda \in (0, 1-\beta)$;
(ii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty}\alpha_n = 0$, $\sum_n \alpha_n = \infty$;
(iii) $\{\beta_n\} \subset [0,1)$, $\limsup_{n\to\infty}\beta_n < 1$

Then sequence $\{x_n\}$ converges strongly to a point $x^* \in F(T)$ which is also the unique solution of the variational inequality (6), or equivalently, $x^* = P_{F(T)}(I - A + \gamma f)x^*$.

Proof. From Lemma 2.13 and the condition (i), we have that T_λ is $\frac{1-\beta-\lambda}{\lambda}$-strongly quasi- nonexpansive and $F(T_\lambda) = F(T)$, and from $\lambda(I - T) = I - T_\lambda$, it is obvious that $I - T_\lambda$ is demiclosed at zero. The remaining of the proof is followed from Theorem 3.5. □

Theorem 4.2. *Let H be an infinite dimensional real Hilbert space. Let $T_i : H \to H$ be a τ_i-demicontractive such that $I - T_i$ are demiclosed at zero, for $1 \leq i \leq N$. Suppose that $f : H \to H$ is a Meir-Keeler-type contraction and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $\|A\| \leq 1$, constant $\gamma \leq \bar{\gamma}$ and $\Gamma = \bigcap_{i=1}^N F(T_i) \neq \emptyset$. For an arbitrary $x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n A]T_\lambda x_n.$$

where $T_\lambda = (1-\lambda)I + \lambda\sum_{i=1}^N \mu_i T_i$. Assume that the parameter λ and the sequence $\{\mu_i\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\lambda \in (0, 1-\tau)$, $\tau = \max_{1\leq i\leq N}\{\tau_i\}$;

(i) $\mu_i \in (0,1)$, $\sum_{i=1}^N \mu_i = 1$;

(iii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_n \alpha_n = \infty$;

(iv) $\{\beta_n\} \subset [0,1)$, $\limsup_{n\to\infty} \beta_n < 1$

Then sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$ which is also the unique solution of the variational inequality (6), or equivalently, $x^* = P_\Gamma(I - A + \gamma f)x^*$.

Proof. Set $T = \sum_{i=1}^N \mu_i T_i$. It follows from Lemma 3.2 that T is τ-demicontractive and $F(T) = \bigcap_{1\le i\le N} F(T_i)$. From Lemma 2.13 and the condition (i), we have that T_λ is $\frac{1-\tau-\lambda}{\lambda}$-strongly quasi-nonexpansive and $F(T_\lambda) = F(T)$, and from $I - T_\lambda = \lambda(I - T) = \lambda \sum_{i=1}^N \mu_i(I - T_i)$, it is obvious that $I - T_\lambda$ is demiclosed at zero. The remaining of the proof is followed from Theorem 3.5. □

Remark 4.3. *Since the class of quasi-nonexpansive operators and the class of strictly pseudocontractive operators with nonempty fixed points belong to the class of demicontractive, then we can apply the results of Theorem 4.1 and Theorem 4.2 to quasi-nonexpansive operators and strictly pseudocontractive operators (Lemma 2.11 and Lemma 2.10 to be used).*

Theorem 4.4. *Let H be an infinite dimensional real Hilbert space. Let $U : H \to H$ be a β-demicontractive operator and $T : H \to H$ be a α-strongly quasi-nonexpansive operator with $\beta < \alpha$ such that $I - U$ and $I - T$ are demiclosed at zero. Suppose that $f : H \to H$ is a Meir-Keeler-type contraction and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $\|A\| \le 1$, constant $\gamma \le \bar{\gamma}$ and $\Gamma = F(S) \bigcap F(T) \neq \emptyset$. For an arbitrary $x_1 \in H$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]Sx_n.$$

where $S = (1 - \lambda)I + \lambda UT$ or $S = (1 - \lambda)I + \lambda TU$. Assume that the parameter λ and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\lambda \in (0, 1 - \frac{\alpha\beta}{\alpha-\beta})$;

(ii) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_n \alpha_n = \infty$;

(iii) $\{\beta_n\} \subset [0,1)$, $\limsup_{n\to\infty} \beta_n < 1$

Then sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$ which is also the unique solution of the variational inequality (6), or equivalently, $x^* = P_\Gamma(I - A + \gamma f)x^*$.

