Filomat 38:12 (2024), 4167–4185
https://doi.org/10.2298/FIL2412167Y

Published by Faculty of Sciences and Mathematics,
University of Niš, Serbia
Available at: http://www.pmf.ni.ac.rs/filomat

Generic submanifolds of quaternion Kaehler manifolds and Wintgen type inequality

Amine Yılmaz[a]

[a]*Ministry of Education, 35100, Izmir, Turkey*

Abstract. We define and study generic submanifolds of a quaternion Kaehler manifolds, give examples and obtain integrability conditions for distributions and investigate the geometry of their leaves. Further, we investigate the main properties of both, totally umbilical and totally geodesic generic submanifolds and we study mixed foliate generic submanifolds. Finally, we obtain generalized Wintgen type inequality for generic submanifolds in a quaternionic space form.

1. Introduction

The theory of submanifolds of an almost Hermitian manifold is one of the most interesting topics in differential geometry. We note that submanifolds of a Kaehler manifold are determined by the behaviour of its tangent bundle under the action of the almost complex structure J of the ambient manifold. There are two well-known classes of submanifolds, namely, holomorphic(invariant) submanifolds and totally real(anti-invariant) submanifolds. In the first case the tangent space of the submanifold remains invariant under the action of the almost complex structure J where as in the second case it is mapped into the normal space. In 1978, A.Bejancu [5] introduced the notion of CR-submanifold, which is a generalization of holomorphic submanifolds, and totally real submanifolds. The first detailed research on this subject was investigated by Chen in [8] and [9]. This topic is still a very active field of research in the submanifold theory [16]. On the other hand, the topology of CR-submanifolds was also studied, see: [7], [28], [29] and [30]. Quaternion CR-submanifolds were defined by Barros, Chen and Urbano [3] as an analog of CR-submanifolds in quaternion Kaehler manifolds. A submanifold M of a quaternion Kaehler manifold $\bar{M}$ is called a *quaternion* CR-submanifold if there exist two orthogonal complementary distributions D and $D^{\perp}$ such that D is invariant under quaternion structure, that is, $J_aD = D, i = 1, 2, 3$ and $D^{\perp}$ is totally real, that is, $J_a(D^{\perp}) \subset TM^{\perp}, i = 1, 2, 3$. Such submanifolds have been studied by many authors (see, [1], [6], [18], [19], [21], [22], [23], [27], [26], [25], [34]). Generic submanifold was defined as generalization of the concept of CR-submanifold [11]. These submanifolds are known by relaxing the condition on the complementary distribution of holomorphic distribution. More precisely, if the maximal complex subspaces $D_p = T_pM \cap J(T_pM)$ determine on M a distribution $D : D_p \subset T_pM$, the M is called a generic submanifold of $\bar{M}$. Generic submanifolds have been studied widely by many authors, see[12], [14], [15], [20], [31], for recent papers on this topic.
The present paper is organized as follows: In section 2, we recall basic notions and results of quaternion

2020 *Mathematics Subject Classification*. Primary 53C15; Secondary 53C42
Keywords. Quaternion Kaehler manifold; Generic submanifold; Wintgen type inequality; Quaternionic space form
Received: 18 April 2023; Revised: 25 October 2023; Accepted: 27 October 2023
Communicated by Ljubica Velimirović
Email address: amineyilmaz2020@hotmail.com (Amine Yılmaz)

Kaehler manifolds. In section 3, we introduce and study generic submanifolds, a new submanifold that also includes quaternion totally real and quaternion CR-submanifolds, give examples and obtain integrability condition for two distributions. In section 4, we obtain some basic lemmas for later use and for totally umbilical generic submanifold of a quaternion Kaehler manifold, we investigate integrability conditions for distributions and we show that there exist no proper totally umbilical generic submanifolds in positively or negatively curved quaternion Kaehler manifold. In section 5, we study mixed foliate generic submanifolds. The classical Wintgen inequality is a geometric inequality established in [32]. Later, this inequality was extended independently by Rouxel [24] and Gaudalupe and Rodriguez [17]. It was conjectured by P.J.De Smet, F. Dillen, L.Vestraelen and L.Vrancken [13] that the following inequality holds at every point of an n-dimensional Riemann submanifold M^n into a real space form $\bar{M}^{n+m}(c)$ of constant sectional curvature c:

$$\rho + \rho^{\perp} \leq \|H\|^2 + c$$

In section 6, we obtain generalized Wintgen type inequality for generic submanifolds in a quaternionic space form.

2. Preliminaries

In this section we recall some basic notions from [4], [5] and [33] for later sections. Let $\bar{M}$ be a 4m-dimensional manifold with the Riemann metric $<,>$ on $\bar{M}$, $m \geq 1$. Then $\bar{M}$ is called quaternion Kaehler manifold if there exist a 3-dimensional vector bundle σ of type (1,1) with local basis of almost Hermitian structures J_1, J_2, J_3 (that is,$< J_aX, J_aY >=< X, Y >, a = 1, 2, 3$) satisfying

$$J_1 \circ J_2 = -J_2 \circ J_1 = J_3. \tag{1}$$

Also, for a quaternionic Kaehler manifold, we have

$$\bar{\nabla}_X J_a = \sum_{b=1}^{3} Q_{ab}(X) J_b, a = 1, 2, 3, \forall X \in (T\bar{M}), \tag{2}$$

where Q_{ab} are certain 1-forms locally defined on $\bar{M}$ such that $Q_{ab} + Q_{ba} = 0$.
Let $\bar{M}$ be a Riemann manifold and M a Riemann submanifold of $\bar{M}$ with the Riemann metric induced by the metric of $\bar{M}$. Let ∇ and $\bar{\nabla}$ be the covariant differentiations on M and $\bar{M}$, respectively. We denote by TM and $TM^{\perp}$ the tangent and normal bundle respectively. The Gauss and Weingarten formulae are given, respectively by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{3}$$

$$\bar{\nabla}_X \xi = -A_\xi X + \nabla^{\perp}_X \xi \tag{4}$$

for any vector fields X, Y tangent to M and any vector field ξ normal to M, where h denotes the second fundamental form, $\nabla^{\perp}_X$ is the normal connection on the $TM^{\perp}$ and A_ξ is the fundamental tensor of Weingarten with respect to the normal section. We also have the relation

$$< h(X, Y), \xi >=< A_\xi X, Y > . \tag{5}$$

Let X be a unit vector in $\bar{M}$. Then the 4-plane spanned by $\{X, J_1X, J_2X, J_3X\}$ denoted by $Q(X)$ is called a quaternionic 4-plane. Any 2-plane in $Q(X)$ is called a quaternionic plane. The sectional curvature of a quaternionic plane is a quaternionic sectional curvature. A quaternionic Kaehler manifold is called a quaternionic space form if its quaternionic sectional curvature is constant and a quaternionic space form is denoted by $\bar{M}(c)$. In this case, the curvature tensor of $\bar{M}(c)$ is given by

$$\begin{aligned} \bar{R}(X, Y)Z &= \frac{c}{4}\{< Y, Z > X- < X, Z > Y + \sum_{a=1}^{3} < Z, J_aY > J_aX \\ &\quad - < Z, J_aX > J_aY + 2 < X, J_aY > J_aZ\} \end{aligned} \tag{6}$$

$\forall X, Y, Z \in \Gamma(TM)$, [33].For give a space the second fundamental form h, the covariant derivation is defined by

$$(\nabla_X h)(Y, Z) = \nabla_X^{\perp} h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{7}$$

and the Gauss, Codazzi and Ricci equations of M are then given by

$$R(X, Y, Z, W) = \bar{R}(X, Y, Z, W) + < h(X, W), h(Y, Z) > - < h(X, Z), h(Y, W) >, \tag{8}$$

$$(\bar{R}(X, Y)Z)^{\perp} = (\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) \tag{9}$$

$$\bar{R}(X, Y, \xi, \eta) = R^{\perp}(X, Y, \xi, \eta) + < \left[A_{\xi}, A_{\eta}\right] X, Y >, \tag{10}$$

$\forall X, Y, Z, W \in \Gamma(TM)$ and $\xi, \eta \in \Gamma(TM)^{\perp}$.
The mean curvature vector H of M in $\bar{M}$ is defined by $H = (\frac{1}{n})trace\ h$,where n denotes the dimension of M. If we have

$$h(X, Y) = < X, Y > H \tag{11}$$

for any $X, Y \in \Gamma(TM)$, M is called totally umbilical submanifold. Finally M is called totally geodesic if $h(X, Y) = 0$ identically on M.

