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1WG inverse of square matrices

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Abstract. In this paper, we introduce a new generalized inverse, which is called 1WG inverse of complex square matrices. We investigate the existence and uniqueness for the 1WG inverse and give some characterizations, representations, and properties of it. Next, by using the core-EP decomposition, we discuss the relationships between the 1WG inverse and other generalized inverses. Successive matrix squaring algorithm is considered for calculating the 1WG inverse. In the end, we present a binary relation for the 1WG inverse.

1. Introduction

The generalized inverses of matrices are closely related to linear equations, minimum-norm least-square solutions and numerous matrix factorizations [29]. In the past half century, the study of the generalized inverse has been explosively increased and various generalized inverses were introduced such as core inverse [1], Drazin inverse [4], core-EP inverse [20], DMP inverse [14], (B,C)-inverse [6], MPWG inverses [28], etc. The WG inverse was established in [27] and the application of the WG inverse in solving constrained systems of linear equations was discussed in [18, 31]. Recently, Jiang and Zuo [13] has extended the m-weak group in the ring to complex matrices and gave some new characterizations.

Let $\mathbb{C}_{n,n}$ be the set of $n \times n$ complex matrices. For $A \in \mathbb{C}_{n,n}$, the symbols A^*, $\text{rank}(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ represent the conjugate transpose, the rank, the range space and the kernel of A, respectively. As usual, I_n stands for the $n \times n$ identity matrix. The index of a complex square matrix A, denoted by $Ind(A)$, is the smallest positive integer k such that $rank(A^{k+1}) = rank(A^k)$. Clearly, $Ind(A) = 0$ if and only if A is nonsingular. The symbol $\mathbb{C}_n^{CM}$ stands for a set of $n \times n$ matrices of index less than or equal to 1. $P_{T,S}$ represents the projector (idempotent) on the subspace S along the subspace T. If S is orthogonal to the subspace T , this notation will be reduced to P_T.

The Moore-Penrose inverse of $A \in \mathbb{C}_{m,n}$, where $\mathbb{C}_{m,n}$ is the set of $m \times n$ complex matrices, is the unique matrix $A^\dagger = X \in \mathbb{C}_{m,n}$ which satisfies the four Penrose equations

$$(1)\ AXA = A, \quad (2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (4)\ (XA)^* = XA.$$

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A matrix X satisfying (1) is called an inner inverse or {1}-inverse of A and is denoted as A^-. The outer inverse of $A \in \mathbb{C}_{m,n}$ with prescribed range T and null space S, denoted by $A^{(2)}_{T,S}$, satisfies (2) and two additional properties: $\mathcal{R}(X) = T$ and $\mathcal{N}(X) = S$ [10].

The Drazin inverse of $A \in \mathbb{C}_{n,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ such that

$$(1^k)\ A^{k+1}X = A^k, \quad (2)\ XAX = X, \quad (5)\ AX = XA,$$

and denoted by A^D, where $k = Ind(A)$. If $Ind(A) = 1$, the matrix X is called the group inverse of A, which is denoted by $A^\#$.

The core-EP inverse has extended the core inverse of a square matrix of index one to a square matrix of arbitrary index in [20]. For $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$, the core-EP inverse $A^{\oplus}$ of A is the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the following conditions [12]

$$(2)\ XAX = X, \quad (3)\ (AX)^* = AX, \quad (1^l)\ XA^{l+1} = A^l,$$

and $R(X) \subseteq R(A^l)$, where $l \geq Ind(A)$. It is actually the {2}-inverse of A with prescribed range $\mathcal{R}\left(A^k\right)$ and nullspace $\mathcal{N}\left(\left(A^k\right)^*\right)$. When $A \in \mathbb{C}_n^{CM}$, $A^{\oplus}$ is called the core inverse of A and is denoted by $A^{\#\!\!\!\bigcirc}$. Moreover, the core inverse of A is given by $A^{\#\!\!\!\bigcirc}=A^\# AA^\dagger$.