Proof. It follows from Lemma 2.12 and 3.1 that UT and TU is $\frac{\alpha\beta}{\alpha-\beta}$-demicontractive and $F(UT) = F(TU) = F(U) \bigcap F(T)$. From Lemma 2.13 and the condition (i), we have that S is $\frac{1-\frac{\alpha\beta}{\alpha-\beta}-\lambda}{\lambda}$-strongly quasi-nonexpansive and $F(S) = F(UT) = F(TU)$,

Next, we show that the operator $I - S$ is demiclosed at zero, where$S = (1 - \lambda)I + \lambda UT$. Let $\{x_n\}$ be a sequence such that $x_n - Sx_n \to 0$ and $x_n \rightharpoonup x$. Since $x_n - Sx_n = \lambda(x_n - UTx_n)$, we have that $x_n - UTx_n \to 0$. Picking any $p \in F(S)$, we have that

$$\begin{aligned}
\|UTx_n - p\|_2 &\le \|Tx_n - p\|_2 + \beta\|Tx_n - UTx_n\|^2 \\
&\le \|x_n - p\|_2 - \alpha\|x_n - Tx_n\|^2 + \beta\|Tx_n - UTx_n\|^2 \\
&= \|x_n - p\|_2 - \alpha\|x_n - Tx_n\|^2 + \beta\|Tx_n - x_n + x_n - UTx_n\|^2 \\
&= \|x_n - p\|_2 - \alpha\|x_n - Tx_n\|^2 + \beta\|Tx_n - x_n\|^2 + 2\beta\langle Tx_n - x_n, x_n - UTx_n\rangle + \beta\|x_n - UTx_n\|^2.
\end{aligned}$$

Then we get that

$$\begin{aligned}
(\alpha - \beta)\|Tx_n - x_n\|^2 &= \|x_n - p\|_2 - \|UTx_n - p\|_2 + 2\beta\langle Tx_n - x_n, x_n - UTx_n\rangle + \beta\|x_n - UTx_n\|^2 \\
&= \|x_n - p\|_2 - \|UTx_n - p\|_2 + 2\beta\|Tx_n - x_n\|\|x_n - UTx_n\| + \beta\|x_n - UTx_n\|^2.
\end{aligned}$$

From the condition $\alpha > \beta$, we can obtain that $Tx_n - x_n \to 0$. Since $x_n \rightharpoonup x$ and the demiclosedness of $I - T$, we get $x \in F(T)$.

Observe that

$$\|UTx_n - Tx_n\| \leq \|UTx_n - x_n\| + \|x_n - Tx_n\| \to 0.$$

Since $Tx_n \rightharpoonup x$ and the demiclosedness of $I - U$, we get $x \in F(U)$. Therefore, $x \in F(S)$. That is, $I - S$ is demiclosed at zero. When the case $S = (1 - \lambda)I + \lambda TU$, The method of proof is similar to that of the above case. The remaining of the proof is followed from Theorem 3.5. □

4.2. Applications to solving the split common fixed point problems of nonlinear operators

Theorem 4.5. *Let H_1 and H_2 be an infinite dimensional real Hilbert space. Let $U : H_2 \to H_2$ be a β-demicontractive operator and $T : H_1 \to H_1$ be a α-demicontractive operator such that $I - U$ and $I - T$ are demiclosed at zero. Suppose that $f : H_1 \to H_1$ is a Meir-Keeler-type contraction, $B : H_1 \to H_2$ be a bounded linear operator with $L = \|B^*B\|$, and A is a strongly positive bounded linear self-adjoint operator on H_1 with coefficient $\overline{\gamma} > 0$. Assume that $\|A\| \leq 1$, constant $\gamma \leq \overline{\gamma}$ and $\Gamma = \{x \in F(T) : Bx \in F(U)\} \neq \emptyset$. For an arbitrary $x_1 \in H_1$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n A]S_\lambda x_n.$$

where $S_\lambda = (1 - \lambda)I + \lambda S$, $S = T[I + \xi B^*(U - I)B]$. Assume that the parameter ξ, λ and the sequence $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\xi \in (0, \frac{(1-\alpha)(1-\beta)}{L})$;
(ii) $\lambda \in (0, 1 - \frac{\alpha_1 \alpha}{\alpha_1 - \alpha})$, *where* $\alpha_1 = \frac{1-\beta-\xi L}{\xi L}$;
(iii) $\{\alpha_n\} \subset (0, 1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_n \alpha_n = \infty$;
(iv) $\{\beta_n\} \subset [0, 1)$, $\limsup_{n\to\infty} \beta_n < 1$

Then sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$ which is also the unique solution of the variational inequality (6), *or equivalently, $x^* = P_\Gamma(I - A + \gamma f)x^*$.*

