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Novel solution bounds for the continuous and discrete algebraic Riccati equations in Hilbert space

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Abstract. In this study, we define new solution bounds for the infinite-dimensional algebraic Riccati equation. We first suggest an upper bound solution in the continuous case. After that, we develop upper and lower bounds for the algebraic Riccati equation solution in discrete time. We provide a robust control application.

1. Introduction

The algebraic Riccati and Lyapunov equations are frequently used in many branches of engineering, including control theory and signal processing. These equations are crucial in the analysis of control systems because they help to determine the best controllers, estimate transient behavior, and more. Over the last twenty years, there has been a lot of research done on the subject of determining bounds for the solution of these equations, see [7–9, 20–22].

Because the bounds are used to solve numerous control problems, including convergence of numerical algorithms [1], stability analysis [16], robust stabilization problem [17], estimation of the minimal cost and the suboptimal controller design [19], time-delay system controller design [27] and others, the problem of estimating upper and lower bounds of these equations has attracted considerable attention of the control community. Most papers in this area deal with systems in finite dimensions. In this study, we address systems in Hilbert space.

The infinite-dimensional Riccati equation is one of the most deeply studied equations arising in optimal control (see, e.g., [2, 5, 14, 15, 23–26]), $H\infty$ control and robust control (see, e.g., [11, 12, 28]). The computation of the positive definite solution of the Riccati equation is of some difficulty especially when the dimension is infinite. This paper aims to give new bounds for the operator algebraic Riccati equation's solution. We consider both cases, the continuous one and the discrete-time one. To our knowledge, this paper seems to be the first where bounds for such type of Riccati equation are proposed.

Numerous control systems are subject to parameter uncertainty-based perturbations. The stability radius is an important quantitative measure of the robustness of a system's stability to such perturbations. In [12] important results have been established for studying the robust stabilization of infinite dimensional

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systems subjected to stochastic perturbations. The results were given via a Riccati equation. We show how we can get a bound for the stability radius using the solution bounds of the Riccati equation.

The following is how the paper is arranged: We recall some well-known, fundamental results in Section 2. The continuous algebraic equation is the focus of Section 3. We define an upper bound for the CARE problem's solution and provide an example for the robust stabilization problem. On the discrete-time Riccati equation, see Section 4. We define lower and upper bounds for the solutions. The stability of the operator A leads to the derivation of two upper bounds.

Notations

Let H and U be two real separable Hilbert spaces. Denote by $L(U,H)$ the space of linear operators from U to H and $B(U,H)$ the space of bounded operators from U to H, we write $L(H)$ for $L(H,H)$ and $B(H)$ for $B(H,H)$. The symbols $\|.\|$, $\langle , \rangle$ denote respectively the norm and inner product in H. Let P and Q be two self adjoint operators.

1. By $P > 0$ we mean that P is positive that is $\langle Pz, z\rangle > 0$, for all $z \in H$, $z \neq 0$;
2. By $P > Q$ we mean that $(P - Q)$ is positive;
3. By $P \geq 0$ we mean that P is non-negative that is $\langle Pz, z\rangle \geq 0$, for all $z \in H$;
4. By $P \geq Q$ we mean that $(P - Q)$ is non-negative;
5. By $P >> 0$ we mean that there exists a constant $c > 0$ such that:

$$\langle Pz, z\rangle \geq c\|z\|^2, \quad \text{for all } \; z \in H \text{ with } \; z \neq 0.$$

Define the set $B^+(H)$ by

$$B^+(H) := \{P \in B(H), P \;\text{ is self-adjoint and }\; P \geq 0\}$$

and the set $G^+(H)$ by

$$G^+(H) := \{P \in B(H), P \text{ is self-adjoint and } P >> 0\}.$$

For $A \in L(H)$, we denote by $\lambda_{\min}(A)$ (resp. $\lambda_{\max}(A)$) for the minimal (resp. the maximal) eigenvalue of A. For $A \in B(H)$, we denote by $s_{\min}(A)$ (resp. $s_{\max}(A)$) for the minimal (resp. the maximal) singular value of A. We denote by (CARE) for the following continuous algebraic Riccati equation:

$$A^*P + PA - PBR^{-1}B^*P + Q = 0. \tag{1}$$

Finally, denoting by (DARE) for the following discrete algebraic Riccati equation:

$$P = A^*PA + A^*PB(I - B^*PB)^{-1}B^*PA + Q. \tag{2}$$

2. Preliminaries

Before developing the main results, we shall recall the following Lemmas. First, we collect some important results from the spectral theory of self adjoint operators.

Lemma 2.1. *[6] If A is a self-adjoint operator on H, then $\sigma(A) \subset \mathbb{R}$. Furthermore, if $A \in B(H)$, then we have the following additional properties:*

1. *$\sigma(A) \subset [m, M]$, where $m := \inf_{\|z\|=1} \langle Az, z\rangle$ and $M := \sup_{\|z\|=1} \langle Az, z\rangle$;*
2. *$m, M \in \sigma(A)$;*

3. $\|A\| = \max\{|m|, |M|\}$;

4. $\rho(A) = \|A\|$.

Lemma 2.2. *[4] For any positive definite self-adjoint operator X, the following inequality holds:*

$$\lambda_{\min}(X)I \leq X \leq \lambda_{\max}(X)I.$$

Lemma 2.3. *[4] Let X and Y be two positive definite self-adjoint operators satisfying $X > Y$. Then $X^{-1} < Y^{-1}$.*

Lemma 2.4. *[4] Let A and B be self-adjoint operators. Then*

$$\lambda_i(AB) = \lambda_i(BA), \quad \text{for } i = 1, 2, \cdots$$

From [4, Th. 8, p. 254], we may obtain the following result.