3. Generic Submanifolds and Integrability of the Distributions

In this section, we introduce generic submanifolds and investigate the integrability of the totally real distribution $\tilde{D}$ and the quaternion distribution D.

Definition 3.1. *Let M be a real submanifold of a quaternion Kaehler manifold $\bar{M}$ and TM the tangent bundle of M. Suppose there are two subbundles D and $\tilde{D}$ with constant ranks on M such that*

$$D = J_a TM \cap TM \tag{12}$$

and $\tilde{D}$ is complementary orthogonal to D, then M is called generic submanifold of $\bar{M}$.

From definition, we have

$$TM = D \oplus \tilde{D},\ J_a(D) = D,\ a \in \{1, 2, 3\}.$$

For $X \in \Gamma(\tilde{D})$, we have

$$J_a X = T_a X + F_a X \tag{13}$$

where $T_a X \in \Gamma(\tilde{D})$ and $F_a X \in \Gamma(\vartheta^{\perp})$.
On the other hand, we denote the complementary orthogonal distribution to $\vartheta^{\perp}$ by ϑ then we have $J_a(\vartheta) = \vartheta$.For $V \in \Gamma(\vartheta^{\perp})$, we have

$$J_a V = t_a V + f_a V \tag{14}$$

where $t_a X \in \Gamma(TM)$ and $f_a X \in \Gamma(\vartheta^{\perp})$.
Now we give some examples of generic submanifolds.

Example 3.2. *Every holomorphic submanifold M of a quaternion Kaehler manifold is generic with $\tilde{D} = \{0\}$ and $D = TM$.*

Example 3.3. *Every totally real submanifold M of a quaternion Kaehler manifold is generic with $D = \{0\}$ and $\tilde{D} = TM$.*

Example 3.4. *Every real hypersurface of a quaternion Kaehler manifold is generic submanifold with* $\vartheta = \{0\}$ *and* $\tilde{D} = Sp\{J_a N\}$*, where* N *is the unit normal vector field of the hypersurface.*

Example 3.5. *Every CR-submanifold [3] of a quaternion Kaehler manifold is generic submanifold such that* D *and* $\tilde{D}$ *is orthogonal.*

We now present an elementary example of generic submanifolds.

Example 3.6. *For any* $\alpha \in (0, \frac{\pi}{2})$*, let* M *be a submanifold of* R^{16} *given by*

$$x_1 = x_2 = x_3 = x_4 = const,$$

$$x_5 = u_2 - u_3 \cos\alpha - u_4 \sin\alpha$$

$$x_6 = -u_1 + u_3 \sin\alpha - u_4 \cos\alpha$$

$$x_7 = u_1 \cos\alpha - u_2 \sin\alpha - u_4$$

$$x_8 = u_1 \sin\alpha + u_2 \cos\alpha + u_3$$

$$x_9 = u_5 + u_6 + u_7$$

$$x_{10} = u_5 - u_6 + u_7$$

$$x_{11} = u_5 + u_6 - u_7$$

$$x_{12} = -u_5 + u_6 + u_7$$

$$x_{13} = -u_6 \sin\alpha - u_7 \cos\alpha + u_9 \cos\alpha - u_{10} \sin\alpha$$

$$x_{14} = u_6 \cos\alpha - u_7 \sin\alpha - u_9 \sin\alpha + u_{10} \cos\alpha$$

$$x_{15} = -u_8 \sin\alpha$$

$$x_{16} = -u_8 \cos\alpha.$$

Then TM *is spanned by*

$$Z_1 = -\partial x_6 + \cos\alpha \partial x_7 + \sin\alpha \partial x_8$$

$$Z_2 = \partial x_5 - \sin\alpha \partial x_7 + \cos\alpha \partial x_8$$

$$Z_3 = -\cos\alpha \partial x_5 + \sin\alpha \partial x_6 + \partial x_8$$

$$Z_4 = -\sin\alpha \partial x_5 - \cos\alpha \partial x_6 - \partial x_7$$

$$Z_5 = \partial x_9 + \partial x_{10} + \partial x_{11} - \partial x_{12}$$

$$Z_6 = \partial x_9 - \partial x_{10} + \partial x_{11} + \partial x_{12} - \sin\alpha \partial x_{13} + \cos\alpha \partial x_{14}$$

$$Z_7 = \partial x_9 + \partial x_{10} - \partial x_{11} + \partial x_{12} - \cos\alpha \partial x_{13} - \sin\alpha \partial x_{14}.$$

$$Z_8 = -\sin\alpha \partial x_{15} - \cos\alpha \partial x_{16}$$

$$Z_9 = \cos\alpha \partial x_{13} - \sin\alpha \partial x_{14}$$

$$Z_{10} = -\sin\alpha \partial x_{13} + \cos\alpha \partial x_{14}.$$

And $TM^{\perp}$ *is spanned by*

$$V_1 = -\partial x_3 - \partial x_4$$

$$V_2 = \partial x_3 - \partial x_4$$

$$V_3 = \partial x_1 - \partial x_2$$

$$V_4 = \partial x_1 + \partial x_2$$

$$V_5 = \partial x_9 - \partial x_{10} - \partial x_{11} - \partial x_{12}$$
$$V_6 = -\cos\alpha\partial x_{15} + \sin\alpha\partial x_{16}.$$

We have $J_1(Z_1) = Z_2$, $J_2(Z_1) = Z_3$ and $J_3(Z_1) = Z_4$, thus $D = Span\{Z_1, Z_2, Z_3, Z_4\}$ is an invariant distribution in TM and

$$J_1(Z_5) = -V_5, \qquad J_2(Z_5) = -Z_7 - Z_9, \qquad J_3(Z_5) = Z_6 - Z_{10},$$
$$J_1(Z_6) = Z_7, \qquad J_2(Z_6) = -V_5 + Z_8, \qquad J_3(Z_6) = -Z_5 - V_6,$$
$$J_1(Z_7) = -Z_6, \qquad J_2(Z_7) = Z_5 + V_6, \qquad J_3(Z_7) = -V_5 + Z_8,$$
$$J_1(Z_8) = -V_6, \qquad J_2(Z_8) = -Z_{10}, \qquad J_3(Z_8) = Z_9,$$
$$J_1(Z_9) = Z_{10}, \qquad J_2(Z_9) = -V_6, \qquad J_3(Z_9) = -Z_8,$$
$$J_1(Z_{10}) = -Z_9, \qquad J_2(Z_{10}) = Z_8, \qquad J_3(Z_{10}) = -V_6,$$

that is $\tilde{D} = Span\{Z_5, Z_6, Z_7, Z_8, Z_9, Z_{10}\}$.Also, we have

$$J_1(V_5) = Z_5, \qquad J_2(V_5) = Z_6 - Z_{10}, \qquad J_3(V_5) = Z_7 + Z_9,$$
$$J_1(V_6) = Z_8, \qquad J_2(V_6) = Z_9, \qquad J_3(V_6) = Z_{10}.$$

Hence M is generic submanifold of R^{16} with $\vartheta = Span\{V_1, V_2, V_3, V_4\}$ and $\vartheta^\perp = Span\{V_5, V_6\}$.

We now give the following definition by adjusting notion in [5].

Definition 3.7. *Let M be generic submanifold of a quaternion Kaehler manifold. M is called D-geodesic if $h(X, Y) = 0$, $\forall X, Y \in \Gamma(D)$*

We now investigate the integrability of distributions on a generic submanifold M.

Theorem 3.8. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. The the following assertions are equivalent:*
(i) the second fundamental form of M satisfies

$$h(X, J_aY) = h(Y, J_aX) \tag{15}$$

for any $X, Y \in \Gamma(D)$ and $a \in \{1, 2, 3\}$,
(ii)M is D-geodesic,
(iii) the distribution D is integrable.

