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Reverse order laws and absorption laws of the weak group inverse

Mengmeng Zhou[a]

[a]*College of Information Engineering, Nanjing Xiaozhuang University, Nanjing 211171, China*

Abstract. In a proper $*$-ring, reverse order laws of the weak group inverse are investigated under certain conditions. Some new equivalent characterizations which ensure that reverse order laws of the weak group inverse hold are presented. Finally, absorption laws of the weak group inverse are studied.

1. Introduction

Let R be a unitary ring with an involution. An involution $a \mapsto a^*$ in ring R is an anti-isomorphism of degree 2, that is, $(a+b)^* = a^* + b^*$, $(ab)^* = b^*a^*$, $(a^*)^* = a$, for any $a,\ b \in R$. We denote by $\mathbb{N}^+$ the set of all positive integers.

In 1958, Drazin [6] introduced pseudo inverses in associative rings and semigroups. Later, this type of generalized inverse is called the Drazin inverse. Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that

$$xa^{k+1} = a^k, \quad xax = x, \quad xa = ax,$$

then x is called the Drazin inverse of a. It is unique and denoted by a^D when the Drazin inverse exists. If k is the smallest positive integer such that above equations hold, then k is the Drazin index of a and denoted by $\mathrm{ind}(a) = k$. When $k = 1$, the Drazin inverse is reduced to the group inverse and denoted by $a^\#$.

In 2014, Manjunatha Prasad et al. [14] introduced core-EP inverses of complex matrices. In 2017, Gao et al. [9] generalized the core-EP inverse to rings with involution, and characterized the core-EP inverse by three equations. Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ such that

$$xa^{k+1} = a^k, \quad ax^2 = x, \quad (ax)^* = ax,$$

then x is called the pseudo core inverse. The smallest positive integer k satisfying above equations is called the pseudo core index of a. They proved that the pseudo core index of a equals to its Drazin index. If x exists, then it is unique and denoted by $a^{\textcircled{D}}$. When the pseudo core index of a is 1, the pseudo core inverse is the core inverse and denoted by $a^{\textcircled{\#}}$ [1, 20, 23].

In 2018, Wang et al. [21] defined weak group inverses of complex matrices by the core-EP inverse. Recall that a ring R with involution is called proper $*$-ring if $a^*a = 0$ implies $a = 0$, for arbitrary $a \in R$. If the

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Email address: mmz9209@163.com (Mengmeng Zhou)

involution is defined as conjugate transpose, then the complex matrix ring is a proper $*$-ring. In [25], Zhou et al. extended this generalized inverse to proper $*$-rings and characterized it by three equations. Let $a \in R$. If there exist $x \in R$ and $k \in \mathbb{N}^+$ satisfying three equations:

$$xa^{k+1} = a^k,\ ax^2 = x,\ (a^k)^*a^2x = (a^k)^*a,$$

then x is called the weak group inverse of a. If such x exists, then it is unique and denoted by $a^{ⓦ}$. When k is the smallest positive integer such that above equations hold, the integer k is called the weak group index of a. When $k = 1$ and R is a proper $*$-ring , the weak group inverse is the group inverse. The authors showed that the weak group index of a is equal to the Drazin index of a. For more details of the weak group inverse, readers refer to [8, 18, 19, 21, 24, 26, 27].

The symbols R^{-1}, R^D and $R^{ⓦ}$ denote the sets of all invertible, Drazin invertible and weak group invertible elements in R, respectively.

Let $a,\ b \in R^{-1}$. As it is well known, we have $a^{-1}(a+b)b^{-1} = a^{-1} + b^{-1}$ which is called the absorption law. Also, we have $(ab)^{-1} = b^{-1}a^{-1}$ which is called reverse order law. However, absorption laws and reverse order laws of generalized inverses may not hold in general. In 2011, Deng [5] studied some necessary and sufficient conditions about the reverse order law of the group inverse of linear bounded operators in Hilbert spaces. In 2013, Mosić et al. [17] investigated reverse order laws of the group inverse in rings. In 2015, Mary [15] studied reverse order law for the group inverse in semigroups and rings. At the same year, Jin et al. [11] presented the absorption law of the group inverse in rings. Recently, Gao et al. [10] studied absorption laws and reverse order laws for the pseudo core inverse. For more details about absorption laws and reverse order laws of generalized inverses, readers can see [2–4, 13, 16, 22, 28, 29] for example.

We know that the group inverse is a special case of the weak group inverse. Motivated by above discussion, we exploit reverse order laws and absorption laws for the weak group inverse.

The paper is organized as follows. In Section 2, some auxiliary lemmas are given. In Section 3, we investigate necessary and sufficient conditions which ensure that reverse order laws for the weak group inverse hold. In Section 4, some equivalent characterizations about absorption laws of the weak group inverse are studied.

2. Preliminaries

In the rest of this paper, we restrict R is a proper $*$-ring. We will write $aR = \{ax : x \in R\}$ and $Ra = \{xa : x \in R\}$. The right annihilator of a is defined by $a^\circ = \{x \in R : ax = 0\}$. Similarly, the set ${}^\circ a = \{x \in R : xa = 0\}$ represents the left annihilator of a. In the following, some necessary lemmas and definitions are presented.