Lemma 2.5. *[4] Let A and B be two positive semi-definite operators, then*

$$\sum_{i=1}^{r} \lambda_i(AB) \leq \sum_{i=1}^{r} \lambda_i(A)\lambda_i(B), \qquad r = 1, 2, \cdots.$$

Also, from [4, inequality (7), p. 247], we have the following Lemma.

Lemma 2.6. *[4] Let A and B be two positive semi-definite self-adjoint operators, then*

$$\sum_{j=1}^{r} \lambda_j(A+B) \leq \sum_{j=1}^{r} \lambda_j(A) + \sum_{j=1}^{r} \lambda_j(B), \quad r = 1, \cdots.$$

Eventually, for further use, we compile some pertinent findings on the infinitesimal generator of the C_0-semigroups.

Definition 2.7. *A C_0-semigroup $(T(t))_{t\geq 0}$, on a Hilbert space H is exponentially stable if there exist positive constants M and ω such that*

$$\|T(t)\| \leq Me^{-\omega t}, \quad t \geq 0.$$

Lemma 2.8. *[6] Assume that A is the infinitesimal generator of the C_0-semigroup $T(t)$ on the Hilbert space H. Then $T(t)$ is exponentially stable if and only if there exists a positive operator $P \in L(H)$ such that:*

$$\langle Az, Pz\rangle + \langle Pz, Az\rangle = -\langle z, z\rangle \text{ for all } z \in D(A).$$

Proposition 2.9. *[18] Let A be a closed, densely defined linear operator on the real separable Hilbert space H. Then there exists $\alpha \in \mathbb{R}$ such that*

$$\langle x, Ax\rangle \leq \alpha\|x\|^2 \text{ for all } x \in D(A),$$

and

$$\langle x, A^*x\rangle \leq \alpha\|x\|^2 \text{ for all } x \in D(A^*),$$

if and only if A generates a strongly continuous semigroup $T(t)$ such that $\|T(t)\| \leq e^{\alpha t}, t \geq 0$, for some number $\alpha \in \mathbb{R}$. In particular, if $\alpha < 0$, i.e., both A and its adjoint A^* are strictly dissipative, the semigroup $T(t), t \geq 0$, is then exponentially stable.

Proposition 2.10. *Define the lower and upper stability indices for the generator A of $T(t), t \geq 0$:*

$$\gamma(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\},$$

and

$$\Gamma(A) := \inf\left\{\mu, \|T(t)\| \leq Me^{\mu t} \text{ for some } M \geq 1 \text{ and for all } t \geq 0\right\}.$$

Then

$$\gamma(A) \leq \Gamma(A), \tag{3}$$

and therefore, if $(T(t))_{t\geqslant 0}$ is exponentially stable, $\gamma(A) < 0$. Moreover, if $T(t_0)$ is a compact operator for some $t_0 > 0$, then $\gamma(A) = \Gamma(A)$ and consequently (3) implies the exponential stability.

This section concludes with the following significant findings.

Lemma 2.11. *Let $X_i \in L(H)$, $i \geq 1$, then the series $\sum_{i=1}^{\infty} X_i$ converges if there is an operator norm $\|.\|$ on $L(H)$ such that the numerical series $\sum_{i=1}^{\infty} \|X_i\|$ converges.*

Lemma 2.12. *[3]. Let (X, d) be a complete metric space and f be a strict contraction, i.e.,a map satisfying*

$$d(f(x), f(y)) \leq ad(x, y), \quad \text{for all } \ x, y \in X, \ \text{ and for some constant } \ 0 \leq a < 1.$$

Then f has a unique fixed point in X.

Lemma 2.13. *Let $(B, \|.\|)$ be a real Banach space, and $\Omega \subset B$ be a convex closed and bounded subset and $f : \Omega \to \Omega$ be a contraction map, i.e., a map satisfying*

$$\|f(P_1) - f(P_2)\| \leq p\|P_1 - P_2\|, \quad \text{for all } \ P_1, P_2 \in \Omega, \ \text{ where } \ 0 \leq p < 1.$$

Then f has a unique fixed point in Ω.

3. Upper solution bounds for the CARE

This subsection focuses on the continuous algebraic equation of the form

$$A^*P + PA - PBR^{-1}B^*P + Q = 0,$$

where A is the infinitesimal generator of C_0-semigroup $(T(t))_{t\geq 0}$ on the real separable Hilbert space H. Assume that $B \in B(U, H)$, $Q \in B^+(H)$ and $R \in G^+(U)$. Moreover, assume that (A, B) is stabilizable. We will provide an extension of some results of [20] in the Hilbert space H.

Theorem 3.1. *Let $P \in B(H)$ be the positive semi definite solution of CARE (1). Assume that $BR^{-1}B^* >> 0$. Choose $\alpha > 0$ such that $\alpha > \Gamma(A)$. Thus,*

$$P \leq \frac{Q + \alpha^2(BR^{-1}B^*)^{-1}}{-2\Gamma(\widehat{A})} = P_{u_1}, \quad \text{where } \ \widehat{A} = A - \alpha I. \tag{4}$$

Proof. We have

$$\langle BR^{-1}B^*(P - \alpha(BR^{-1}B^*)^{-1})x, (P - \alpha(BR^{-1}B^*)^{-1})x\rangle = \langle BR^{-1}B^*y, y\rangle, \ \text{ with } \ y = (P - \alpha(BR^{-1}B^*)^{-1})x.$$

Letting

$$S := (P - \alpha(BR^{-1}B^*)^{-1})^* BR^{-1}B^*(P - \alpha(BR^{-1}B^*)^{-1}),$$
$$\widehat{Q} := Q + \alpha^2(BR^{-1}B^*)^{-1}.$$