Proof. Let $V = I + \xi B^*(U - I)B$. From Lemma 2.14, we have that V is a α_1-strongly quasi-nonexpansive. From the condition (i), we know $\alpha < \alpha_1$, and from Lemma 2.12, we get S is $\frac{\alpha_1 \alpha}{\alpha_1 - \alpha}$-demicontractive operator. From Lemma 2.13 and the condition (ii), we have that S_λ is $\frac{1-\frac{\alpha_1\alpha}{\alpha_1-\alpha}-\lambda}{\lambda}$-strongly quasi-nonexpansive. Also, we have

$$\begin{aligned}
\Gamma &= \{x \in H_1 : x \in F(T) \text{ and } Bx \in F(U)\} \\
&= \{x \in H_1 : x \in F(T) \text{ and } x \in F(V)\} \\
&= F(TV) = F(S) = F(S_\lambda).
\end{aligned}$$

Next, we show that the operator $I - S_\lambda$ is demiclosed at zero, that is, we need to show that the operator $I - S$ is demiclosed at zero. Let $\{x_n\}$ be a sequence such that $x_n - Sx_n \to 0$ and $x_n \rightharpoonup x$. By the method similar to Theorem 4.4, we can obtain that $Vx_n - x_n \to 0$, then, we have $\|(U - I)Bx_n\| \to 0$. Since $Bx_n \rightharpoonup Bx$ and the demiclosedness of $I - U$, we get $Bx \in F(U)$.

Observe that

$$\|TVx_n - Vx_n\| \leq \|TVx_n - x_n\| + \|x_n - Vx_n\| \to 0.$$

From $Vx_n - x_n \to 0$ and $x_n \rightharpoonup x$, we have $Vx_n \rightharpoonup x$, and by the demiclosedness of $I - T$, we get $x \in F(T)$. Therefore, $x \in F(S) = F(S_\lambda)$. That is, $I - S_\lambda$ is demiclosed at zero. The remaining of the proof is followed by Theorem 3.5. □

Theorem 4.6. *Let H_1 and H_2 be an infinite dimensional real Hilbert space. Let $U_i : H_2 \to H_2$ be a τ_i-demicontractive operator, for $1 \le i \le N_1$ and $T_j : H_1 \to H_1$ be a σ_j-demicontractive operator, for $1 \le j \le N_2$ such that $I - U_i$ and $I - T_j$ are demiclosed at zero. Suppose that $f : H_1 \to H_1$ is a Meir-Keeler-type contraction, $B : H_1 \to H_2$ be a bounded linear operator with $L = \|B^*B\|$, and A is a strongly positive bounded linear self-adjoint operator on H with coefficient $\overline{\gamma} > 0$. Assume that $\|A\| \le 1$, constant $\gamma \le \bar{\gamma}$ and $\Gamma = \{x \in \bigcap_{j=1}^{N_2} F(T_j) : Bx \in \bigcap_{i=1}^{N_1} F(U_i)\} \neq \emptyset$. For an arbitrary $x_1 \in H_1$, let $\{x_n\}$ be a sequence generated by the following algorithm:*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n A]S_\lambda x_n.$$

where $S_\lambda = (1-\lambda)I + \lambda S$, $S = T[I + \xi B^*(U - I)B]$, $U = \sum_{i=1}^{N_1} \mu_i U_i$, $T = \sum_{j=1}^{N_2} \nu_j T_j$. Set $\tau = \max_{1\le i\le N_1}\{\tau_i\}$, $\sigma = \max_{1\le j\le N_2}\{\sigma_j\}$. Assume that the parameter ξ, λ and the sequence $\{\mu_i\}$, $\{\nu_j\}$, $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying the following conditions:

(i) $\xi \in (0, \frac{(1-\sigma)(1-\tau)}{L})$;
(ii) $\lambda \in (0, 1 - \frac{\alpha_1\sigma}{\alpha_1 - \sigma})$, *where* $\alpha_1 = \frac{1-\tau-\xi L}{\xi L}$;
(iii) $\mu_i \in (0,1)$, $\sum_{i=1}^{N_1} \mu_i = 1$, $\nu_j \in (0,1)$, $\sum_{j=1}^{N_2} \nu_j = 1$;
(iv) $\{\alpha_n\} \subset (0,1)$, $\lim_{n\to\infty} \alpha_n = 0$, $\sum_n \alpha_n = \infty$;
(v) $\{\beta_n\} \subset [0,1)$, $\limsup_{n\to\infty} \beta_n < 1$

Then sequence $\{x_n\}$ converges strongly to a point $x^* \in \Gamma$ which is also the unique solution of the variational inequality (6), *or equivalently, $x^* = P_\Gamma(I - A + \gamma f)x^*$.*

Proof. The method of the proof is the same as Theorem 4.5, and we omit it. □

5. Numerical examples

In this section, we give a numerical example of Theorem 4.1 to illustrate the implementation of the algorithm (14). All codes were written in Matlab 2010b and run on Dell i - 5 Dual-Core laptop.