Proof. $(i) \Rightarrow (ii)$ By (1) and (15) we obtain

$$h(J_3X, Y) = h(X, J_3Y) = h(X, (J_1 \circ J_2)Y) = h(J_1X, J_2Y) = -h(J_3X, Y). \tag{16}$$

From (16), M is D-geodesic.
$(ii) \Rightarrow (iii)$ The distribution D is integrable if and only if

$$J_a < [X, Y], Z >= 0$$

for any $X, Y \in \Gamma(D)$, $Z \in \Gamma(\vartheta^\perp)$. By (2) and (3) we have

$$< h(X, J_aY) - h(Y, J_aX), Z >=< J_a[X, Y], Z > . \tag{17}$$

Since M is D-geodesic, we find

$$< J_a[X, Y], Z >= 0,$$

which implies $[X, Y] \in \Gamma(D)$. Thus D is integrable.
$(iii) \Rightarrow (i)$ This implication follows from (17). □

Theorem 3.9. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. The distribution $\tilde{D}$ is integrable if and only if*

$$\nabla_U T_a V - \nabla_V T_a U - A_{F_a V} U + A_{F_a U} V \in \Gamma(D^{\perp}), \tag{18}$$

for all $U, V \in \Gamma(\tilde{D})$.

Proof. By (2), (3), (4) and (13) we obtain

$$\nabla_U T_a V + h(U, T_a V) - A_{F_a V} U + \nabla^{\perp}_U F_a V = Q_{ab}(U) J_b V + Q_{ac}(U) J_c V + J_a(\nabla_U V + h(U, V)).$$

For any $X \in \Gamma(D)$ we have

$$< \nabla_U T_a V - \nabla_V T_a U - A_{F_a V} U + A_{F_a U} V, X >=< T_a[U, V], X >= 0$$

where $T_a V$, $T_a U$ (resp., $F_a V$, $F_a U$) are the tangential (resp. the normal) component of $J_a V$ and $J_a U$. Thus proof is completes. □

4. Totally Umbilical Generic Submanifolds

We first give several lemmas for later use in this section and other section.

Lemma 4.1. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we have*
(i)

$$< h(J_a X, Z), \xi >=< \nabla^{\perp}_X F_a Z, \xi > + < h(X, T_a Z), \xi >=< J_a h(X, Z), \xi >, \tag{19}$$

(ii)

$$< h(D, D), \vartheta >= 0 \tag{20}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(\tilde{D})$ and $\xi \in \Gamma(\vartheta)$.

Proof. (i) From (2) we obtain

$$Q_{ab}(X) J_b Z + Q_{ac}(X) J_c Z = \bar{\nabla}_X J_a Z - J_a \bar{\nabla}_X Z. \tag{21}$$

From (3), (4) and (13) we have

$$\begin{aligned} Q_{ab}(X) J_b Z + Q_{ac}(X) J_c Z \quad = \quad & \nabla_X T_a Z + h(X, T_a Z) - A_{F_a Z} X + \nabla^{\perp}_X F_a Z \\ & - J_a(\nabla_X Z + h(X, Z)). \end{aligned} \tag{22}$$

For any $\xi \in \Gamma(\vartheta)$ we get

$$< J_a h(X, Z), \xi >=< \nabla^{\perp}_X F_a Z, \xi > + < h(X, T_a Z), \xi > . \tag{23}$$

In a similar way, for $X \in \Gamma(D)$, $Z \in \Gamma(\tilde{D})$ by using (3), we obtain

$$< h(J_a X, Z), \xi >=< J_a h(X, Z), \xi > . \tag{24}$$

From (23) and (24) we obtain (i).
(ii) By using (2) and (3) we have

$$Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y = \nabla_X J_a Y + h(X, J_a Y) - J_a(\nabla_X Y + h(X, Y)).$$

For $\xi \in \Gamma(\vartheta)$, we obtain

$$< h(X, Y), J_a \xi >= - < h(X, J_a Y), \xi > . \tag{25}$$

Replacing X by $J_b X$, from (1) and (25), we have

$$< h(X, Y), J_c \xi >= 0.$$

Thus proof is completed. □

For any $X, Y \in \Gamma(D)$ we put

$$\nabla_X Y = \dot{\nabla}_X Y + \dot{h}(X, Y), \tag{26}$$

where $\dot{\nabla}_X Y$ and $\dot{h}(X, Y)$ are D and $\tilde{D}$ components of $\nabla_X Y$ respectively.

Lemma 4.2. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. The distribution D is integrable if and only if*

$$\dot{h}(X, J_a Y) = \dot{h}(Y, J_a X)$$

for all $X, Y \in \Gamma(D)$.

Proof. From (2) and (3) we have

$$Q_{ab}(X) J_b Y + Q_{ac}(X) J_c Y = \nabla_X J_a Y + h(X, J_a Y) - J_a(\nabla_X Y + h(X, Y)).$$

For $Z \in \Gamma(\tilde{D})$, by using (13), we obtain

$$- < \nabla_X Y, T_a Z > - < h(X, Y), F_a Z > = < \nabla_X J_a Y, Z > .$$

From (26), we get

$$< \nabla_X Y, T_a Z > = - < \dot{h}(X, J_a Y), Z > - < h(X, Y), F_a Z > . \tag{27}$$

By interchanging X and Y in (27) we obtain

$$< [X, Y], T_a Z > = < \dot{h}(Y, J_a X) - \dot{h}(X, J_a Y), Z >$$

which proves the assertion. □

Lemma 4.3. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. If $\tilde{D}$ is integrable and its leaves are totally geodesic in M, then we have*

$$< h(D, \tilde{D}), F_a \tilde{D} > = 0. \tag{28}$$

Proof. Under the hypothesis, for $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D})$ we have $\nabla_Z X \in \Gamma(D)$. From (2) and (3) we have

$$Q_{ab}(Z) J_b X + Q_{ac}(Z) J_c X = \nabla_Z J_a X + h(Z, J_a X) - J_a(\nabla_Z X + h(Z, X)).$$

For $W \in \Gamma(\tilde{D})$, we obtain

$$< \nabla_Z X, W > = < \nabla_Z J_a X, T_a W > + < h(Z, J_a X), F_a W >$$

$$< h(Z, J_a X), F_a W > = 0.$$

Thus the proof is completed. □

Lemma 4.4. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. If D is integrable and its leaves are totally geodesic in M, then we have*

$$< h(D, D), F_a \tilde{D} > = 0. \tag{29}$$

Proof. Similar to proof of the Lemma 4.3 by using (2), (3), (4) we obtain the assertion. □

Definition 4.5. *A real submanifold M of a quaternion Kaehler manifold is called a generic product if it is locally the Riemann product $M^T \times M^\perp$ where M^T(resp.$M^\perp$) is a leaf of D(resp.$\tilde{D}$) if and only if both distributions D and $\tilde{D}$ are integrable and their leaves are totally geodesic in M [11].*

Lemma 4.6. *Let M be a generic submanifold of a quaternion Kaehler manifold $\bar{M}$. If M is a generic product submanifold of a quaternion Kaehler manifold $\bar{M}$, then we have*

$$A_{F_a\tilde{D}}D = 0. \tag{30}$$

Proof. From Lemma 4.3 and Lemma 4.4 we proves assertion of the Lemma 4.6. □

We now start to examine the geometry of totally umbilical generic submanifolds.

Theorem 4.7. *Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we have (i) the distribution $\tilde{D}$ is involutive if and only if*

$$\nabla_U T_a V - \nabla_V T_a U \in \Gamma(\tilde{D}), \tag{31}$$

for all $U, V \in \Gamma(\tilde{D})$
(ii) the distribution D is involutive if and only if M is totally geodesic.

Proof. From Theorem 3.8, Theorem 3.9 and by using (11), we obtain the assertion. □

Lemma 4.8. *Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifold $\bar{M}$. Then we have the following expression;*

$$\nabla_X Z = Q_{ab}(X)T_c Z - Q_{ac}(X)T_b Z - T_a\nabla_X T_a Z - t_a\nabla^{\perp}_X F_a Z + J_a A_{F_aZ}X \tag{32}$$

$X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D})$.