Note that $BR^{-1}B^* >> 0$, so $S \geq 0$.
Since

$$S = PBR^{-1}B^*P - 2\alpha P + \alpha^2(BR^{-1}B^*)^{-1},$$

we get

$$PBR^{-1}B^*P = S + 2\alpha P - \alpha^2(BR^{-1}B^*)^{-1}.$$

Also,

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle - \langle PBR^{-1}B^*Px, x\rangle + \langle Qx, x\rangle = 0, \ \ x \in D(A).$$

Hence

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle - \langle Sx, x\rangle - 2\alpha\langle Px, x\rangle + \alpha^2\langle (BR^{-1}B^*)^{-1}x, x\rangle + \langle Qx, x\rangle = 0, \ \ x \in D(A)$$

and hence

$$\langle (A-\alpha I)x, Px\rangle + \langle Px, (A-\alpha I)x\rangle - \langle Sx, x\rangle + \alpha^2\langle (BR^{-1}B^*)^{-1}x, x\rangle + \langle Qx, x\rangle = 0, \ \ x \in D(A).$$

Or,

$$\langle (A-\alpha I)x, Px\rangle + \langle Px, (A-\alpha I)x\rangle = \langle Sx, x\rangle - \alpha^2\langle (BR^{-1}B^*)^{-1}x, x\rangle - \langle Qx, x\rangle, \ \ x \in D(A).$$

So, for $x \in D(A)$, we have

$$\begin{aligned}
&\langle (P_{u_1} - P)x, (A-\alpha I)x\rangle + \langle (P_{u_1} - P)(A-\alpha I)x, x\rangle \\
&= -\langle Sx, x\rangle + \alpha^2\langle (BR^{-1}B^*)^{-1}x, x\rangle + \langle Qx, x\rangle \\
&+\langle P_{u_1}x, (A-\alpha I)x\rangle + \langle P_{u_1}(A-\alpha I)x, x\rangle. \\
&= -\langle Sx, x\rangle + \langle \widehat{Q}x, x\rangle + \langle P_{u_1}x, \widehat{A}x\rangle + \langle P_{u_1}\widehat{A}x, x\rangle \\
&= -\langle Sx, x\rangle + \langle \widehat{Q}x, x\rangle + \langle \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}x, \widehat{A}x\rangle + \langle \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}\widehat{A}x, x\rangle.
\end{aligned}$$

Also,

$$\begin{aligned}
&\langle \widehat{Q}x, x\rangle + \langle \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}x, \widehat{A}x\rangle + \langle \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}\widehat{A}x, x\rangle \\
&= \frac{1}{-2\Gamma(\widehat{A})}\left(\langle \widehat{Q}x, \widehat{A}x\rangle + \langle \widehat{A}x, \widehat{Q}x\rangle - 2\langle \widehat{Q}x, \Gamma(\widehat{A})x\rangle\right) \\
&= \frac{1}{-2\Gamma(\widehat{A})}\left(\langle \widehat{Q}x, (\widehat{A} - \Gamma(\widehat{A})I)x\rangle + \langle (\widehat{A} - \Gamma(\widehat{A})I)x, \widehat{Q}x\rangle\right)
\end{aligned}$$

Since $\alpha > \Gamma(A)$, it follows that $\Gamma(\widehat{A}) < 0$, from which we deduce that $\widehat{A}$ generates an exponentially stable semigroup $S(t) = e^{-\alpha t}T(t), t \geq 0$. Using the fact that $\widehat{Q} > 0$, we obtain

$$\langle \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}x, \widehat{A}x\rangle + \langle \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}\widehat{A}x, x\rangle + \langle \widehat{Q}x, x\rangle \leq 0, \tag{5}$$

and so,

$$\langle (P_{u_1} - P)x, (A-\alpha I)x\rangle + \langle (A-\alpha I)x, (P_{u_1} - P)x\rangle \leq 0, \ \ x \in D(A).$$

It follows from [6] that $(P_{u_1} - P) \geq 0$ and therefore $P \leq P_{u_1}$. □

Remark 3.2. *If $A \in B(H)$, we choose $\alpha > 0$ such that*

$$A + A^* < 2\alpha I.$$

In fact,

$$\begin{aligned}
A + A^* < 2\alpha I &\iff \langle (A+A^*)x, x\rangle < 2\alpha\langle x, x\rangle, \ \ \textit{for all} \ \ x \in H \\
&\iff \frac{\langle (A+A^*)x, x\rangle}{2\|x\|^2} < \alpha, \ \ \textit{for all} \ \ x \neq 0 \\
&\iff \frac{\langle \frac{A+A^*}{2}x, x\rangle}{\|x\|^2} < \alpha, \ \ \textit{for all} \ \ x \neq 0 \\
&\iff \sup_{x\neq 0} \frac{\langle \frac{A+A^*}{2}x, x\rangle}{\|x\|^2} < \alpha.
\end{aligned}$$

So, it is sufficient to choose $\alpha > \sup_{x\neq 0} \dfrac{\langle \frac{A+A^*}{2}x, x\rangle}{||x||^2}$.