For $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$, A is said to be the weak group (WG) inverse, if there exists a unique matrix $X = A^{Ⓦ} \in \mathbb{C}_{n,n}$ such that the following conditions hold [27]

$$AX^2 = X, \quad AX = A^{\oplus}A.$$

The (B,C)-inverse of $A \in \mathbb{C}_{m,n}$, denoted by $A^{(B,\ C)}$ [6], is the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying $XAB = B$, $CAX = C$, $\mathcal{N}(X) = \mathcal{N}(C)$ and $\mathcal{R}(X) = \mathcal{R}(B)$, where $B, C \in \mathbb{C}_{n,m}$. [3, 6]. The DMP inverse of $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$, denoted by $A^{D,\dagger}$, was defined by Malik and Thome [14] as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying $XAX = X$, $XA = A^DA$, $A^kX = A^kA^\dagger$. And it was proved that $A^{D,\dagger}=A^DAA^\dagger$. The weak core inverse $A^{Ⓦ,\dagger}$ [7] of $A \in \mathbb{C}_{n,n}$ presents a unique solution to the matrix system $X = XAX$, $AX = CA^\dagger$, $XA = A^DC$, where C is the weak core part of A with $C = AA^{Ⓦ}A$. And the dual weak core inverse $A^{\dagger,Ⓦ} = A^\dagger AA^{Ⓦ}$. The BT-inverse of $A \in \mathbb{C}_{n,n}$, written by $A^\diamond$, which was defined in [2] can be represented by $(AP_A)^\dagger$. For $A \in \mathbb{C}_{n,n}$, the CMP inverse of A is expressed by $A^{c,\dagger}=A^\dagger AA^DAA^\dagger$ [16]. Several characterizations and representations of the CMP inverse were given in [19]. For $m = 2$, the m-weak group coincides with the generalized group (GG) inverse [9] $A^{Ⓦ_2} = (A^{\oplus})^3A^2$. From projectors, 1MP and MP1 generalized inverses and their corresponding partial orders were introduced in [11].

Recently, two new generalized inverses have emerged by combining the inner inverse and Drazin inverse, which are the 1Drazin inverse (in short, 1D) and Drazin1 inverse (in short, D1) [23]. A matrix $X = A^{-,D} \in \mathbb{C}_{n,n}$ is called the 1Drazin inverse of A if it satisfies

$$XAX = X, \quad XA^k = A^-A^k, \quad AX = AA^D,$$

where A^- is a fixed inner inverse of A. Notice that $A^{-,D} = A^-AA^D$ and $A^{D,-} = A^DAA^-$. Similarly, the ICEP inverse and CEPI inverse of A can be denoted by $A^{-,\oplus}$ and $A^{\oplus,-}$, respectively [24].

The paper is organized as follows. Some preliminaries are given in Sect.2. Definition, characterizations and representations of the 1WG inverse are presented in Sect.3. In Sect.4, we discuss the relationships between the 1WG inverse and other generalized inverses by the core-EP decomposition. A variant of the SMS algorithm for computing the 1WG inverse is given in Sect.5. In Sect.6, we use the 1WG inverse to solving some systems of linear equations. In Sect.7, we give the binary relation for the 1WG inverse.

2. Preliminaries

In this section, we present some preliminary results.

Lemma 2.1. *[18, 28] Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$. Then*
(i) $A^{\textcircled{w}}$ is an outer inverse of A.
(ii) $A^{\textcircled{w}}A^{k+1} = A^k$.
(iii) $A^{\textcircled{w}} = A^k(A^{\textcircled{w}})^{k+1}$.
(iv) $\mathcal{R}\left(A^{\textcircled{w}}\right) = \mathcal{R}\left(A^k\right)$.
(v) $A^{\textcircled{w}} = A^kZ$ for some matrix Z.
(vi) $AA^{\textcircled{w}} = A^kW$ for some matrix W.

Lemma 2.2. *[27] Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$. Then*

$$A^{\textcircled{w}} = (AA^{\textcircled{\dagger}}A)^{\#} = (A^{\textcircled{\dagger}})^2A = (A^2)^{\textcircled{\dagger}}A.$$

Lemma 2.3. *[27] Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$. Then*

$$A^{\textcircled{w}} = A^k(A^{k+2})^{\textcircled{\#}}A = (A^2P_{A^k})^{\dagger}A.$$

Lemma 2.4. *[26] (Core-EP Decomposition) Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$, $rank(A^k) = t$. Then A can be written as the sum of matrices A_1 and A_2, i.e., $A = A_1 + A_2$, where*

$$A_1 \in \mathbb{C}_n^{CM},\ \ A_2^k = 0 \ \text{ and } \ A_1A_2 = A_2A_1 = 0.$$

Here one or both of A_1 and A_2 can be null.

Lemma 2.5. *[26] Let the core-EP decomposition of $A \in \mathbb{C}_{n,n}$ be as in Lemma 2.4. Then there exists a unitary matrix U such that*

$$A_1 = U\begin{bmatrix} T & S \\ 0 & 0 \end{bmatrix}U^*,\ A_2 = U\begin{bmatrix} 0 & 0 \\ 0 & N \end{bmatrix}U^*, \tag{1}$$

where $T \in \mathbb{C}_{t,t}$ is non-singular, and $N \in \mathbb{C}_{(n-t),(n-t)}$ is nilpotent.