Definition 2.1. [12] *An element $a \in R$ is left $*$-cancellable if $a^*ax = a^*ay$ implies $ax = ay$, it is right $*$-cancellable if $xaa^* = yaa^*$ implies $xa = ya$, and $*$-cancellable if it is both left and right $*$-cancellable.*

It is easy to see that R is a proper $*$-ring if and only if every element in R is $*$-cancellable.

Lemma 2.2. [9] *Let $a \in R$. If there exists $x \in R$ such that*

$$\text{(i) } xa^{k+1} = a^k \ \ for\ some\ positive\ integer\ k, \ \ \text{(ii) } ax^2 = x,$$

then we have

(i) *$ax = a^m x^m$ for arbitrary positive integer m;*

(ii) *$xax = x$;*

(iii) *$axa^m = a^m$ for any $m \geq k$;*

(iv) *a is Drazin invertible, $a^D = x^{k+1}a^k$ and $\mathrm{ind}(a) \leq k$.*

Lemma 2.3. [6] *Let $a,\ x,\ y \in R$. If there exists $m \in \mathbb{N}^+$ such that $a^m = xa^{m+1} = a^{m+1}y$, then $a \in R^D$.*

Lemma 2.4. [7] *Let* $a_1,\ a_2 \in R^D$ *and* $x \in R$. *If* $a_1x = xa_2$, *then* $a_1^Dx = xa_2^D$.

Lemma 2.5. [26] *If* $a \in R^{\textcircled{w}}$ *with* $ab = ba$ *and* $a^*b = ba^*$, *then* $a^{\textcircled{w}}b = ba^{\textcircled{w}}$.

Lemma 2.6. [26] *If* $a,\ b \in R^{\textcircled{w}}$ *with* $ab = ba$ *and* $a^*b = ba^*$, *then* $ab \in R^{\textcircled{w}}$ *with* $(ab)^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$.

Lemma 2.7. [27] *If each idempotent element in R is left* $*$*-cancellable and* $a \in R^D$, *then a is weak group invertible if and only if there exists* $x \in R$ *such that* $(a^D)^*a = (a^D)^*a^Dx$.

Lemma 2.8. [20] *Let* $a, b \in R$. *Then*

(i) *If* $aR \subseteq bR$, *then* ${}^\circ b \subseteq {}^\circ a$;

(ii) *If* $Ra \subseteq Rb$, *then* $b^\circ \subseteq a^\circ$.

3. Reverse order laws

In this section, some sufficient and necessary conditions, which ensure that reverse order laws of the weak group inverse hold, are presented.

Proposition 3.1. *Let* $a,\ b,\ ab \in R^{\textcircled{w}}$. *Then the following statements are equivalent:*

(i) $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$;

(ii) $(ab)^{\textcircled{w}}a = b^{\textcircled{w}}a^{\textcircled{w}}a$ *and* $(ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}aa^{\textcircled{w}}$;

(iii) $b(ab)^{\textcircled{w}} = bb^{\textcircled{w}}a^{\textcircled{w}}$ *and* $(ab)^{\textcircled{w}} = b^{\textcircled{w}}b(ab)^{\textcircled{w}}$.

Proof. (i) ⇒ (ii), (iii): According to $a^{\textcircled{w}}aa^{\textcircled{w}} = a^{\textcircled{w}}$ and $b^{\textcircled{w}}bb^{\textcircled{w}} = b^{\textcircled{w}}$, it is obvious.
(ii) ⇒ (i): Since $(ab)^{\textcircled{w}}a = b^{\textcircled{w}}a^{\textcircled{w}}a$ and $(ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}aa^{\textcircled{w}}$, we have $(ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}aa^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}aa^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$.
(iii) ⇒ (i): Since $b(ab)^{\textcircled{w}} = bb^{\textcircled{w}}a^{\textcircled{w}}$ and $(ab)^{\textcircled{w}} = b^{\textcircled{w}}b(ab)^{\textcircled{w}}$,
we obtain that $(ab)^{\textcircled{w}} = b^{\textcircled{w}}b(ab)^{\textcircled{w}} = b^{\textcircled{w}}bb^{\textcircled{w}}a^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$. □

In the following theorem, we exploit equivalent characterizations of reverse order laws for the weak group inverse under certain conditions. First of all, an auxiliary lemma is given.

Lemma 3.2. *Let* $b \in R$ *and* $a \in R^{\textcircled{w}}$ *with* $aba = a^2b$ *and* $aba^* = a^*ab$. *Then* $ab \in R^{\textcircled{w}}$ *if and only if* $a^{\textcircled{w}}ab \in R^{\textcircled{w}}$. *In this case, the following statements hold:*

(i) $(a^{\textcircled{w}}ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}a = a(ab)^{\textcircled{w}}$;

(ii) $(ab)^{\textcircled{w}} = (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}}ab)^{\textcircled{w}}$.