Theorem 3.3. *Let $P \in L(H)$ be the positive semi-definite solution of CARE (1). Assume that $BR^{-1}B^* >> 0$ and choose $\alpha > 0$ such that $\alpha > \Gamma(A)$. Let β be a positive constant. Then:*

$$P \leq ((\widehat{A} - \beta I)^*)^{-1}(\widehat{A} + \beta I)^* P_{u_1}(\widehat{A} + \beta I)(\widehat{A} - \beta I)^{-1} + 2\beta(\widehat{A} - \beta I)^*)^{-1}(Q + \alpha^2(BR^{-1}B^*)^{-1})(\widehat{A} - \beta I)^{-1} \equiv P_{u_2}. \quad (6)$$

Proof. Let $\beta > 0$, for $x \in D(A)$, we have

$$\langle P(\widehat{A} - \beta I)x, (\widehat{A} - \beta I)x\rangle = \langle P\widehat{A}x, \widehat{A}x\rangle - \beta\langle Px, \widehat{A}x\rangle - \beta\langle P\widehat{A}x, x\rangle + \beta^2\langle Px, x\rangle.$$

Also,

$$\langle P(\widehat{A} + \beta I)x, (\widehat{A} + \beta I)x\rangle = \langle P\widehat{A}x, \widehat{A}x\rangle + \beta\langle Px, \widehat{A}x\rangle + \beta\langle P\widehat{A}x, x\rangle + \beta^2\langle Px, x\rangle.$$

Then

$$\begin{aligned}
&\langle P(\widehat{A} - \beta I)x, (\widehat{A} - \beta I)x\rangle - (\langle P(\widehat{A} + \beta I)x, (\widehat{A} + \beta I)x\rangle \\
&= -2\beta(\langle Px, \widehat{A}x\rangle + \langle P\widehat{A}x, x\rangle) \\
&= -2\beta(\langle Px, Ax\rangle + \langle PAx, x\rangle - 2\alpha\langle Px, x\rangle) \\
&= -2\beta\langle (PBR^{-1}B^*P - Q - 2\alpha P)x, x\rangle \\
&= -2\beta\langle [(P - \alpha(BR^{-1}B^*)^{-1})^*(BR^{-1}B^*)(P - \alpha(BR^{-1}B^*)^{-1})]x, x\rangle \\
&+ 2\beta\langle (Q + \alpha^2(BR^{-1}B^*)^{-1})x, x\rangle.
\end{aligned}$$

Since $BR^{-1}B^*$ is positive, we get

$$(P - \alpha(BR^{-1}B^*)^{-1})^*(BR^{-1}B^*)(P - \alpha(BR^{-1}B^*)^{-1} \geq 0.$$

So,

$$\langle P(\widehat{A} - \beta I)x, (\widehat{A} - \beta I)x\rangle - (\langle P(\widehat{A} + \beta I)x, (\widehat{A} + \beta I)x\rangle) \leq 2\beta\langle (Q + \alpha^2(BR^{-1}B^*)^{-1})x, x\rangle.$$

Since $\beta > 0 > \Gamma(A) - \alpha$, Lemma 2.1.11 of [6, p. 24] shows that $(\beta I - \widehat{A})^{-1}$ exist. Therefore

$$P \leq ((\widehat{A} - \beta I)^*)^{-1}(\widehat{A} + \beta I)^* P(\widehat{A} + \beta I)(\widehat{A} - \beta I)^{-1} + 2\beta((\widehat{A} - \beta I)^*)^{-1}(Q + \alpha^2(BR^{-1}B^*)^{-1})(\widehat{A} - \beta I)^{-1}.$$

According to Theorem 3.1, we get

$$P \leq ((\widehat{A} - \beta I)^*)^{-1}(\widehat{A} + \beta I)^* P_{u_1}(\widehat{A} + \beta I)(\widehat{A} - \beta I)^{-1} + 2\beta(\widehat{A} - \beta I)^*)^{-1}(Q + \alpha^2(BR^{-1}B^*)^{-1})(\widehat{A} - \beta I)^{-1}.$$

This completes the proof. □

Corollary 3.4. *The following comparison holds*

$$P_{u_2} \leq P_{u_1}.$$

Proof. We have

$$\begin{aligned}
P_{u_2} - P_{u_1} &= ((\widehat{A} - \beta I)^*)^{-1}(\widehat{A} + \beta I)^* \frac{Q + \alpha^2(BR^{-1}B^*)^{-1}}{-2\Gamma(\widehat{A})}(\widehat{A} + \beta I)(\widehat{A} - \beta I)^{-1} \\
&+ ((\widehat{A} - \beta I)^*)^{-1} 2\beta(Q + \alpha^2(BR^{-1}B^*)^{-1}(\widehat{A} - \beta I)^{-1} - \frac{Q + \alpha^2(BR^{-1}B^*)^{-1}}{-2\Gamma(\widehat{A})} \\
&= ((\widehat{A} - \beta I)^*)^{-1}\left[(\widehat{A} + \beta I)^* \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}(\widehat{A} + \beta I) + 2\beta\widehat{Q} - (\widehat{A} - \beta I)^* \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}(\widehat{A} - \beta I)\right](\widehat{A} - \beta I)^{-1} \\
&= 2\beta((\widehat{A} - \beta I)^*)^{-1}\left[(\widehat{A})^* \frac{\widehat{Q}}{-2\Gamma(\widehat{A})} + \frac{\widehat{Q}}{-2\Gamma(\widehat{A})}\widehat{A} + \widehat{Q}\right](\widehat{A} - \beta I)^{-1}.
\end{aligned}$$

According to the equation (5), we get the desired result. □

Now, we will apply our results on the following example considered in [6, Ex. 9.2.14, p. 421].