Proof. From (2), we have

$$\bar{\nabla}_X J_a Z = Q_{ab}(X)J_b Z + Q_{ac}(X)J_c Z + J_a\bar{\nabla}_X Z.$$

By (3), (4) and (13) we obtain

$$\nabla_X T_a Z + h(X, T_a Z) - A_{F_aZ}X + \nabla^{\perp}_X F_a Z = Q_{ab}(X)J_b Z - Q_{ac}(X)J_c Z + J_a(\nabla_X Z + h(X, Z)).$$

Since M is totally umbilical we find

$$\nabla_X T_a Z - A_{F_aZ}X + \nabla^{\perp}_X F_a Z = Q_{ab}(X)J_b Z - Q_{ac}(X)J_c Z + J_a\nabla_X Z.$$

Thus using quaternion structures J_a and taking the tangential parts, we obtain

$$\nabla_X Z = Q_{ab}(X)T_c Z - Q_{ac}(X)T_b Z - T_a\nabla_X T_a Z - t_a\nabla^{\perp}_X F_a Z + J_a A_{F_aZ}X.$$

□

For a quaternionic space form, we have the following result.

Theorem 4.9. *Let M be a totally umbilical generic submanifold of a quaternionic space form with $c \neq 0$. Then we have*
(i) M is totally geodesic,
(ii) M is generic product.

Proof. (i)For $\forall X, Y \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D})$ from (6), (7) and (9) we get

$$(\nabla_X h)(Y, Z) - (\nabla_Y h)(X, Z) = \frac{c}{2}g(X, J_a Y)F_a Z.$$

$$\begin{aligned}\nabla^{\perp}_X h(Y, Z) - \nabla^{\perp}_Y h(X, Z) - h([X, Y], Z) - h(Y, \nabla_X Z) + h(X, \nabla_Y Z)\\ = \frac{c}{2}g(X, J_a Y)F_a Z.\end{aligned}$$

Since M is totally umbilical we have

$$-g([X,Y],Z)H - g(Y,\nabla_X Z)H + g(X,\nabla_Y Z)H = \frac{c}{2}g(X,J_aY)F_aZ.$$

From (32) we obtain

$$\begin{aligned}\frac{c}{2}g(X,J_aY)F_aZ &= -g([X,Y],Z)H + g(Y,T_a\nabla_X T_aZ)H - g(Y,J_aA_{F_aZ}X)H \\ &\quad -g(X,T_a\nabla_Y T_aZ)H + g(X,J_aA_{F_aZ}Y)H.\end{aligned}$$

Then by direct calculations we get

$$\begin{aligned}&-g(\nabla_X J_aY,T_aZ)H + g(\nabla_Y J_aX,T_aZ)H - g(J_aY,\nabla_X T_aZ)H \\ &+g(J_aX,\nabla_Y T_aZ)H = \frac{c}{2}g(X,J_aY)F_aZ.\end{aligned}$$

Hence, for $X = J_aY$ we have

$$\begin{aligned}&-g(\nabla_X X,T_aZ)H + g(\nabla_{-J_aX} J_aX,T_aZ)H - g(X,\nabla_X T_aZ)H \\ &+g(J_aX,\nabla_{-J_aX} T_aZ)H = \frac{c}{2}g(X,X)H.\end{aligned}$$

This implies

$$0 = \frac{c}{2}g(X,X)H.$$

Hence we have $H = 0$, that is, M is totally geodesic in $\bar{M}$.
(*ii*) If M is a proper generic quaternion submanifold and M is totally umbilical, then we have

$$h(X,Y) =< X,Y > H.$$

Thus from (20), $< H,\xi >= 0$, that is, $H \in \vartheta^{\perp}$ for any vector X,Y tangent to M. From (11), we have

$$< h(D,\tilde{D}),F_a\tilde{D} >=< D,\tilde{D} >< H,F_a\tilde{D} >= 0. \tag{33}$$

Since M is totally geodesic, from Theorem 4.7 (ii) we obtain

$$< h(D,D),F_a\tilde{D} >= 0. \tag{34}$$

From (33) and (34) proof is completed. □

In the sequel, we show that there are some restrictions for the existence of totally umbilical generic submanifolds. we first present the following general expression.

Theorem 4.10. *Let M be a totally umbilical generic submanifold of a quaternionic space form with $c = 0$. Then we have*

$$K_M(X\wedge Y) = ||H||^2$$

for any unit vectors $X \in \Gamma(D)$, $Y \in \Gamma(\tilde{D})$.

Proof. From (6) and (8) we have

$$\begin{aligned}R(X,Y,Z,W) &= \frac{c}{4}\{< Y,Z >< X,W > - < X,Z >< Y,W > + \sum_{a=1}^{3} < Z,J_aY >< J_aX,W > \\ &\quad - < Z,J_aX >< J_aY,W > +2 < X,J_aY >< J_aZ,W >\} \\ &\quad + < h(X,W),h(Y,Z) > - < h(X,Z),h(Y,W) > .\end{aligned}$$

For $X = Z \in \Gamma(D)$, $Y = W \in \Gamma(\tilde{D})$ we obtain

$$R(X,Y,X,Y) = -\frac{c}{4} < X,X >< Y,Y > + < h(X,Y), h(Y,X) > - < h(X,X), h(Y,Y) > .$$

From (11) and for $c = 0$ we get

$$K_M(X \wedge Y) = \|H\|^2 .$$

From Theorem 4.10 we have

Corollary 4.11. *There exist no proper totally umbilical negatively curved generic submanifold of a quaternion space form.*

Corollary 4.12. *Let M be a totally umbilical generic submanifold of a quaternionic Euclidean space form. Then any proper totally geodesic generic submanifold of constant sectional curvature is flat.*

Theorem 4.13. *There exist no proper totally umbilical generic submanifolds in positively or negatively curved quaternion Kaehler manifolds.*

Proof. Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifolds. Thus (7) and (11) implies

$$\begin{aligned} \bar{\nabla}_X h(Y,Z) &= \nabla^{\perp}_X(< Y,Z > H) - < \nabla_X Y, Z > H - < Y, \nabla_X Z > H && (35) \\ &= \nabla^{\perp}_X < Y,Z > H + < Y,Z > \nabla^{\perp}_X H - < \nabla_X Y, Z > H - < Y, \nabla_X Z > H \\ &= < Y,Z > \nabla^{\perp}_X H. && (36) \end{aligned}$$

For any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(\tilde{D})$, by using the equation (36) in (9), we obtain

$$\bar{R}(X,Y,Z,F_a W) =< Y,Z >< \nabla^{\perp}_X H, F_a W > - < X,Z >< \nabla^{\perp}_Y H, F_a W >= 0,$$

which is a contradiction.

Lemma 4.14. *Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifold $\bar{M}$. If the distribution $\tilde{D}$ is integrable, then we have*

$$A_{F_a W} X = A_{F_a X} W \qquad (37)$$

for all $X, W \in \Gamma(\tilde{D})$.

Proof. From (2) we have

$$\begin{aligned} (\bar{\nabla}_X J_a) W &= \bar{\nabla}_X J_a W - J_a \bar{\nabla}_X W \\ Q_{ab}(X) J_b W + Q_{ac}(X) J_c W &= \bar{\nabla}_X J_a W - J_a \bar{\nabla}_X W. \end{aligned}$$

By using (3), (4), (13), we have

$$\begin{aligned} &\nabla_X T_a W + h(X, T_a W) - A_{F_a W} X + \nabla^{\perp}_X F_a W && (38) \\ &= Q_{ab}(X) J_b W + Q_{ac}(X) J_c W + J_a(\nabla_X W + h(X,W)). \end{aligned}$$

By interchanging X and W in (38) and for $J_a Y \in \Gamma(D)$, we obtain

$$< A_{F_a W} X - A_{F_a X} W, J_a Y >=< \nabla_X T_a W - \nabla_W T_a X, J_a Y > - < [X,W], Y > .$$

From Theorem 4.7 this completes proof. □

Theorem 4.15. *Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifold $\bar{M}$. If the distribution $\tilde{D}$ is integrable, then we have*
(i)$\nabla^{\perp}_Y H \in \Gamma(\vartheta)$, $Y \in \Gamma(D)$ and $J_a \nabla^{\perp}_Y H = h(Y, T_a H) + \nabla^{\perp}_Y F_a H$.
(ii) the totally real distribution is one dimensional, that is,dim$(\tilde{D}) = 1$ or $H \perp F_a X$, $X \in \Gamma(\tilde{D})$.