Example 5.1. *Let $H = R$. For each $x \in R$, define the mappings T and f as follows:*

$$T(x) = \begin{cases} x, & x \in (-\infty, 0), \\ -2x, & x \in [0, \infty). \end{cases} \qquad \text{and} \qquad f(x) = \begin{cases} 0, & x \in (-\infty, 0), \\ \frac{1}{2}x, & x \in [0,1], \\ \frac{\sqrt{x}}{2}, & x \in (1, +\infty). \end{cases}$$

we have that T is $\frac{1}{3}$-demicontractive mapping(for details, see [14]). Also we define $Ax = \frac{3}{4}x$, put $\gamma = \frac{1}{2}$, $\alpha_n = \frac{1}{n+1}$ and $\beta_n = \frac{1}{2} - \frac{1}{4n}$. Obviously, T, A, f, γ α_n, β_n satisfy all the conditions of Theorem 4.1. Then we have the following algorithm:

$$x_{n+1} = \frac{1}{2(n+1)} f(x_n) + (\frac{1}{2} - \frac{1}{4n})x_n + (\frac{1}{2} + \frac{1}{4n} - \frac{3}{4(n+1)})T_\lambda x_n. \tag{15}$$

Next, we will analyze the convergence of the algorithm (15) *from two aspects.*

First, we take three initial points randomly generated with parameter $\lambda = \frac{1}{2}$, then we have the following numerical results in Figure 1. We can observe that the sequence $\{x_n\}$ generated by the algorithm (15) *converges to the same solution $0 \in F(T)$.*

In addition, we take different parameter λ with $x_1 = 1$ to test the convergence of this algorithm (15), *Figure 2 presents the behaviour of x_n by choosing three different values $\lambda = \frac{1}{2}$, $\lambda = \frac{1}{3}$, $\lambda = \frac{1}{4}$.*

Remark 5.2. *In fact, we can easily observe that the mapping f is a contraction in the above example. Because the Meir-Keeler-type contraction is not easy to find, and the class of contraction belongs to the class of Meir-Keeler-type contraction, so, we use a contraction to demonstrate the convergence of our algorithm. In addition, for the contraction, the research results in this paper are novel and have not been studied before from a theoretical point of view.*

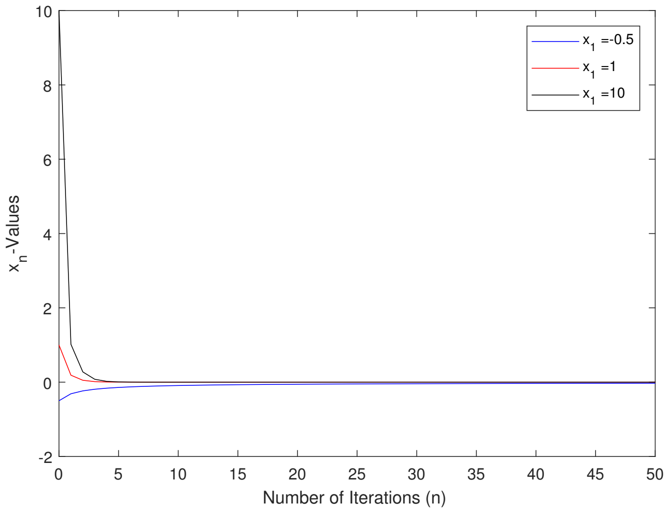

Figure 1: Behaviours of x_n with three random initial point x_1

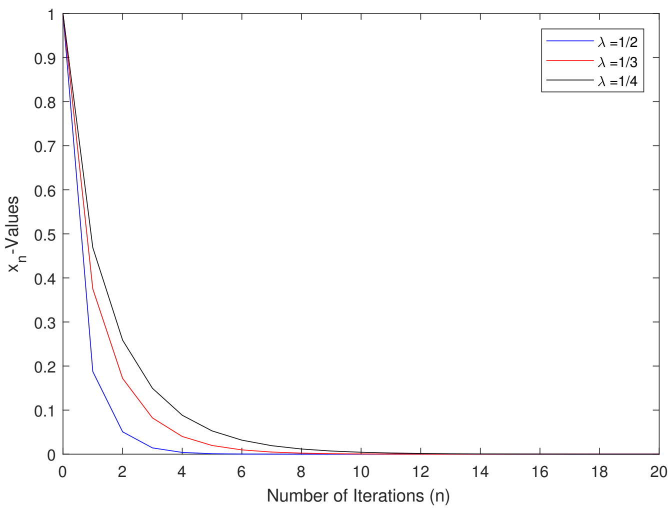

Figure 2: Behaviours of x_n with three different parameter λ

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