Example 3.5. *Consider the system*

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) = \frac{\partial^2 y}{\partial x^2}(x,t) + u(t), \ \ t > 0, \ 0 < x < 1, \\ \frac{\partial y}{\partial x}(0,t) = \frac{\partial y}{\partial x}(1,t) = 0, \ \ t > 0, \\ y(x,0) = y_0(x), \ \ 0 < x < 1. \end{cases} \tag{7}$$

We define the operator A in $H = L^2(0,1)$ by setting $Ah = \frac{d^2h}{dx^2}$ with domain

$$D(A) = \{h \in H; h, \frac{dh}{dx} \text{are absolutely continuous}, \ \frac{d^2h}{dx^2} \in H \ \text{and} \ \frac{dh}{dx}(0) = \frac{dh}{dx}(1) = 0\}.$$

A is the infinitesimal generator of a C_0-semigroup $(T(t))_{t\geq 0}$. The eigenvalues and the eigenvectors of A are given by ([6, Example. 3.2.15, p. 98])

$$\lambda_n = -n^2\pi^2; \psi_n(x) = \sqrt{2}cos(n\pi x), \ \ n \geq 0,$$

and we have

$$\|T(t)\| \leq 1, \ \ t \geq 0,$$

We seek to minimize

$$J(u) = \int_0^\infty (\int_0^1 \|y(t,x)\|^2 dx + \|u(t)\|^2) dt.$$

In the abstract from system (7) can be presented as follows

$$\begin{cases} dz(t) = Az(t)dt + Bu(t), \\ z(0) = z_0, \end{cases} \tag{8}$$

with $B = I_H$. (A, B) is stabilizable. The algebraic Riccati equation has the form

$$2\langle Pz, Az\rangle - \langle z, PBB^*Pz\rangle + \langle z, z\rangle = 0.$$

By Theorem 2.1, we have

$$P_{u_1} = \frac{Q + \alpha^2(BR^{-1}B^*)^{-1}}{-2\Gamma(\widehat{A})},$$

with $\widehat{A} = A - \alpha I$. Let $\alpha > \Gamma(A) = 0$. Thus

$$P_{u_1} = \frac{I + \alpha^2 I}{-2\Gamma(\widehat{A})} = \frac{(1+\alpha^2)I}{-2(\Gamma(A) - \alpha)} = \frac{(1+\alpha^2)I}{-2(0-\alpha)}.$$

For $\alpha = \Gamma(A) + 1$, we obtain

$$P_{u_1} = \frac{(1+\alpha^2)I}{2\alpha} = I.$$

Example 3.6. *Consider the system*

$$\begin{cases} \frac{\partial y}{\partial t}(x,t) = \frac{\partial^2 y}{\partial x^2}(t,x) + u(t), t > 0, 0 < x < 1, \\ y(t,0) = y(t,1) = 0, \\ y(0,x) = y_0(x), \end{cases} \tag{9}$$

We define the operator A in $H = L^2(0,1)$ as follows: $Ah := \frac{d^2h}{dx^2}$ with the domain

$$D(A) := \{h \in H; h, \frac{dh}{dx} \text{are absolutely continuous}, \ \frac{d^2h}{dx^2} \in H \ \text{and} \ h(0) = h(1) = 0\}.$$

A is self-adjoint and $\langle Az, z\rangle \leq -\pi^2\|z\|^2$, $z \in D(A)$ (see [10], page 151). A generates a compact semigroup $(S(t))_{t\geq 0}$. The eigenvalues and the eigenvectors of A are given by (see [6])

$$\lambda_n = -n^2\pi^2; \psi_n(x) = \sqrt{2}\sin(n\pi x), \ n \geq 1.$$

$(S(t))$ is exponentially stable with $\|S(t)\| \leq e^{-\pi^2 t}, t \geq 0$.
Consider the Riccati equation

$$2\langle Pz, Az\rangle - \varepsilon^{-2}\langle Pz, Pz\rangle + \sum_{i=1}^{N} a_i^2 \langle z, z\rangle = 0.$$

Thus $R = \varepsilon^{-1}I$, with $\varepsilon > 0$ $B = I$. Set $\beta = \sum_{i=1}^{N} a_i^2$.
According to Theorem 2.1, we have

$$P_{u_1} = \frac{Q + \alpha^2(BR^{-1}B^*)^{-1}}{-2\Gamma(\widehat{A})} = \frac{(\beta + \alpha^2\varepsilon^{-1})I}{-2\Gamma(\widehat{A})},$$

with $\widehat{A} = A - \alpha I$. Let $\alpha > \Gamma(A) = -\pi^2$. Thus

$$P_{u_1} = \frac{(\beta + \alpha^2\varepsilon^{-1})I}{-2\Gamma(\widehat{A})} = \frac{(\beta + \alpha^2\varepsilon^{-1})I}{-2(\Gamma(A) - \alpha)} = \frac{(\beta + \alpha^2\varepsilon^{-1})I}{2(\pi^2 + \alpha)}$$

For $\alpha = \Gamma(A) + \varepsilon$, we obtain

$$P_{u_1} = \frac{(\beta + (-\pi^2 + \varepsilon)^2\varepsilon^{-1})I}{2(\pi^2 + \alpha)}.$$

Therefore

$$P_{u_1} = \frac{(\beta + (-\pi^2 + \varepsilon)^2\varepsilon^{-1})I}{2\varepsilon}.$$

Thus

$$P_{u_1} = \frac{(\beta + (-\pi^2 + \varepsilon)^2\varepsilon^{-1})I}{2\varepsilon}.$$

3.1. Application to robust stabilization

In this subsection, we will employ the previous section's conclusions to derive a lower bound for the stability radius. To begin, we recall the following Theorem.

Theorem 3.7. *[12] Suppose that there exist $a = (a_i)_{i\in\overline{N}} \in (0, +\infty)^N$, $P \in L^+(H)$ satisfying*

$$2\langle Px, Ax\rangle + \langle E(a)x, E(a)x\rangle - \varepsilon^{-2}\langle x, PBB^*Px\rangle = 0, \ \ x \in \mathcal{D}(A) \tag{10}$$

and

$$I - (\frac{\sigma}{a_j})^2\theta_j D_j^* P D_j \geq 0, \ \ j \in \overline{N}, \tag{11}$$

where

$$\langle E(a)x, E(a)x\rangle = \sum_{i=1}^{N} a_i^2 \langle E_i x, E_i x\rangle.$$

Then

$$r^w(A; (D_i, E_i))_{i\in\overline{N}}) \geq \sigma.$$

Proposition 3.8. *Suppose that there exist $\varepsilon > 0, a = (a_i)_{i\in\overline{N}} \in (0, +\infty)^N$, $P \in L^+(H)$ satisfying*

$$2\langle Px, Ax\rangle + \langle E(a)x, E(a)x\rangle - \varepsilon^{-2}\langle x, PBB^*Px\rangle = 0, \quad x \in D(A). \tag{12}$$

Let σ such that

$$(\frac{\sigma}{a_j})^{-2}(\theta_j)^{-1} \geq \|P_u\| \cdot \left\|D_j\right\|^2, \quad j = 1, \cdots, N.$$

Then $r^w(A; (D_i, E_i))_{i\in\overline{N}}) \geq \sigma$.