Proof. (*i*) We take $Y \in \Gamma(D)$, $V \in \Gamma(\vartheta^{\perp})$. Since M is a totally umbilical generic submanifold, from (9), we have

$$\bar{R}(J_1Y, J_2Y, J_3Y, V) =< J_2Y, J_3Y >< \nabla^{\perp}_{J_1Y}H, V > - < J_1Y, J_3Y >< \nabla^{\perp}_{J_2Y}H, V >= 0 \tag{39}$$

and

$$\begin{aligned} \bar{R}(J_1Y, J_2Y, J_3Y, V) &= < \bar{R}(J_3Y, V)J_1Y, J_2Y > \\ &= - < J_2\bar{R}(J_3Y, V)J_1Y, Y > \\ &=< \bar{R}(J_3Y, V)J_3Y, Y > \\ &= \bar{R}(J_3Y, Y, J_3Y, V) \\ &= -\bar{R}(Y, J_3Y, J_3Y, V). \end{aligned} \tag{40}$$

From (9) and (11) we get

$$\begin{aligned} \bar{R}(Y, J_3Y, J_3Y, V) &= - < J_3Y, J_3Y >< \nabla^{\perp}_{Y}H, V > + < Y, J_3Y >< \nabla^{\perp}_{J_3Y}H, V > \\ &= - \ \|Y\|^2 < \nabla^{\perp}_{Y}H, V > . \end{aligned}$$

From (39) and (41) we obtain $\nabla^{\perp}_{Y}H \in \Gamma(\vartheta)$, $Y \in \Gamma(D)$.
From (2), for $H \in \Gamma(\vartheta^{\perp})$

$$\begin{aligned} &\nabla_Y T_aH + h(Y, T_aH) - A_{F_aH}Y + \nabla^{\perp}_{Y}F_aH \\ &= Q_{ab}(Y)J_bH + Q_{ac}(Y)J_cH + J_a(-A_HX + \nabla^{\perp}_{X}H). \end{aligned}$$

For $J_a\xi \in \Gamma(\vartheta)$, we obtain

$$< h(Y, T_aH) + \nabla^{\perp}_{Y}F_aH, J_a\xi >=< J_a\nabla^{\perp}_{X}H, J_a\xi > .$$

Thus, we have

$$J_a\nabla^{\perp}_{Y}H = h(Y, T_aH) + \nabla^{\perp}_{Y}F_aH.$$

(*ii*) From Lemma 4.14 we have

$$A_{F_aW}X = A_{F_aX}W.$$

For $X \in \Gamma(\tilde{D})$ we get

$$< A_{F_aX}W, X >=< A_{F_aW}X, X > .$$

Since M is a totally umbilical generic submanifold, we obtain

$$< W, X >< H, F_aX >=< X, X >< H, F_aW > . \tag{41}$$

By interchanging X and W in (41), we have

$$< X, W >< H, F_aW >=< W, W >< H, F_aX > \tag{42}$$

This together with (41) gives

$$< H, F_aX >= \frac{< X, X >< W, W >}{< X, W >^2} < H, F_aX > .$$

Thus the distribution $\tilde{D}$ is one dimensional or $H \perp F_aX$. □

5. Mixed Foliate Generic Submanifolds

Definition 5.1. *Let M be generic submanifold of a quaternion Kaehler manifold. Then we say that*
i) M is mixed geodesic if $h(X,Y) = 0$, $\forall X \in \Gamma(D)$, $Y \in \Gamma(\tilde{D})$
ii) M is mixed foliated if M is mixed geodesic and D is integrable[5].

In this section we study an important class of generic submanifolds of a quaternion Kaehler manifolds. We give the following preparatory lemma.

Lemma 5.2. *Let M be a foliate generic submanifold of a quaternion Kaehler manifold $\bar{M}$. If the leaves of the distribution $\tilde{D}$ are totally geodesic, then we have*

$$A_{F_aZ}J_aX = -J_aA_{F_aZ}X \tag{43}$$

for any $X \in \Gamma(D)$, $Z \in \Gamma(\tilde{D})$.

Proof. Lemma 4.3 gives

$$< h(D,\tilde{D}), F_a\tilde{D} >= 0.$$

Thus, we have

$$A_{F_a\tilde{D}}D \in \Gamma(D).$$

Since the distribution D is foliate, we have

$$h(X, J_aY) = h(J_aX, Y).$$

For $Z \in \Gamma(\tilde{D})$, from (13) we find

$$< h(X, J_aY), F_aZ >=< h(J_aX, Y), F_aZ >$$

$$< A_{F_aZ}X, J_aY >=< A_{F_aZ}J_aX, Y >$$

$$< A_{F_aZ}J_aX + J_aA_{F_aZ}X, Y >= 0.$$

□

Let M be a mixed foliate generic submanifold of a quaternion Kaehler manifold $\bar{M}$ and the leaves of the distribution $\tilde{D}$ are totally geodesic. From Lemma 4.3 we have

$$A_{F_a\tilde{D}}\tilde{D} \in \Gamma(\tilde{D}), A_{F_a\tilde{D}}D \in \Gamma(D). \tag{44}$$

By the equation of Codazzi, for any $X, Y \in \Gamma(D)$ and $Z, W \in \Gamma(\tilde{D})$

$$\begin{aligned}\bar{R}(X,Y,Z,F_aW) =&< \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z), F_aW > \\ &- < \nabla^{\perp}_Y h(X,Z) - h(\nabla_Y X, Z) - h(X, \nabla_Y Z), F_aW > .\end{aligned} \tag{45}$$

By using (2), (3), (4), Lemma 4.3 and Lemma 4.1, we find

$$< \nabla^{\perp}_X h(Y,Z), F_aW >=< h(J_aY,Z), h(X,W) > + < h(Y,Z), h(T_aZ,X) > .$$

Hence, for $Y = J_aX$ and $W = Z$, we have

$$< \nabla^{\perp}_X h(J_aX,Z), F_aZ >= -\|h(X,Z)\|^2 + < J_ah(X,Z), h(T_aZ,X) > . \tag{46}$$

Similarly we may also prove that

$$< \nabla^{\perp}_{J_aX} h(X,Z), F_aZ >= \|h(X,Z)\|^2 - < J_ah(X,Z), h(T_aZ,X) > . \tag{47}$$

By using (2), (3), (4), (5) and Lemma 5.2, we obtain

$$< h(Y, \nabla_X Z), F_a W >=< J_a A_{F_a W} Y, \nabla_X T_a Z > - < J_a A_{F_a W} Y, A_{F_a Z} X > .$$

Thus, for $Y = J_a X$ and $W = Z$, from Lemma 5.2 we have

$$< h(J_a X, \nabla_X Z), F_a Z >= - \left\| A_{F_a Z} X \right\|^2 + < A_{F_a Z} X, \nabla_X T_a Z > . \tag{48}$$

Similarly we may also prove that

$$< h(X, \nabla_{J_a X} Z), F_a Z >= \left\| A_{F_a Z} X \right\|^2 + < J_a A_{F_a Z} X, \nabla_{J_a X} T_a Z > . \tag{49}$$

On the other hand, from (44) $< J_a A_{F_a Z} X, T_a Z >= 0$,

$$< J_a A_{F_a Z} X, \nabla_{J_a X} T_a Z > + < \nabla_{J_a X} J_a A_{F_a Z} X, T_a Z >= 0.$$

By using (2) and (3), we find

$$< [A_{F_a Z} X, X], T_a Z > + < \nabla_X A_{F_a Z} X, T_a Z >= 0.$$

Hence, we obtain

$$< J_a A_{F_a Z} X, \nabla_{J_a X} T_a Z >= - < A_{F_a Z} X, \nabla_X T_a Z > . \tag{50}$$

Substituting (50) equation into (49) we have

$$< h(X, \nabla_{J_a X} Z), F_a Z >= \left\| A_{F_a Z} X \right\|^2 - < A_{F_a Z} X, \nabla_X T_a Z > . \tag{51}$$

Combining (45), (46), (47), (48) and (51) we get

$$\begin{aligned} &\bar{R}(X, J_a X, Z, F_a Z) = -2 \|h(X, Z)\|^2 + 2 < J_a h(X, Z), h(T_a Z, X) > \\ &+2 \left\| A_{F_a Z} X \right\|^2 - 2 < A_{F_a Z} X, \nabla_X T_a Z > . \end{aligned} \tag{52}$$

Thus we have the following result.