Proof. Suppose that $ab \in R^{\textcircled{w}}$. Since $aba = a^2b$, we have $(ab)^n = a^nb^n$ for any $n \in \mathbb{N}^+$. From $aba = a^2b$ and $aba^* = a^*ab$, by Lemma 2.5 and Lemma 2.6, we get that

$$aba^{\textcircled{w}} = a^{\textcircled{w}}ab,$$

$$(ab)^*a^{\textcircled{w}} = a^{\textcircled{w}}(ab)^*,$$

$$(ab)^{\textcircled{w}}a = a(ab)^{\textcircled{w}},$$

$$(ab)^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}(ab)^{\textcircled{w}}.$$

Suppose that $m = \max\{\mathrm{ind}(a), \mathrm{ind}(ab)\}$. Since

$$\begin{aligned}(ab)^{\textcircled{w}}a(a^{\textcircled{w}}ab)^{m+1} &= (ab)^{\textcircled{w}}a(a^{\textcircled{w}})^{m+1}(ab)^{m+1} = (ab)^{\textcircled{w}}(a^{\textcircled{w}})^m(ab)^{m+1}\\ &= (a^{\textcircled{w}})^m(ab)^{\textcircled{w}}(ab)^{m+1} = (a^{\textcircled{w}})^m(ab)^m = (a^{\textcircled{w}}ab)^m,\end{aligned}$$

$$\begin{aligned} a^{\textcircled{w}}ab((ab)^{\textcircled{w}}a)^2 &= a^{\textcircled{w}}a(ab)^m((ab)^{\textcircled{w}})^m(ab)^{\textcircled{w}}a = a^{\textcircled{w}}a^{m+1}b^m((ab)^{\textcircled{w}})^m(ab)^{\textcircled{w}}a \\ &= a^mb^m((ab)^{\textcircled{w}})^m(ab)^{\textcircled{w}}a = ab(ab)^{\textcircled{w}}(ab)^{\textcircled{w}}a = (ab)^{\textcircled{w}}a \end{aligned}$$

and

$$\begin{aligned} ((a^{\textcircled{w}}ab)^m)^*(a^{\textcircled{w}}ab)^2(ab)^{\textcircled{w}}a &= ((a^{\textcircled{w}}ab)^m)^*a^{\textcircled{w}}aba^{\textcircled{w}}a(ab)^m((ab)^{\textcircled{w}})^m \\ &= ((a^{\textcircled{w}}ab)^m)^*a^{\textcircled{w}}aba^{\textcircled{w}}a^{m+1}b^m((ab)^{\textcircled{w}})^m \\ &= ((a^{\textcircled{w}}ab)^m)^*a^{\textcircled{w}}aba^mb^m((ab)^{\textcircled{w}})^m \\ &= ((a^{\textcircled{w}}ab)^m)^*a^{\textcircled{w}}(ab)^2(ab)^{\textcircled{w}} \\ &= ((a^{\textcircled{w}})^m)^*a^{\textcircled{w}}((ab)^m)^*(ab)^2(ab)^{\textcircled{w}} \\ &= ((a^{\textcircled{w}})^m)^*a^{\textcircled{w}}((ab)^m)^*ab = ((a^{\textcircled{w}}ab)^m)^*a^{\textcircled{w}}ab, \end{aligned}$$

we get that $a^{\textcircled{w}}ab \in R^{\textcircled{w}}$ and $(a^{\textcircled{w}}ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}a = a(ab)^{\textcircled{w}}$. Thus, (i) holds.

Conversely, suppose that $a^{\textcircled{w}}ab \in R^{\textcircled{w}}$. Set $m = \max\{\mathrm{ind}(a), \mathrm{ind}(a^{\textcircled{w}}ab)\}$. Since $aba = a^2b, (ab)^n = a^nb^n, aba^{\textcircled{w}} = a^{\textcircled{w}}ab$, we obtain that

$$\begin{aligned} (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}}(ab)^{m+1} &= (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}}a^{m+1}b^{m+1} \\ &= (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}}(a^{\textcircled{w}})^ma^{2m+1}b^{m+1} \\ &= (a^{\textcircled{w}}ab)^{\textcircled{w}}(a^{\textcircled{w}})^{m+1}(ab)^{m+1}a^m \\ &= (a^{\textcircled{w}}ab)^{\textcircled{w}}(a^{\textcircled{w}}ab)^{m+1}a^m = (a^{\textcircled{w}}ab)^ma^m \\ &= (a^{\textcircled{w}})^m(ab)^ma^m = (a^{\textcircled{w}})^ma^{2m}b^m = a^mb^m = (ab)^m \end{aligned}$$

and

$$\begin{aligned} (ab)^{m+1}(a^{\textcircled{w}})^{m+1}(a^{\textcircled{w}}ab)^Da^m &= (a^{\textcircled{w}}ab)^{m+1}(a^{\textcircled{w}}ab)^Da^m = (a^{\textcircled{w}}ab)^ma^m \\ &= (a^{\textcircled{w}})^m(ab)^ma^m = (a^{\textcircled{w}})^ma^m(ab)^m \\ &= (a^{\textcircled{w}})^ma^{2m}b^m = a^mb^m = (ab)^m. \end{aligned}$$