Proof. Let $P \in L^+(H)$ be a solution of equation (12). Choose σ such that

$$(\frac{\sigma}{a_j})^{-2}(\theta_j)^{-1} \geq \|P_u\| \cdot \left\|D_j\right\|^2, \quad j = 1, \cdots, N.$$

We have

$$\left\langle D_j^*PD_jx, x\right\rangle = \left\langle PD_jx, D_jx\right\rangle \leq \left\langle P_uD_jx, D_jx\right\rangle, \quad \text{for all } \; x \neq 0,$$

then

$$\sup_{x\neq 0} \frac{\left\langle D_j^*PD_jx, x\right\rangle}{\|x\|^2} \leq \|P_u\| \cdot \left\|D_j\right\|^2.$$

Thus,

$$(\frac{\sigma}{a_j})^{-2}(\theta_j)^{-1} \geq \frac{\left\langle D_j^*PD_jx, x\right\rangle}{\|x\|^2}, \quad \text{for all } \; x \neq 0.$$

We conclude that

$$\|x\|^2 \geq (\frac{\sigma}{a_j})^2\theta_j \left\langle D_j^*PD_jx, x\right\rangle, \quad \text{for all } \; x \neq 0,$$

and finally that

$$I - (\frac{\sigma}{a_j})^2\theta_jD_j^*PD_j \geq 0.$$

Consequently, Theorem 3.7 proves the desired result. □

4. Solution bounds for the DARE

This section aims to establish lower and upper solution bounds for the discrete-time Riccati equation; we give infinite-dimension counterparts of the results of [29]. Before proceeding further, we recall a result concerning the stability of discrete-time systems.

Consider the infinite-dimensional linear discrete-time invariant system described by the following autonomous homogeneous difference equation in H:

$$\begin{cases} x(t+1) = Ax(t) \;\; \text{for all } \; t \in \mathbb{N}, \; \text{with } \; t \geq t_0 \; \text{ and } \; A \in B(H), \\ x(0) = x_0 \in H. \end{cases} \tag{13}$$

The system (13) is said to be Schur stable if $\rho(A) < 1$, where $\rho(A)$ is the spectral radius of A. Using [13], we can deduce the following conditions for the stability of (13).

Theorem 4.1. *The following assertions are equivalent:*

1. $\rho(A) < 1$;

2. *For every positive operator $Q \in G^+(H)$ there exists a unique positive operator $P \in G^+(H)$ solution of the Lyapunov equation*

$$A^*PA - P + Q = 0, \tag{14}$$

and it is provided by

$$P = \sum_{n=0}^{+\infty} (A^*)^n QA^n;$$

3. *There exist $\beta > 0$ and $a \in (0,1)$ such that*

$$\|A^n\|^2 \leq \beta a^n, \quad n \geq 0.$$

We state that A is stable when the system (13) is Schur stable.

Consider the following Riccati equation

$$P = A^*PA - A^*PB(I + B^*PB)^{-1}B^*PA + Q, \tag{15}$$

where $A \in B(H)$, $B \in B(U, H)$, $Q \in G^+(H)$. Using the fact that

$$(I + YZ)^{-1} = I - Y(I + ZY)^{-1}Z \tag{16}$$

with Y, Z are self adjoint operators of $B(H)$, the DARE (15) can be written as

$$P = A^*(I + PBB^*)^{-1}PA + Q, \tag{17}$$

$$P = A^*(P^{-1} + BB^*)^{-1}A + Q. \tag{18}$$

4.1. Lower solution bounds for the DARE

First, we recall the following useful Lemma.

Lemma 4.2. *Let A, R, S, T be operators on H such that $R \geq 0, S \geq T \geq 0$, then*

$$A^*(I + SR)^{-1}SA \geq A^*(I + TR)^{-1}TA$$

provided that $(I + SR)$ and $(I + TR)$ are invertible operators.

We can establish the following lower bound for the (DARE) solution using this Lemma. To this end, assume that $Q \in B^+(H)$.

Theorem 4.3. *Let $P \in B^+(H)$ be the solution of the DARE (15). Then,*

$$P \geq A^*(I + QBB^*)^{-1}QA + Q = P_{l1}.$$

Proof. In fact,

$$P = A^*(I + PBB^*)^{-1}PA + Q \geq Q. \tag{19}$$

So,

$$P \geq A^*(I + QBB^*)^{-1}QA + Q.$$

□

Now, if $Q > 0$, we get the following result:

Theorem 4.4. *Let $P \in G^+(H)$ be the positive definite solution of the DARE (15), then P has the lower operator bound*

$$P \geq A^* \left\{ \left[A^*(Q^{-1} + BB^*)^{-1}A + Q \right]^{-1} + BB^* \right\}^{-1} A + Q \equiv P_l. \tag{20}$$

Proof. Using (18) we get

$$P = A^*(P^{-1} + BB^*)^{-1}A + Q = A^* \left\{ \left[A^*(P^{-1} + BB^*)^{-1}A + Q \right]^{-1} + BB^* \right\}^{-1} A + Q. \tag{21}$$