Theorem 5.3. *Let M be a mixed foliate generic submanifold of a quaternionic space form $\bar{M}(c)$. If the leaves of $\tilde{D}$ are totally geodesic in M, then we have*

$$-\frac{c}{4} \sum_{a=1}^{3} \|F_a Z\|^2 = \left\| A_{F_a Z} X \right\|^2 .$$

Proof. If $\bar{M}$ is quaternionic space form , then, by (6), we find

$$\bar{R}(X, J_a X, Z, F_a Z) = -\frac{c}{2} \sum_{a=1}^{3} < X, X >< F_a Z, F_a Z > \tag{53}$$

for any unit vector $X \in \Gamma(D)$.

Since M is mixed geodesic, from equation (52) thus gives

$$\bar{R}(X, J_a X, Z, F_a Z) = 2 \left\| A_{F_a Z} X \right\|^2 . \tag{54}$$

Substituting (53) equation into (54) this completes proof. □

From Theorem 5.3 we have

Corollary 5.4. *Let M be a mixed foliate generic submanifold of a quaternionic space form with $c = 0$. If the leaves of $\tilde{D}$ are totally geodesic in M, then M is generic product.*

Corollary 5.5. *Let M be a mixed foliate generic submanifold of a quaternionic space form with $c \neq 0$. If the leaves of $\tilde{D}$ are totally geodesic in M, then there exist no holomorphic submanifold in $\bar{M}$.*

Let $QP^m(4)$ be the quaternion projective space of quaternion sectional curvature 4. If M is mixed foliate generic submanifold of $QP^m(4)$ such that the leaves of $\tilde{D}$ are totally geodesic in M, then (6) and (52) imply

$$\|h(X,Z)\|^2 + < A_{F_aZ}X, \nabla_X T_a Z >= 1 + \left\|A_{F_aZ}X\right\|^2 . \tag{55}$$

Since M is mixed foliate, we have

$$\|h(X,Z)\|^2 = 1 + \left\|A_{F_aZ}X\right\|^2 . \tag{56}$$

Theorem 5.6. *Let M be the mixed foliate generic submanifold in $QP^m(4)$. If the leaves of $\tilde{D}$ are totally geodesic in M, then for any unit vectors $X \in \Gamma(D)$ and $Z \in \Gamma(\tilde{D})$ we have*

$$K(X,Z) \leq 0.$$

The equality sign holds if and only if M is a generic product.

Proof. From the Gauss equation we get

$$K(X,Z) = 1+ < h(X,X), h(Z,Z) > - \|h(X,Z)\|^2 .$$

Thus by (56) we have

$$K(X,Z) =< h(X,X), h(Z,Z) > - \left\|A_{F_aZ}X\right\|^2 .$$

Since the distribution D is integrable, we obtain

$$K(X,Z) = -\left\|A_{F_aZ}X\right\|^2 \leq 0. \tag{57}$$

Therefore the equality of (57) holds if and only if the M is a Riemann product. □

Theorem 5.7. *Let M be a generic product in $QP^m(4)$. Then we have*
(i) $\|h(X,Z)\| = 1$, for any unit vectors $X \in \Gamma(D)$, $Z \in \Gamma(\tilde{D})$,
(ii) $m \geq h + p + hp$, where $h = dimD$, $p = dim\tilde{D}$.

Proof. From the equation of Gauss and (19), we have

$$\bar{R}(X, J_aX, Z, T_aZ) = R(X, J_aX, Z, T_aZ) - 2 < J_a h(X,Z), h(X, T_aZ) > . \tag{58}$$

Combining (52) and (58) we obtain

$$\bar{R}(X, J_aX, Z, T_aZ) = R(X, J_aX, Z, T_aZ) - 2\|h(X,Z)\|^2 + 2\left\|A_{F_aZ}X\right\|^2 - 2 < A_{F_aZ}X, \nabla_X T_a Z > . \tag{59}$$

By the Lemma 4.6, this gives

$$\bar{R}(X, J_aX, Z, T_aZ) = R(X, J_aX, Z, T_aZ) - 2\|h(X,Z)\|^2 . \tag{60}$$

Since M is the Riemann product of M^T and $M^\perp$, $R(X, J_aX, Z, T_aZ) = 0$. This gives

$$\|h(X,Z)\| =< h(X,Z), h(X,Z) >= 1. \tag{61}$$

Thus by linearity we obtain

$$< h(X_i, Z), h(X_j, Z) >= 0, i \neq j, \tag{62}$$

where $X_1, ..., X_{4h}$ and $Z_1, ..., Z_p$ are orthonormal bases for D and $\tilde{D}$ respectively. We see that for any $X, Y \in \Gamma(D)$, $Z, W \in \Gamma(\tilde{D})$,

$$< h(X, Z), h(Y, W) > + < h(X, W), h(Y, Z) >= 0. \tag{63}$$

On the other hand, by (6) and (8) we obtain

$$< h(X, Z), h(Y, W) > - < h(X, W), h(Y, Z) >= 0. \tag{64}$$

From (61), (62), (63) and (64), we see that

$$\left\{ h(X_i, Z_\beta) | i = 1, ..., 4h, \beta = 1, ..., p \right\}$$

are orthonormal vectors in ϑ. From Lemma 4.3, these vectors are perpendicular to $F_a\tilde{D}$, $a = 1, 2, 3$. Thus we conclude that the quatenion dimension of $QP^m(4)$ is greater than or equal to $h + p + hp$. □

6. Generalized Wintgen Inequality for Generic Submanifolds of Quaternionic Space Form

A generalized Wintgen inequality for quaternionic CR-submanifolds obtained in [2]. In this section, we obtain generalized Wintgen type inequality for generic submanifolds in a quaternionic space form.
Let M be a generic submanifold of real dimension n in quaternionic space form $\bar{M}(c)$ of quaternion dimension $m + n$. In the following, let $\{e_1, ..., e_n\}$ and $\{\xi_1, ..., \xi_m\}$ are orthonormal bases of the tangent and the normal bundle respectively.
The squared norm of T_a in (13) is

$$||T_a||^2 = \sum_{i,j=1}^{n} < T_a e_i, e_j >^2, \tag{65}$$

while the mean curvature vector field is given by

$$H = \frac{1}{n} \sum_{i=1}^{n} h(e_i, e_i). \tag{66}$$

Hence we get

$$||H||^2 =< H, H >= \frac{1}{n^2} \sum_{r=1}^{m} (\sum_{i=1}^{n} h_{ii}^r)^2. \tag{67}$$

We also set

$$h_{ij}^r =< h(e_i, e_j), \xi_r >, i, j = 1, ..., n, r = 1, ..., m \tag{68}$$

and

$$||h||^2 = \sum_{i,j=1}^{n} < h(e_i, e_j), h(e_i, e_j) > \tag{69}$$

[10]. We finally recall the following normalized scalar normal curvature from [10].

$$\rho_N = \frac{2}{n(n-1)} \sqrt{\sum_{1 \le i < j \le n} \sum_{1 \le r < s \le m} (\sum_{k=1}^{n} (h_{jk}^r h_{ik}^s - h_{ik}^r h_{jk}^s))^2}. \tag{70}$$

Let R be the curvature tensor of M, $R^{\perp}$ is the normal curvature tensor of the immersion, $\{e_1, ..., e_n\}$ and $\{\xi_1, ..., \xi_m\}$ are orthonormal bases of the tangent and the normal bundle respectively. If we denote by τ the scalar curvature, then the normalized scalar curvature ρ of M can be expressed as [2]

$$\rho = \frac{2\tau}{n(n-1)} = \frac{2}{n(n-1)} \sum_{1\leq i<j\leq n} R(e_i, e_j, e_j, e_i). \tag{71}$$

On the other hand, the normalized normal scalar curvature of M is given by [33]

$$\rho^{\perp} = \frac{2\tau^{\perp}}{n(n-1)} = \frac{2}{n(n-1)} \sqrt{\sum_{1\leq i<j\leq n} \sum_{1\leq r<s\leq m} < R^{\perp}(e_i, e_j)\xi_r, \xi_s >^2}. \tag{72}$$

The following theorem is the main theorem of this section.