Thus, $(ab)^m \in (ab)^{m+1}R \cap R(ab)^{m+1}$. By Lemma 2.3, we get that $ab \in R^D$. From

$$\begin{aligned} (ab)^D &= (ab)^m((ab)^D)^{m+1} = (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}}(ab)^{m+1}((ab)^D)^{m+1} \\ &= (a^{\textcircled{w}}ab)^m((a^{\textcircled{w}}ab)^{\textcircled{w}})^{m+1}a^{\textcircled{w}}ab(ab)^D, \end{aligned}$$

we have $(ab)^D = (a^{\textcircled{w}}ab)^mt$, where $t = ((a^{\textcircled{w}}ab)^{\textcircled{w}})^{m+1}a^{\textcircled{w}}ab(ab)^D$. So,

$$\begin{aligned} ((ab)^D)^*(ab)^D(ab)^{m+2}(a^{\textcircled{w}})^{m+1}((a^{\textcircled{w}}ab)^{\textcircled{w}})^ma &= ((ab)^D)^*(a^{\textcircled{w}}ab)^{m+1}((a^{\textcircled{w}}ab)^{\textcircled{w}})^ma \\ &= ((a^{\textcircled{w}}ab)^mt)^*(a^{\textcircled{w}}ab)^2(a^{\textcircled{w}}ab)^{\textcircled{w}}a \\ &= t^*((a^{\textcircled{w}}ab)^m)^*(a^{\textcircled{w}}ab)^2(a^{\textcircled{w}}ab)^{\textcircled{w}}a \\ &= t^*((a^{\textcircled{w}}ab)^m)^*a^{\textcircled{w}}aba \\ &= ((a^{\textcircled{w}}ab)^mt)^*a^{\textcircled{w}}aba \\ &= ((ab)^D)^*a^{\textcircled{w}}aba. \end{aligned}$$

By Lemma 2.4, we obtain that

$$\begin{aligned} (ab)^D((ab)^D)^*a^{\textcircled{w}}aba &= a^{\textcircled{w}}a(ab)^D((ab)^D)^*ab = a^{\textcircled{w}}a(ab)^m((ab)^D)^{m+1}((ab)^D)^*ab \\ &= a^{\textcircled{w}}a^{m+1}b^m((ab)^D)^{m+1}((ab)^D)^*ab \\ &= (ab)^m((ab)^D)^{m+1}((ab)^D)^*ab = (ab)^D((ab)^D)^*ab, \end{aligned}$$

that is, $(ab)^D((ab)^D)^*(ab)^D(ab)^{m+2}(a^{\textcircled{w}})^{m+1}((a^{\textcircled{w}}ab)^{\textcircled{w}})^ma = (ab)^D((ab)^D)^*a^{\textcircled{w}}aba = (ab)^D((ab)^D)^*ab$. Since R is a proper $*$-ring, we know that $(ab)^D$ is $*$-cancellable. Hence,

$$((ab)^D)^*(ab)^D(ab)^{m+2}(a^{\textcircled{w}})^{m+1}((a^{\textcircled{w}}ab)^{\textcircled{w}})^ma = ((ab)^D)^*ab.$$

That is, $((ab)^D)^*ab \in ((ab)^D)^*(ab)^DR$. By Lemma 2.7, we obtain that $ab \in R^{\textcircled{w}}$.

Now, we prove that $(ab)^{\textcircled{w}} = (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}}ab)^{\textcircled{w}}$. According to (i), it is sufficient to prove that $(ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}aa^{\textcircled{w}} = a^{\textcircled{w}}a(ab)^{\textcircled{w}}$. Set $k = \mathrm{ind}(ab)$. Since

$$ab((ab)^{\textcircled{w}}aa^{\textcircled{w}})^2 = ab(ab)^{\textcircled{w}}(ab)^{\textcircled{w}}aa^{\textcircled{w}} = (ab)^{\textcircled{w}}aa^{\textcircled{w}},$$

$$(ab)^{\textcircled{w}}aa^{\textcircled{w}}(ab)^{k+1} = (ab)^{\textcircled{w}}aa^{\textcircled{w}}a^{k+1}b^{k+1} = (ab)^{\textcircled{w}}(ab)^{k+1} = (ab)^k$$

and

$$\begin{aligned}((ab)^k)^*(ab)^2(ab)^{\textcircled{w}}aa^{\textcircled{w}} &= ((ab)^k)^*(ab)^2aa^{\textcircled{w}}(ab)^{\textcircled{w}}\\ &= ((ab)^k)^*aa^{\textcircled{w}}(ab)^2(ab)^{\textcircled{w}}\\ &= ((ab)^k)^*aa^{\textcircled{w}}a^{k+1}b^{k+1}((ab)^{\textcircled{w}})^k\\ &= ((ab)^k)^*(ab)^{k+1}((ab)^{\textcircled{w}})^k\\ &= ((ab)^k)^*(ab)^2(ab)^{\textcircled{w}} = ((ab)^k)^*ab,\end{aligned}$$

we have $(ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}aa^{\textcircled{w}} = (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}}$. Similarly, we get that $(ab)^{\textcircled{w}} = a^{\textcircled{w}}a(ab)^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}}ab)^{\textcircled{w}}$. Hence, (ii) is proved. □

Theorem 3.3. *Let $a,\ b \in R^{\textcircled{w}}$ with $aba = a^2b$ and $aba^* = a^*ab$. Then the following statements are equivalent:*

(i) *$ab \in R^{\textcircled{w}}$ and $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$;*

(ii) *$a^{\textcircled{w}}ab \in R^{\textcircled{w}}$ and $(a^{\textcircled{w}}ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}a$.*

Proof. (i) ⇒ (ii): Suppose that $m = \max\{\mathrm{ind}(a), \mathrm{ind}(ab)\}$. From $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$, it is sufficient to prove that $(a^{\textcircled{w}}ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}a$. The rest proof is the same as the proof of Lemma 3.2 (i).