So,

$$P > 0 \Rightarrow P^{-1} > 0,$$

hence

$$(P^{-1} + BB^*)^{-1} > 0,$$

and hence

$$P = A^*(P^{-1} + BB^*)^{-1}A + Q \geq Q.$$

Thus, $P^{-1} \leq Q^{-1}$. Therefore

$$P \geq A^* \left\{ \left[A^*(Q^{-1} + BB^*)^{-1}A + Q \right]^{-1} + BB^* \right\}^{-1} A + Q. \tag{22}$$

□

4.2. Upper solution bounds for the DARE

We investigate two distinct situations. First, under the context when A is stable, we establish a solution bound for the DARE. For this purpose, assume that $\rho(A) < 1$. By Theorem 4.1, A is stable and so that (A, B) is stabilizable. Moreover, (A, Q) is detectable. Thus, there exists a unique solution of the Riccati equation (DARE).

Theorem 4.5. *Let $P \in B^+(H)$ be the solution of the DARE (15). So, P has the upper operator bound*

$$P \leq \sum_{m=0}^{\infty} (A^*)^m QA^m \equiv P_{u1}. \tag{23}$$

Proof. From (18) we obtain

$$P = A^*(P^{-1} + BB^*)^{-1}A + Q.$$

Since

$$(P^{-1} + BB^*) \geq P^{-1},$$

we have

$$\begin{aligned} P &= A^*(P^{-1} + BB^*)^{-1}A + Q, \\ &\leq A^*(P^{-1})^{-1}A + Q. \\ &= A^*PA + Q. \end{aligned} \tag{24}$$

So,

$$\begin{aligned} P \leq A^*PA + Q &\leq A^*(A^*PA + Q)A + Q \\ &= (A^*)^2PA^2 + A^*QA + Q \\ &\leq (A^*)^2(A^*PA + Q)A^2 + A^*QA + Q \\ &\leq \cdots \leq \sum_{m=0}^{n} (A^*)^m QA^m + (A^*)^{n+1} PA^{n+1}. \end{aligned} \tag{25}$$

Combining Lemma 2.4 and Lemma 2.5, we get

$$\begin{aligned}
\sum_{m=0}^{\infty} \sigma_{\max}\left[(A^*)^m Q A^m\right] &= \sum_{m=0}^{\infty} \lambda_{\max}\left[(A^*)^m Q A^m\right] \\
&= \sum_{m=0}^{\infty} \lambda_{\max}\left[A^m (A^*)^m Q\right] \\
&\leq \sum_{m=0}^{\infty} \lambda_{\max}\left[A^m (A^*)^m\right] \lambda_{\max}(Q) \\
&\leq \sum_{m=0}^{\infty} \lambda_{\max}^m(AA^*) \lambda_{\max}(Q) \\
&\leq \lambda_{\max}(Q) \sum_{m=0}^{\infty} \left[\sigma_{\max}^2(A)\right]^m \\
&= \frac{\lambda_{\max}(Q)}{1 - \sigma_{\max}^2(A)}.
\end{aligned}$$

Since $\sigma_1^2(A) < 1$, it follows that $\sum_{m=0}^{\infty} \sigma_1\left[(A^*)^m Q A^m\right]$ is convergent. By Lemma 2.11, $\sum_{m=0}^{\infty} (A^*)^m Q A^m$ is also convergent. In addition, since $\sigma_1^2(A) < 1$, the Theorem 4.1 gives

$$\lim_{n\to\infty} \left[(A^*)^{n+1} P A^{n+1}\right] = 0.$$

Taking the limit in (25), we obtain

$$\begin{aligned}
P &= \lim_{n\to\infty} P \\
&\leq \sum_{m=0}^{n} (A^*)^m\, Q A^m + \lim_{n\to\infty} \left[(A^*)^{n+1} P A^{n+1}\right] \\
&= \sum_{m=0}^{\infty} (A^*)^m\, Q A^m.
\end{aligned}$$

□

The following upper bound is obtained if A is not assumed to be stable:

Theorem 4.6. *Let $P \in B(H)$ be the positive definite solution of the DARE (15). If $\sigma_{\min}^2(B) > 0$, then P has the upper operator bound*

$$\begin{aligned}
P &\leq \frac{\eta_1}{1 + \lambda_{\max}(P_l)\sigma_{\min}^2(B)} A^* A + Q \\
&\equiv P_u,
\end{aligned} \tag{26}$$

where P_l is defined by (20) and

$$\eta_1 = \frac{1}{2\sigma_{\min}^2(B)} \left\{\lambda_{\max}(Q)\sigma_{\min}^2(B) + \sigma_{\max}^2(A) - 1 + \Lambda_{\lambda,\sigma}(A, B, Q)\right\}, \tag{27}$$

with

$$\Lambda_{\lambda,\sigma}(A, B, Q) := \left[\left(\lambda_{\max}(Q)\sigma_{\min}^2(B) + \sigma_{\max}^2(A) - 1\right)^2 + 4\lambda_{\max}(Q)\sigma_{\min}^2(B)\right]^{\frac{1}{2}}.$$

Proof. From Lemma 2.6 and Lemma 2.2, we get

$$\begin{aligned} P &= A^*(P^{-1}+BB^*)^{-1}A+Q \\ &\leq \lambda_{\max}(P^{-1}+BB^*)^{-1}A^*A+Q \\ &= \frac{1}{\lambda_{\min}(P^{-1}+BB^*)}A^*A+Q \\ &\leq \frac{1}{\lambda_{\min}(P^{-1})+\sigma^2_{\min}(B)}A^*A+Q \\ &= \frac{1}{\frac{1}{\lambda_{\max}(P)}+\sigma^2_{\min}(B)}A^*A+Q \\ &= \frac{\lambda_{\max}(P)}{1+\lambda_{\max}(P)\sigma^2_{\min}(B)}A^*A+Q. \end{aligned} \tag{28}$$