3. The defintion, characterizations and representations of the 1WG inverse

In this section, we introduce a new generalized inverse of $A \in \mathbb{C}_{n,n}$ by using the inner inverse and the WG inverse of A.

Theorem 3.1. *Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$. Suppose A^- is a fixed inner inverse of A. The system of equations*

$$XAX = X,\ \ AX = A^{\textcircled{w}}A \ \text{ and } \ XA^k = A^-A^k \tag{2}$$

is consistent and its unique solution is $X = A^-A^{\textcircled{w}}A$.

Proof. Let $X = A^-A^{\textcircled{w}}A$. It is evident that $A^{\textcircled{w}} = A(A^{\textcircled{w}})^2$ implies $A^{\textcircled{w}} = A^k(A^{\textcircled{w}})^{k+1}$. Then

$$XAX = A^-A^{\textcircled{w}}AAA^-A^{\textcircled{w}}A = A^-A^{\textcircled{w}}AAA^-A^k(A^{\textcircled{w}})^{k+1}A = A^-A^{\textcircled{w}}AA^{\textcircled{w}}A = X,$$

$$AX = AA^-A^{\textcircled{w}}A = AA^-A^k(A^{\textcircled{w}})^{k+1}A = A^k(A^{\textcircled{w}})^{k+1}A = A^{\textcircled{w}}A,$$

$$XA^k = A^-A^{\textcircled{w}}A^{k+1} = A^-A^k.$$

Hence, $X = A^-A^{\textcircled{w}}A$ satisfies the three equations in the system (2)

For the uniqueness, we assume that Y is another solution of the system (2). Now, we obtain

$$Y = YAY = YA^{\textcircled{w}}A = YA^k(A^{\textcircled{w}})^{k+1}A = A^-A^k(A^{\textcircled{w}})^{k+1}A = A^-A^{\textcircled{w}}A = X,$$

which implies that the system (2) has the unique solution. □

According to Theorem 3.1, we give the definition of the 1WG inverse.

Definition 3.2. *Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$. Suppose A^- is a fixed inner inverse of A. The 1WG inverse of A is defined as $A^{-,Ⓦ} = A^- A^{Ⓦ} A$.*

Remark 3.3. *We know that every fixed inner inverse A^- of A may produce a different 1WG inverse of A. Therefore, if we mention the 1WG inverse of A, then it is the inner inverse that we have fixed previously.*

We observe that the 1WG inverse provide new classes of generalized inverses by the following example.

Example 3.4. *Let*

$$A = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^- = \begin{pmatrix} 0 & -1 & -1 & -1 & 1 \\ 0 & 1 & 0 & -1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}.$$

It is easy to check that $Ind(A) = 3$. It can be obtained by calculation that the Moore-Penrose inverse, the WG inverse are

$$A^\dagger = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ \frac{1}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \ A^{Ⓦ} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The MPWG inverse, the weak core inverse and its dual inverse are

$$A^{\dagger,WG} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 1 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{Ⓦ,\dagger} = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\dagger,Ⓦ} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The generalized group (GG) inverse, the DMP inverse and the CMP inverse are

$$A^{Ⓦ_2} = \begin{pmatrix} 1 & 0 & 1 & 1 & 2 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{D,\dagger} = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{C,\dagger} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

We also get the 1D inverse, the D1 inverse, the ICEP inverse, the CEPI inverse and the 1WG inverse are

$$A^{-,D} = \begin{pmatrix} 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{D,-} = \begin{pmatrix} 1 & 0 & 1 & 1 & 6 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

$$A^{-,\oplus} = \begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{\oplus,-} = \begin{pmatrix} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \ A^{-,Ⓦ} = \begin{pmatrix} 0 & -1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Obviously, this example shows that $A^{-,Ⓦ}$ is different from other generalized inverses.

Theorem 3.5. *Let $A \in \mathbb{C}_{n,n}$ with Ind$(A) = k$. Then*
(i) $A^{-,\textcircled{w}} = A^{(2)}_{\mathcal{R}(A^-A^k),\ \mathcal{N}((A^k)^*A^2)}$.
(ii) $AA^{-,\textcircled{w}}$ *is a projector onto* $\mathcal{R}\left(A^k\right)$ *along* $\mathcal{N}\left(\left(A^k\right)^* A^2\right)$.
(iii) $A^{-,\textcircled{w}}A$ *is a projector onto* $\mathcal{R}\left(A^-A^k\right)$ *along* $\mathcal{N}\left(\left(A^k\right)^* A^3\right)$.