(ii) ⇒ (i): According to Lemma 3.2, we know that $(ab)^{\textcircled{w}} = (a^{\textcircled{w}}ab)^{\textcircled{w}}a^{\textcircled{w}}$. Since $(a^{\textcircled{w}}ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}a$, we have $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}aa^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$. □

Applying Lemma 3.2, the equivalent condition of $(ab)^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}$ is obtained.

Theorem 3.4. *Let $a,\ b \in R^{\textcircled{w}}$ with $aba = a^2b$ and $aba^* = a^*ab$. Then the following statements are equivalent:*

(i) *$ab \in R^{\textcircled{w}}$ and $(ab)^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}$;*

(ii) *$a^{\textcircled{w}}ab \in R^{\textcircled{w}}$ and $(a^{\textcircled{w}}ab)^{\textcircled{w}} = aa^{\textcircled{w}}b^{\textcircled{w}}$.*

Proof. The proof is analogous to the proof of Theorem 3.3. □

According to Theorem 3.3 and 3.4, we have the following corollary.

Corollary 3.5. *Let $a,\ b \in R^{\textcircled{w}}$ with $aba = a^2b$ and $aba^* = a^*ab$. Then the following statements are equivalent:*

(i) *$ab \in R^{\textcircled{w}}$ and $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}$;*

(ii) *$a^{\textcircled{w}}ab \in R^{\textcircled{w}}$ and $(a^{\textcircled{w}}ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}a = aa^{\textcircled{w}}b^{\textcircled{w}}$.*

Dually, we have the following results.

Lemma 3.6. *Let $a \in R$ and $b \in R^{\textcircled{w}}$ with $bab = ab^2$ and $abb^* = b^*ab$. Then $ab \in R^{\textcircled{w}}$ if and only if $abb^{\textcircled{w}} \in R^{\textcircled{w}}$. In this case, the following statements hold:*

(i) *$(abb^{\textcircled{w}})^{\textcircled{w}} = b(ab)^{\textcircled{w}} = (ab)^{\textcircled{w}}b$;*

(ii) *$(ab)^{\textcircled{w}} = b^{\textcircled{w}}(abb^{\textcircled{w}})^{\textcircled{w}} = (abb^{\textcircled{w}})^{\textcircled{w}}b^{\textcircled{w}}$.*

Proof. Suppose that $ab \in R^{\textcircled{w}}$. Since $bab = ab^2$ and $abb^* = b^*ab$, we have $abb^{\textcircled{w}} = b^{\textcircled{w}}ab$, $(ab)^{\textcircled{w}}b = b(ab)^{\textcircled{w}}$ and $(ab)^{\textcircled{w}}b^{\textcircled{w}} = b^{\textcircled{w}}(ab)^{\textcircled{w}}$. By induction, we get that

$$(ab)^n = a^n b^n \text{ and } (abb^{\textcircled{w}})^n = a^n bb^{\textcircled{w}}$$

for any $n \in \mathbb{N}^+$. Set $m = \max\{\mathrm{ind}(b),\ \mathrm{ind}(ab)\}$. Applying Lemma 2.2, we have

$$\begin{aligned}
abb^{\textcircled{w}}b(ab)^{\textcircled{w}} &= b^{\textcircled{w}}bab(ab)^{\textcircled{w}} = b^{\textcircled{w}}b(ab)^{m+1}((ab)^{\textcircled{w}})^{m+1}\\
&= b^{\textcircled{w}}ba^{m+1}b^{m+1}((ab)^{\textcircled{w}})^{m+1} = b^{\textcircled{w}}ba^{m+1}bb^{\textcircled{w}}b^{m+1}((ab)^{\textcircled{w}})^{m+1}\\
&= b^{\textcircled{w}}b(abb^{\textcircled{w}})^{m+1}b^{m+1}((ab)^{\textcircled{w}})^{m+1} = b^{\textcircled{w}}b(b^{\textcircled{w}})^{m+1}(ab)^{m+1}b^{m+1}((ab)^{\textcircled{w}})^{m+1}\\
&= (b^{\textcircled{w}})^{m+1}(ab)^{m+1}b^{m+1}((ab)^{\textcircled{w}})^{m+1} = (abb^{\textcircled{w}})^{m+1}b^{m+1}((ab)^{\textcircled{w}})^{m+1}\\
&= a^{m+1}bb^{\textcircled{w}}b^{m+1}((ab)^{\textcircled{w}})^{m+1} = a^{m+1}b^{m+1}((ab)^{\textcircled{w}})^{m+1}\\
&= ab(ab)^{\textcircled{w}}.
\end{aligned}$$

The rest of proof is analogous to the proof of Lemma 3.2. □

Theorem 3.7. *Let $a,\ b \in R^{\textcircled{w}}$ with $bab = ab^2$ and $abb^* = b^*ab$. Then the following statements are equivalent:*

(i) $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}}$;

(ii) $(abb^{\textcircled{w}})^{\textcircled{w}} = bb^{\textcircled{w}}a^{\textcircled{w}}$.