Theorem 6.1. *Let M be a generic submanifold of real dimension n in a quaternionic space form $\bar{M}(c)$ of quaternion dimension $4m$. Then we have*

$$\rho \leq ||H||^2 - \rho_N + \frac{c}{4} + \frac{9qc}{n(n-1)} + \frac{9c}{4n(n-1)}||T_a||^2, \tag{73}$$

where

$$||T_a||^2 = \sum_{i,j=4q+1}^{n} < T_a e_j, e_i >^2$$

.

Proof. Let $\left\{e_1, ..., e_{4q}, e_{4q+1}, ..., e_n\right\}$ be orthonormal frame on TM such that $\left\{e_1, ..., e_{4q}\right\}$ is in D, $\left\{e_{4q+1}, ..., e_n\right\}$ is in $\tilde{D}$ and let $\{\xi_1, ..., \xi_{4m-n}\}$ be orthonormal frame on $TM^{\perp}$.
From (6), (8) and (13) we derive

$$\begin{aligned} R(X, Y, Z, W) = \frac{c}{4}\{&< Y, Z >< X, W > - < X, Z >< Y, W > \\ &+ \sum_{a=1}^{3} < Z, J_a Y >< J_a X, W > \\ &- < Z, J_a X >< J_a Y, W > +2 < X, J_a Y >< J_a Z, W >\} \\ &+ < h(X, W), h(Y, Z) > - < h(X, Z), h(Y, W) > . \end{aligned} \tag{74}$$

$\forall X, Y, Z, W \in \Gamma(TM)$.
Taking $X = W = e_i$, $Y = Z = e_j$ in (74) and summing over i and j from 1 to n, we get

$$\begin{aligned} R(e_i, e_j, e_j, e_i) \quad = \quad & \frac{c}{4}\{< e_j, e_j >< e_i, e_i > - < e_i, e_j >< e_i, e_j > \\ &+ \sum_{a=1}^{3} < e_j, J_a e_j >< J_a e_i, e_i > \\ &- < e_j, J_a e_i >< J_a e_j, e_i > +2 < e_i, J_a e_j >< J_a e_j, e_i >\} \\ &+ < h(e_i, e_i), h(e_j, e_j) > - < h(e_i, e_j), h(e_i, e_j) > \end{aligned} \tag{75}$$

$$\begin{aligned} \sum_{i,j=1}^{n} R(e_i, e_j, e_j, e_i) = 2\tau \quad = \quad & \frac{c}{4}[n(n-1) + 3\sum_{i,j=1}^{n} \sum_{a=1}^{3} < e_i, T_a e_j >^2] \\ &+ \sum_{i,j=1}^{n} < h(e_i, e_i), h(e_j, e_j) > - \sum_{i,j=1}^{n} < h(e_i, e_j), h(e_i, e_j) > . \end{aligned} \tag{76}$$

Taking now into account that $T_a e_j = J_a e_j \in D$ for $i = 1, ..., 4q$ and $T_a e_j \in \tilde{D}$ for $i = 4q+1, ..., n$ and using (67) and (69) in (76) we obtain

$$2\tau = \frac{c}{4}n(n-1) + 9qc + \frac{9}{4}c\|T_a\|^2 + n^2H^2 - h^2 \tag{77}$$

where

$$\|T_a\|^2 = \sum_{i,j=4q+1}^{n} < e_i, T_a e_j >^2 .$$

Thus (71) and (77) implies

$$\frac{2\tau}{n(n-1)} = \rho = \frac{c}{4} + \frac{9qc}{n(n-1)} + \frac{9c}{4n(n-1)}\|T_a\|^2 + \frac{2}{n(n-1)}\sum_{r=1}^{4m-n}\sum_{1\le i<j\le n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2]. \tag{78}$$

Further, from [2], we have

$$n^2\|H\|^2 - n^2\rho_N \ge \frac{2n}{n-1}\sum_{r=1}^{4m-n}\sum_{1\le i<j\le n}[h^r_{ii}h^r_{jj} - (h^r_{ij})^2] \tag{79}$$

.Combining Equations (78) and (79),we find

$$n^2[\rho - \frac{c}{4} - \frac{9qc}{n(n-1)} - \frac{9c}{4n(n-1)}\|T_a\|^2] \le n^2H^2 - n^2\rho_N.$$

□

For particular cases, we have the following results.

Corollary 6.2. *Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifold $\bar{M}$. Then*

$$\rho + \rho_N \le \frac{c}{4} + \frac{9qc}{n(n-1)} + \frac{9c}{4n(n-1)}\|T_a\|^2.$$

Corollary 6.3. *Let M^n be a quaternionic generic submanifold of the quaternionic Euclidean space H^m. Then*

$$\rho + \rho_N \le \|H\|^2.$$

Corollary 6.4. *Let M^n be a quaternionic generic submanifold of the quaternionic projective space HP^m of constant quaternionic sectional curvature 4. Then*

$$\rho + \rho_N - 1 - \frac{36q}{n(n-1)} - \frac{36}{n(n-1)}\|T_a\|^2 \le \|H\|^2.$$

Corollary 6.5. *Let M^n be a quaternionic generic submanifold of the quaternionic hyperbolic space HH^m of constant quaternionic sectional curvature -4. Then*

$$\rho + \rho_N + 1 + \frac{36q}{n(n-1)} + \frac{36}{n(n-1)}\|T_a\|^2 \le \|H\|^2.$$

Theorem 6.6. *Let M^n be a generic submanifold of quaternionic space form $\bar{M}^{4m}(c)$ with minimal codimension. Then*

$$(\rho^\perp)^2 \le 3[\|H\|^2 - \rho + \frac{c}{4} + \frac{9qc}{n(n-1)} + \frac{9c}{4n(n-1)}\|T_a\|^2]^2 + \frac{9q(q-1)c^2}{8n^2(n-1)^2} + \frac{27c^2}{n^2(n-1)^2}\|T_a\|^2\|F_a\|^2 \tag{80}$$

where

$$\|T_a\|^2 = \sum_{i,j=4q+1}^{n} < T_a e_j, e_i >^2, \|F_a\|^2 = \sum_{r,s=1}^{4m-n} < J_a\xi_r, \xi_s >^2$$

.

Proof. Let $\{e_1, ..., e_{4q}, e_{4q+1}, ..., e_n = e_{4q+p}\}$ be orthonormal bases on TM such that $\{e_1, ..., e_{4q}\}$ is in D, $\{e_{4q+1}, ..., e_{4q+p}\}$ is in $\tilde{D}$ and let $\{\xi_1, ..., \xi_{4m-n}\}$ of $TM^\perp$.
The Ricci equation implies

$$\begin{aligned} R^\perp(e_i, e_j, \xi_r, \xi_s) &= \frac{c}{4}\sum_{a=1}^{3}\{< \xi_r, J_a e_j >< J_a e_i, \xi_s > - < \xi_r, J_a e_i >< J_a e_j, \xi_s > \\ &+2 < e_i, J_a e_j >< J_a \xi_r, \xi_s >\} - < [A_{\xi_r}, A_{\xi_s}] e_i, e_j > \\ &= \frac{c}{4}\sum_{a=1}^{3}\{-(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) + 2 < e_i, J_a e_j >< J_a \xi_r, \xi_s >\} - < [A_{\xi_r}, A_{\xi_s}] e_i, e_j > \end{aligned} \tag{81}$$

for all $i, j \in \{1, ..., n\}, r, s \in \{1, ..., 4m - n\}$. Then from (72) we have

$$\begin{aligned} (\tau^\perp)^2 &= \sum_{1\le i<j\le n}\sum_{1\le r<s\le 4m-n} < R^\perp(e_i, e_j)\xi_r, \xi_s >^2 \\ &= \sum_{1\le i<j\le n}\sum_{1\le r<s\le 4m-n} [\frac{c}{4}\sum_{a=1}^{3}\{-(\delta_{ir}\delta_{js} - \delta_{is}\delta_{jr}) + 2 < e_i, J_a e_j >< J_a \xi_r, \xi_s >\} \\ &- < [A_{\xi_r}, A_{\xi_s}] e_i, e_j >]^2. \end{aligned} \tag{82}$$