Proof. *(i) Clearly,* $A^{-,\textcircled{w}}$ *is an outer inverse of A. From the expressions*

$$A^{-,\textcircled{w}} = A^-A^{\textcircled{w}}A = A^-A^k(A^{\textcircled{w}})^{k+1}A,$$

and $A^-A^k = A^{-,\textcircled{w}}A^k$, *we have* $\mathcal{R}\left(A^{-,\textcircled{w}}\right) = \mathcal{R}\left(A^-A^k\right)$. *On the other hand,*

$$\mathcal{N}\left(A^{\textcircled{w}}A\right) \subseteq \mathcal{N}\left(AA^{\textcircled{w}}A\right) = \mathcal{N}\left(A^{\textcircled{\#}}A^2\right) \subseteq \mathcal{N}\left((A^{\textcircled{\#}})^2A^2\right) = \mathcal{N}\left(A^{\textcircled{w}}A\right),$$

$$\mathcal{N}\left(A^{-,\textcircled{w}}\right) = \mathcal{N}\left(AA^{-,\textcircled{w}}\right) = \mathcal{N}\left(A^{\textcircled{w}}A\right) = \mathcal{N}\left(\left(A^k\right)^* A^2\right).$$

Therefore, $A^{-,\textcircled{w}} = A^{(2)}_{\mathcal{R}(A^-A^k),\ \mathcal{N}((A^k)^*A^2)}$.
(ii) From Definition 3.2, we have

$$\mathcal{R}\left(AA^{-,\textcircled{w}}\right) = \mathcal{R}\left(A^{\textcircled{w}}A\right) = \mathcal{R}\left(A^{\textcircled{w}}\right) = \mathcal{R}\left(A^k\right).$$

Also, we know $\mathcal{N}\left(AA^{-,\textcircled{w}}\right) = \mathcal{N}\left(\left(A^k\right)^* A^2\right)$.

(iii) Clearly, $x \in \mathcal{N}\left(A^{-,\textcircled{w}}A\right)$ *if and only if* $Ax \in \mathcal{N}\left(A^{-,\textcircled{w}}\right) = \mathcal{N}\left(\left(A^k\right)^* A^2\right)$. *Therefore,* $x \in \mathcal{N}\left(A^{-,\textcircled{w}}A\right)$ *if and only if* $x \in \mathcal{N}\left(\left(A^k\right)^* A^3\right)$. *Thus, we have* $\mathcal{N}\left(A^{-,\textcircled{w}}A\right) = \mathcal{N}\left(\left(A^k\right)^* A^3\right)$. *We observe that,* $\mathcal{R}\left(A^{-,\textcircled{w}}A\right) = \mathcal{R}\left(A^{-,\textcircled{w}}\right) = \mathcal{R}\left(A^-A^k\right)$. □

Theorem 3.6. *Let $A \in \mathbb{C}_{n,n}$ with Ind$(A) = k$. Then*
(i) $A^{-,\textcircled{w}} = A^-(A^{\textcircled{\#}})^2A^2 = A^-(A^2)^{\textcircled{\#}}A^2$.
(ii) $A^{-,\textcircled{w}} = A^-(AA^{\textcircled{\#}}A)^{\#}A$.
(iii) $A^{-,\textcircled{w}} = A^-A^k(A^{k+2})^{\textcircled{\dagger}}A^2$.
(iv) $A^{-,\textcircled{w}} = A^-(A^2P_{A^k})^{\dagger}A^2$.

Proof. *Items (i)-(iv) are direct consequences of* $A^{-,\textcircled{w}} = A^-A^{\textcircled{w}}A$, *Lemma 2.4 and Lemma 2.3.* □

We discuss some characterizations for a matrix to be a 1WG inverse from a geometrical point of view [23].

Theorem 3.7. *Let $A \in \mathbb{C}_{n,n}$ with Ind$(A) = k$. Then the following statements are equivalent:*
(i) $X = A^{-,\textcircled{w}}$.
(ii) $\mathcal{R}(X^*) = \mathcal{R}\left((A^2)^*A^k\right)$, $\mathcal{N}(X^*) = \mathcal{N}\left((A^-A^k)^*\right)$ *and* $AX = A^{\textcircled{w}}A$.
(iii) $XA^k = A^-A^k$ *and* $\mathcal{N}(X) = \mathcal{N}\left((A^k)^*A^2\right)$.
(iv) $XA^k = A^-A^k$ *and* $\mathcal{R}(X^*) = \mathcal{R}\left((A^2)^*A^k\right)$.