Theorem 3.8. *Let $a,\ b \in R^{\textcircled{w}}$ with $bab = ab^2$ and $abb^* = b^*ab$. Then the following statements are equivalent:*

(i) $(ab)^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}$;

(ii) $(abb^{\textcircled{w}})^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}b$.

Corollary 3.9. *Let $a,\ b \in R^{\textcircled{w}}$ with $bab = ab^2$ and $abb^* = b^*ab$. Then the following statements are equivalent:*

(i) $(ab)^{\textcircled{w}} = b^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}$;

(ii) $(abb^{\textcircled{w}})^{\textcircled{w}} = bb^{\textcircled{w}}a^{\textcircled{w}} = a^{\textcircled{w}}b^{\textcircled{w}}b$.

Applying Lemma 3.6, we exploit the reverse order law $(a^{\textcircled{w}}abb^{\textcircled{w}})^{\textcircled{w}} = b(a^{\textcircled{w}}ab)^{\textcircled{w}}$ and $(a^{\textcircled{w}}ab)^{\textcircled{w}} = b^{\textcircled{w}}(a^{\textcircled{w}}bb^{\textcircled{w}})^{\textcircled{w}}$ in the following proposition.

Proposition 3.10. *Let $a,\ b \in R^{\textcircled{w}}$. If $a^{\textcircled{w}}ab^2 = ba^{\textcircled{w}}ab$ and $a^{\textcircled{w}}abb^* = b^*a^{\textcircled{w}}ab$, then the following statements are equivalent:*

(i) $(a^{\textcircled{w}}abb^{\textcircled{w}})^{\textcircled{w}} = b(a^{\textcircled{w}}ab)^{\textcircled{w}}$;

(ii) $(a^{\textcircled{w}}ab)^{\textcircled{w}} = b^{\textcircled{w}}(a^{\textcircled{w}}abb^{\textcircled{w}})^{\textcircled{w}}$.

In addition, $\mathrm{ind}(a^{\textcircled{w}}abb^{\textcircled{w}}) \leq \max\{\mathrm{ind}(b),\ \mathrm{ind}(a^{\textcircled{w}}ab)\}$.

Proof. According to Lemma 3.6, we obtain that $a^{\textcircled{w}}ab \in R^{\textcircled{w}}$ if and only if $a^{\textcircled{w}}abb^{\textcircled{w}} \in R^{\textcircled{w}}$. Set $d = a^{\textcircled{w}}a$, by using Lemma 3.6 (i) and (ii), it is easy to know that (i)⇔ (ii). □

Similar to Proposition 3.10, we have the following proposition by using Lemma 3.2.

Proposition 3.11. *Let $a,\ b, \in R^{\textcircled{w}}$. If $abb^{\textcircled{w}}a = a^2bb^{\textcircled{w}}$ and $a^*abb^{\textcircled{w}} = abb^{\textcircled{w}}a^*$, then the following statements are equivalent:*

(i) $(a^{\textcircled{w}}abb^{\textcircled{w}})^{\textcircled{w}} = (abb^{\textcircled{w}})^{\textcircled{w}}a$;

(ii) $(abb^{\textcircled{w}})^{\textcircled{w}} = a^{\textcircled{w}}(a^{\textcircled{w}}abb^{\textcircled{w}})^{\textcircled{w}}$.

In addition, $\mathrm{ind}(a^{\textcircled{w}}abb^{\textcircled{w}}) \leq \max\{\mathrm{ind}(a),\ \mathrm{ind}(abb^{\textcircled{w}})\}$.

4. Absorption laws

In this section, we present sufficient and necessary conditions to ensure that absorption laws of the weak group inverse hold.

Theorem 4.1. *Let $a,\ b \in R^{\circledW}$ with $k = \max\{\mathrm{ind}(a),\ \mathrm{ind}(b)\}$. Then the following statements are equivalent:*

(i) $a^{\circledW}(a+b)b^{\circledW} = a^{\circledW} + b^{\circledW}$;

(ii) $aa^{\circledW} = bb^{\circledW}$;

(iii) $a^kR = b^kR$ *and* $Ra^{\circledW} = Rb^{\circledW}$;

(iv) ${}^{\circ}(a^k) = {}^{\circ}(b^k)$ *and* $(a^{\circledW})^{\circ} = (b^{\circledW})^{\circ}$;

(v) $a^kR \subseteq b^kR$ *and* $Rb^{\circledW} \subseteq Ra^{\circledW}$;

(vi) ${}^{\circ}(a^k) \subseteq {}^{\circ}(b^k)$ *and* $(b^{\circledW})^{\circ} \subseteq (a^{\circledW})^{\circ}$;

(vii) $b^{\circledW}(a+b)a^{\circledW} = a^{\circledW} + b^{\circledW}$.