Indeed, by Lemma 2.6, we obtain

$$\begin{aligned} \lambda_{\max}(P) &\leq \lambda_{\max}\left[\frac{\lambda_{\max}(P)}{1+\lambda_{\max}(P)\sigma^2_{\min}(B)}A^*A+Q\right] \\ &\leq \frac{\lambda_{\max}(P)}{1+\lambda_{\max}(P)\sigma^2_{\min}(B)}\sigma^2_{\max}(A)+\lambda_{\max}(Q). \end{aligned}$$

Or,

$$\sigma^2_{\min}(B)\lambda^2_{\max}(P)-\lambda_{\max}(P)\left[\lambda_{\max}(Q)\sigma^2_{\min}(B)+\sigma^2_{\max}(A)-1\right]-\lambda_{\max}(Q)\leq 0. \tag{29}$$

So,

$$\lambda_{\max}(P)\leq\frac{1}{2\sigma^2_{\min}(B)}\left\{\lambda_{\max}(Q)\sigma^2_{\min}(B)+\sigma^2_{\max}(A)-1+\Lambda_{\lambda,\sigma}(A,B,Q)\right\}=\eta_1. \tag{30}$$

Substituting (30) into (28), we get

$$P\leq\frac{\eta_1}{1+\lambda_{\max}(P)\sigma^2_{\min}(B)}A^*A+Q. \tag{31}$$

Theorem 4.4 leads to

$$\lambda_{\max}(P)\geq\lambda_{\max}(P_l). \tag{32}$$

Thus,

$$P\leq\frac{\eta_1}{1+\lambda_{\max}(P_l)\sigma^2_{\min}(B)}A^*A+Q\equiv P_u.$$

□

Remark 4.7. *If $\sigma_{\min}(B)=0$ and $\sigma^2_1(A)<1$, then the DARE (15) has a unique positive definite solution P. Moreover,*

$$P_l\leq P\leq P_{u1}.$$

4.3. On the existence and uniqueness for solution of the DARE

Theorem 4.8. *Let*

$$p=\|A\|^2\cdot\left\|P_l^{-1}\right\|^2\cdot\left(\left\|P_l^{-1}+BB^*\right\|\right)^{-2}.$$

If $\sigma_{min}(B)>0$, $\lambda_1(P_u)\leq\eta_1$, and $0<p<1$, then the DARE (15) has a unique positive definite solution P_0. Moreover,

$$P_l\leq P_0\leq P_u.$$

Proof. Let P be the positive definite solution of DARE (15). From Theorems 4.4 and 4.6, we have $P \in [P_l, P_u]$. Letting

$$\Omega := \{P; P_l \le P \le P_u\},$$

and define on Ω the map f:

$$f(P) = A^*(P^{-1} + BB^*)^{-1}A + Q, \quad \text{for all} \ \ P \in \Omega.$$

Under this assumption, Ω is a convex, closed and bounded set and f is continuous. Consider a norm space $(\Omega, \|.\|)$, with $\|.\|$ is the spectral norm. From the proof of Theorems 4.4 and 4.6, we have $P_l \le f(P) \le P_u$, for all $P \in \Omega$. Thus $f(\Omega) \subseteq \Omega$.
For arbitrary $P_1, P_2 \in \Omega$,

$$\begin{aligned} f(P_1) &= A^*(P_1^{-1} + BB^*)^{-1}A + Q, \\ f(P_2) &= A^*(P_2^{-1} + BB^*)^{-1}A + Q. \end{aligned}$$

Consequently,

$$\begin{aligned} f(P_1) - f(P_2) &= A^*\left[(P_1^{-1} + BB^*)^{-1} - (P_2^{-1} + BB^*)^{-1}\right]A \\ &= A^*(P_1^{-1} + BB^*)^{-1}\left[(P_2^{-1} + BB^*) - (P_1^{-1} + BB^*)\right](P_2^{-1} + BB^*)^{-1}A \\ &= A^*(P_1^{-1} + BB^*)^{-1}\left[P_2^{-1} - P_1^{-1}\right](P_2^{-1} + BB^*)^{-1}A \\ &= A^*(P_1^{-1} + BB^*)^{-1}P_2^{-1}(P_1 - P_2)P_1^{-1}(P_2^{-1} + BB^*)^{-1}A. \end{aligned}$$

Because $P_l \le P \le P_u$, from Lemma 5 we have

$$\begin{aligned} \|f(P_1) - f(P_2)\| &= \left\|A^*(P_1^{-1} + BB^*)^{-1})P_2^{-1}(P_1 - P_2)P_1^{-1}(P_2^{-1} + BB^*)^{-1}A\right\| \\ &\le \|A^*\|.\|A\|.\|P_1^{-1}\|.\|P_2^{-1}\|.\|(P_1^{-1} + BB^*)^{-1}\|.\|(P_2^{-1} + BB^*)^{-1}\|.\|P_1 - P_2\| \\ &\le \|A^*\|.\|A\|.\|P_l^{-1}\|.\|P_l^{-1}\|.(\|P_u^{-1} + BB^*\|)^{-1}.(\|P_u^{-1} + BB^*\|)^{-1}.\|P_1 - P_2\| \\ &= \|A\|^2.\|P_l^{-1}\|^2.(\|P_u^{-1} + BB^*\|)^{-2}.\|P_1 - P_2\| \\ &= p\|P_1 - P_2\|. \end{aligned}$$

Since $p < 1$, the map f is a contraction map in Ω. By Lemma 2.13, the map f has a unique fixed point in Ω. Thus, the DARE (15) has a unique positive definite solution P_0, and $P_l \le P_0 \le P_u$. □

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