Using the $(a + b + c)^2 \le 3(a^2 + b^2 + c^2)$ inequality in (83)

$$(\tau^\perp)^2 \le 3(\frac{n^2(n-1)^2}{4}\rho_N^2 + \frac{3p(p-1)c^2}{32} + \frac{9c^2}{4}||T_a||^2||F_a||^2) \tag{83}$$

$$(\rho^\perp)^2 \le 3\rho_N^2 + \frac{9p(p-1)c^2}{8n^2(n-1)^2} + \frac{27c^2}{n^2(n-1)^2}||T_a||^2||F_a||^2. \tag{84}$$

From (73), we have

$$(\rho^\perp)^2 \le 3[||H||^2 - \rho + \frac{c}{4} + \frac{9qc}{n(n-1)} + \frac{9c}{4n(n-1)}||T_a||^2]^2 + \frac{9p(p-1)c^2}{8n^2(n-1)^2} + \frac{27c^2}{n^2(n-1)^2}||T_a||^2||F_a||^2.$$

□

Corollary 6.7. *Let M be a totally umbilical generic submanifold of a quaternion Kaehler manifold $\bar{M}$. Then*

$$(\rho^\perp)^2 \le [-\rho + \frac{c}{4} + \frac{9qc}{n(n-1)} + \frac{9c}{4n(n-1)}||T_a||^2]^2 + \frac{3p(p-1)c^2}{8n^2(n-1)^2} + \frac{9c^2}{n^2(n-1)^2}||T_a||^2||F_a||^2.$$

References

[1] E.Abedi, G. Haghighatdoost, M. Ilmakchi and Z. Nazari, *Contact 3-structure QR-warped product submanifold in Sasakian space form*, Turkish J. Math. **37**(2) (2013), 340—347.
[2] H. Alodan, B.Y. Chen, S. Deshmukh, G.E. Vilcu, *A generalized Wintgen inequality for quaternionic CR-submanifolds*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. **114**(3), Art 129 (2020), 14 pp.
[3] M. Barros and F. Urbano, *Quaternion CR-submanifolds of quaternion manifolds*, Kodai Math. J. **4**(3) (1981), 399–417.
[4] A. Bejancu, *QR-submanifolds of quaternion Kaehlerian manifolds*, Chinese J. Math. **14**(2), (1986).
[5] A. Bejancu, *Geometry of CR-Submanifolds*, Kluwer, Dortrecht, (1986).
[6] A. Bejancu and H.R. Farran, *On totally umbilical QR-submanifolds of quaternion Kaehlerian manifolds*, Bull. Austral. Math. Soc. **62**(1) (2000), 95–103.
[7] B.Y. Chen,*Cohomology of CR-submanifolds*, Ann. Fac. Sci. Toulouse Math. **3**(2) (1981), 167–172.
[8] B.Y. Chen, *On CR-submanifolds of Kaehler manifold I*, J. Differential Geom. **16**(2) (1981), 305–322.
[9] B.Y. Chen, *On CR-submanifolds of Kaehler manifold III*, J. Differential Geom. **16**(2) (1981), 305–322.
[10] B.Y. Chen, *Recent Developments in Wintgen Inequality and Wintgen Ideal Submanifolds*, Int. Electron. J. Geom. **14**(1) (2021), 6–45.
[11] B.Y. Chen, *Differential geometry of real submanifolds in a Kaehler manifold*, Monatshefte für Mathematik 1981; 91:257-274.

[12] YW Choe, UH Ki, R. Takagi, *Compact minimal generic submanifolds with parallel normal section in a complex projective space*, Osaka Journal of Mathematics 2000; 37 (2): 489-499.
[13] PJ De Smet, F. Dillen, L. Verstraelen and L. Vrancken, *A pointwise inequality in submanifold theory*, Arch. Math. (1999), 35, 115–128.
[14] UC De, AK Sengupta, C. Calin, *Generic submanifolds of quasi-Sasakian manifolds*, Demonstratio Mathematica 2004; 37 (2): 429-437.
[15] UC De, AK Sengupta, *Generic submanifolds of a Lorentzian para-Sasakian manifold*, Soochow Journal of Mathematics 2001; 27 (1): 29-36.
[16] S. Dragomir, M.H. Shahid, F. Al-Solamy and S.M. Hasan, *Geometry of Cauchy-Riemann Submanifolds*, Springer, Singapore, 2016.
[17] I.V. Guadalupe, L. Rodriguez, *Normal curvature of surfaces in space forms*, Pac. J. Math., **14**(1) (1983), 106, 95–103.
[18] R. Güneş, B. Şahin and S. Keleş, *QR-submanifolds and almost contact 3-Structure*, Turkish J. Math **24**(3) (2000), 239–250.
[19] H.S. Kim and J.S. Pak, *Certain QR-submanifolds of maximal QR-dimension in a quaternionic space form*, Balkan J. Geom. Appl. **18**(1) (2013), 31—46.
[20] M. Kon, *On minimal generic submanifolds immersed in* S^{2m+1}, Colloquium Mathematicum 2001; 90 (2): 299-304.
[21] G. Macsim and A. Mihai, *A* δ*-invariant for QR-submanifolds in quaternion space forms*, Int. Electron. J. Geom. **11**(2) (2018), 8—17.
[22] G. Macsim, A. Mihai and I. Mihai, $\delta(2,2)$*-invariant for Lagrangian submanifolds in quaternionic space forms*, Mathematics **8**(4) (2020), 480.
[23] Z. Nazari and E. Abedi, *Ricci solitons on QR-hypersurfaces of a quaternionic space form Qn*, Acta Univ. Sapientiae Math. **8**(2) (2016), 271—281.
[24] B. Rouxel, *Sur une Famille des A-Surfaces d'un Espace Euclidien E 4*, Österreischer Mathematiker Kongress: Insbruck, Austria, (1981).
[25] B. Şahin, R. Güneş, *On some properties of distributions in QR-submanifolds*, Hacettepe Journal of Mathematics and Statistics, **28** (1999), 15–23.
[26] B. Şahin, *On QR-submanifolds of quaternionic space form*, Turkish J. Math **25**(3) (2001), 413–425.
[27] B. Şahin and R. Güneş, *QR-submanifolds of a locally conformal quaternion Kaehler manifold*, Publ. Math. Debrecen **63** (2003), 157–174.
[28] B. Şahin, F. Şahin, *Homology of contact CR-warped product submanifolds of an odd-dimensional unit sphere*, Bull. Korean Math. Soc. **52**(1) (2015), 215—222.
[29] F. Şahin, *On the topology of CR-warped product submanifolds*, Int. J. Geom. Methods Mod. Phys. **15**(02) (2018), 1850032, 7 pp.
[30] F. Şahin, *Homology of submanifolds of six dimensional sphere*, J. Geom. Phys. **145** (2019), 103471, 6 pp.
[31] GE Vilcu, *On generic submanifolds of manifolds endowed with metric mixed 3-structures*, Communications in Contemporary Mathematics 2016; 18 (6): 1550081.
[32] P. Wintgen, *Sur l'inégalité de Chen-Willmore*, C. R. Acad. Sci. Paris (1979), 288, 993—995.
[33] K. Yano and M. Kon,*Structures on Manifolds*, World Scientific (1985).
[34] A. Yılmaz, D. Poyraz and B. Şahin,*Generic QR-submanifolds of Quaternion Kaehler Manifolds*, Acta Math. Univ. Comanianae **XC**(3) (2021), 341–351.