Proof. (i) ⇒ (ii): Pre-multiplying $a^{\circledW}(a+b)b^{\circledW} = a^{\circledW} + b^{\circledW}$ by $aa^{\circledW}a$, we get that $aa^{\circledW}(a+b)b^{\circledW} = aa^{\circledW} + aa^{\circledW}ab^{\circledW}$, i.e., $aa^{\circledW}bb^{\circledW} = aa^{\circledW}$. Then, by Lemma 2.2, we have

$$b^{\circledW}b^2b^{\circledW} = b^{\circledW}b^{k+1}(b^{\circledW})^k = b^k(b^{\circledW})^k = bb^{\circledW}.$$

Post-multiplying $a^{\circledW}(a+b)b^{\circledW} = a^{\circledW}+b^{\circledW}$ by $b^2b^{\circledW}$, it follows that $a^{\circledW}(a+b)bb^{\circledW} = (a^{\circledW}+b^{\circledW})b^2b^{\circledW}$, i.e., $a^{\circledW}abb^{\circledW} = bb^{\circledW}$. Then $a^{\circledW}abb^{\circledW} = aa^{\circledW}a^{\circledW}abb^{\circledW} = aa^{\circledW}bb^{\circledW}$. So, $bb^{\circledW} = a^{\circledW}abb^{\circledW} = aa^{\circledW}bb^{\circledW} = aa^{\circledW}$.

(ii) ⇒ (i): Since $aa^{\circledW} = bb^{\circledW}$, we have $a^{\circledW} = a^{\circledW}aa^{\circledW} = a^{\circledW}bb^{\circledW}$ and

$$a^{\circledW}ab^{\circledW} = a^{\circledW}abb^{\circledW}b^{\circledW} = a^{\circledW}aaa^{\circledW}b^{\circledW} = a^{\circledW}aa^m(a^{\circledW})^mb^{\circledW} = a^m(a^{\circledW})^mb^{\circledW} = aa^{\circledW}b^{\circledW} = bb^{\circledW}b^{\circledW} = b^{\circledW}.$$

Thus, $a^{\circledW}(a+b)b^{\circledW} = a^{\circledW}bb^{\circledW} + a^{\circledW}ab^{\circledW} = a^{\circledW} + b^{\circledW}$.

(ii) ⇔ (vii): It is analogous to (i) ⇔ (ii).

(ii) ⇒ (iii): Since $a^k = a^{\circledW}a^{k+1} = aa^{\circledW}a^k = bb^{\circledW}a^k = b^k(b^{\circledW})^ka^k$, $a^kR \subseteq b^kR$. Similarly, we have $b^kR \subseteq a^kR$. So, $a^kR = b^kR$. Since $aa^{\circledW} = bb^{\circledW}$, $a^{\circledW} = a^{\circledW}bb^{\circledW}$ and $b^{\circledW} = b^{\circledW}aa^{\circledW}$. Thus, $Ra^{\circledW} \subseteq Rb^{\circledW}$ and $Rb^{\circledW} \subseteq Ra^{\circledW}$, i.e., $Ra^{\circledW} = Rb^{\circledW}$.

(iii) ⇒ (iv): It is clear by Lemma 2.8.

(iv) ⇒ (iii): Since $a, b \in R^{\circledW}$, we have $(1-a^{\circledW}a) \in {}^{\circ}(a^k)$, $(1-b^{\circledW}b) \in {}^{\circ}(b^k)$, $(1-aa^{\circledW}) \in (a^{\circledW})^{\circ}$ and $(1-bb^{\circledW}) \in (b^{\circledW})^{\circ}$. Since ${}^{\circ}(a^k) = {}^{\circ}(b^k) \Leftrightarrow {}^{\circ}(a^k) \subseteq {}^{\circ}(b^k)$ and ${}^{\circ}(b^k) \subseteq {}^{\circ}(a^k)$, we obtain that $(1-a^{\circledW}a)b^k = 0$ and $(1-b^{\circledW}b)a^k = 0$, that is, $b^k = a^{\circledW}ab^k = a^k(a^{\circledW})^{k+1}ab^k$ and $a^k = b^{\circledW}ba^k = b^k(b^{\circledW})^{k+1}ba^k$. Hence, $b^kR \subseteq a^kR$ and $a^kR \subseteq b^kR$, that is, $a^kR = b^kR$. Since $(a^{\circledW})^{\circ} = (b^{\circledW})^{\circ} \Leftrightarrow (a^{\circledW})^{\circ} \subseteq (b^{\circledW})^{\circ}$ and $(b^{\circledW})^{\circ} \subseteq (a^{\circledW})^{\circ}$, we have $b^{\circledW}(1-aa^{\circledW}) = 0$ and $a^{\circledW}(1-bb^{\circledW}) = 0$, that is, $b^{\circledW} = b^{\circledW}aa^{\circledW}$ and $a^{\circledW} = a^{\circledW}bb^{\circledW}$. Thus, $Rb^{\circledW} \subseteq Ra^{\circledW}$ and $Ra^{\circledW} \subseteq Rb^{\circledW}$, that is, $Ra^{\circledW} = Rb^{\circledW}$.

(iii) ⇒ (v) and (iv) ⇒ (vi) are obvious.

(v) ⇒ (ii): Since $a^kR \subseteq b^kR$ and $Rb^{\circledW} \subseteq Ra^{\circledW}$, there exist $u,\ v \in R$ such that $a^k = b^ku$ and $b^{\circledW} = va^{\circledW}$. Then $bb^{\circledW}a^k = bb^{\circledW}b^ku = b^ku = a^k$. By Lemma 2.2, we have

$$aa^{\circledW} = a^k(a^{\circledW})^k = bb^{\circledW}a^k(a^{\circledW})^k = bb^{\circledW}aa^{\circledW}.$$

Since $b^{\circledW}aa^{\circledW} = va^{\circledW}aa^{\circledW} = va^{\circledW} = b^{\circledW}$, $bb^{\circledW}aa^{\circledW} = bb^{\circledW}$. Thus, $aa^{\circledW} = bb^{\circledW}$.

(vi) ⇒ (ii): Since ${}^{\circ}(a^k) \subseteq {}^{\circ}(b^k)$, $(1-aa^{\circledW})b^k = 0$, i.e., $aa^{\circledW}bb^{\circledW} = bb^{\circledW}$. Since $(b^{\circledW})^{\circ} \subseteq (a^{\circledW})^{\circ}$, we have $a^{\circledW}(1-bb^{\circledW}) = 0$, i.e., $aa^{\circledW} = aa^{\circledW}bb^{\circledW}$. Hence, $aa^{\circledW} = bb^{\circledW}$. □

Since the group inverse is a special weak group inverse, we have the following corollary.

Corollary 4.2. *Let $a,\ b \in R^{\#}$. Then the following statements are equivalent:*

(i) $a^{\#}(a+b)b^{\#} = a^{\#} + b^{\#}$;

(ii) $aa^{\#} = bb^{\#}$;

(iii) $aR = bR$ *and* $Ra^{\#} = Rb^{\#}$;

(iv) $^{\circ}a = {}^{\circ}b$ *and* $(a^{\#})^{\circ} = (b^{\#})^{\circ}$;

(v) $aR \subseteq bR$ *and* $Rb^{\#} \subseteq Ra^{\#}$;

(vi) $^{\circ}a \subseteq {}^{\circ}b$ *and* $(b^{\#})^{\circ} \subseteq (a^{\#})^{\circ}$;

(vii) $b^{\#}(a+b)a^{\#} = a^{\#} + b^{\#}$.

Remark 4.3. *According to the definition of the group inverse, we know that* $Ra^{\#} = Rb^{\#}$ *is equivalent to* $Ra = Rb$. *So, Corollary 4.2 is the same as [11, Theorem 3.4].*

Finally, we give special case of absorption laws for the weak group inverse.

Theorem 4.4. *Let* $a,\ b \in R^{ⓦ}$. *Then the following statements are equivalent:*

(i) $a^{ⓦ}(a+b)b^{ⓦ} = a^{ⓦ} + b^{ⓦ}$;

(ii) $Ra^{ⓦ} = R(a^{ⓦ}bb^{ⓦ})$ *and* $b^{ⓦ}R = (a^{ⓦ}ab^{ⓦ})R$;

(iii) $Ra^{ⓦ} \subseteq Rb^{ⓦ}$ *and* $b^{ⓦ}R \subseteq a^{ⓦ}R$.

Proof. (i) $\Rightarrow$ (ii): Pre-multiplying $a^{ⓦ}(a+b)b^{ⓦ} = a^{ⓦ} + b^{ⓦ}$ by $a^{ⓦ}a$, we have $a^{ⓦ} = a^{ⓦ}bb^{ⓦ}$. Post-multiplying $a^{ⓦ}(a+b)b^{ⓦ} = a^{ⓦ} + b^{ⓦ}$ by $bb^{ⓦ}$, we get that $b^{ⓦ} = a^{ⓦ}ab^{ⓦ}$. Thus, $Ra^{ⓦ} = R(a^{ⓦ}bb^{ⓦ})$ and $b^{ⓦ}R = (a^{ⓦ}ab^{ⓦ})R$.

(ii) $\Rightarrow$ (iii): It is evident.

(iii) $\Rightarrow$ (i): Since $Ra^{ⓦ} \subseteq Rb^{ⓦ}$ and $b^{ⓦ}R \subseteq a^{ⓦ}R$, there exist $u,\ v \in R$ such that $a^{ⓦ} = ub^{ⓦ}$ and $b^{ⓦ} = a^{ⓦ}v$. Then we have

$$a^{ⓦ}ab^{ⓦ} = a^{ⓦ}aa^{ⓦ}v = a^{ⓦ}v = b^{ⓦ},\ a^{ⓦ}bb^{ⓦ} = ub^{ⓦ}bb^{ⓦ} = ub^{ⓦ} = a^{ⓦ}.$$

So, $a^{ⓦ}(a+b)b^{ⓦ} = a^{ⓦ}ab^{ⓦ} + a^{ⓦ}bb^{ⓦ} = a^{ⓦ} + b^{ⓦ}$. □

Dually, we have the following theorem.

Theorem 4.5. *Let* $a,\ b \in R^{ⓦ}$. *Then the following statements are equivalent:*

(i) $b^{ⓦ}(a+b)a^{ⓦ} = a^{ⓦ} + b^{ⓦ}$;

(ii) $a^{ⓦ}R = (b^{ⓦ}ba^{ⓦ})R$ *and* $Rb^{ⓦ} = R(b^{ⓦ}aa^{ⓦ})$;

(iii) $a^{ⓦ}R \subseteq b^{ⓦ}R$ *and* $Rb^{ⓦ} \subseteq Ra^{ⓦ}$.

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