Proof. (*i*) ⇒ (*ii*) *Let* $X = A^{-,\textcircled{w}}$. *Since* $X^* = (A^-A^{\textcircled{w}}A)^* = (A^-(A^{\textcircled{\#}})^2A^2)^* = (A^2)^*((A^{\textcircled{\#}})^2)^*(A^-)^*$, *then* $\mathcal{R}(X^*) \subseteq \mathcal{R}\left((A^2)^*(A^{\textcircled{\#}})^*\right) = \mathcal{R}\left((A^2)^*A^k\right)$. *Moreover,*

$$\mathcal{R}\left((A^2)^*(A^{\textcircled{\#}})^*\right) = \mathcal{R}\left((AA^{\textcircled{w}}A)^*\right) = \mathcal{R}\left((A^2X)^*\right) \subseteq \mathcal{R}(X^*).$$

Therefore, $\mathcal{R}(X^*) = \mathcal{R}\left((A^2)^*A^k\right)$. *Since*

$$X^* = (A^-A^{\textcircled{W}}A)^* = (A^-A^k(A^{\textcircled{W}})^{k+1}A)^* = ((A^{\textcircled{W}})^{k+1}A)^*(A^-A^k)^*$$

and

$$(A^-A^k)^* = (XA^k)^* = (A^k)^*X^*,$$

then $\mathcal{N}(X^*) = \mathcal{N}\left((A^-A^k)^*\right)$. *Obviously,* $AX = A^{\textcircled{W}}A$.

$(ii) \Rightarrow (iii)$ *From* $\mathcal{N}(X^*) = \mathcal{N}\left((A^-A^k)^*\right)$, *we have* $X = A^-A^kQ$ *for some* $Q \in \mathbb{C}_{n,n}$. *Thus,*

$$XA^k = A^-A^kQA^k = A^-AA^-A^kQA^k = A^-AXA^k = A^-A^{\textcircled{W}}AA^k = A^-A^k.$$

$(iii) \Rightarrow (iv)$ *It is evident since* $\mathcal{R}(X^*) = \mathcal{N}(X)^{\perp}$.

$(iv) \Rightarrow (i)$ *From* $\mathcal{R}(X^*) = \mathcal{R}\left((A^2)^*A^k\right)$, *we obtain* $X = Z(A^k)^*A^2$ *for some* $Z \in \mathbb{C}_{n,n}$. *It follows from* $AA^{\oplus} = A^2(A^{\oplus})^2$,

$$\begin{aligned} X &= Z(A^k)^*A^2 = Z(AA^{\oplus}A^k)^*A^2 = Z(A^k)^*A^2(A^{\oplus})^2A^2 \\ &= XA^{\textcircled{W}}A = XA^k(A^{\textcircled{W}})^{k+1}A = A^-A^{\textcircled{W}}A. \end{aligned}$$

□

Next, from an algebraic and geometrical point of view, we have the following characterizations.

Theorem 3.8. *Let* $A \in \mathbb{C}_{n,n}$ *with* $Ind(A) = k$. *Then the following statements are equivalent:*
(i) $X = A^{-,\textcircled{W}}$.
(ii) $XA = A^-A^{\textcircled{W}}A^2$ *and* $\mathcal{N}\left((A^k)^*A^2\right) \subseteq \mathcal{N}(X)$.
(iii) $AXA = A^{\textcircled{W}}A^2$, $\mathcal{R}(X) \subseteq \mathcal{R}\left(A^-A^k\right)$ *and* $\mathcal{N}\left((A^k)^*A^2\right) \subseteq \mathcal{N}(X)$.
(iv) $AX = A^{\textcircled{W}}A$ *and* $\mathcal{R}(X) \subseteq \mathcal{R}\left(A^-A^k\right)$.

Proof. $(i) \Rightarrow (ii)$ *Let* $X = A^{-,\textcircled{W}}$. *Then* $XA = A^{-,\textcircled{W}}A = A^-A^{\textcircled{W}}A^2$. *Moreover, from Theorem 3.5 we can get* $\mathcal{N}\left((A^k)^*A^2\right) \subseteq \mathcal{N}(X)$.

$(ii) \Rightarrow (iii)$ *Since* $XA = A^-A^{\textcircled{W}}A^2$, *then* $AXA = AA^-A^{\textcircled{W}}A^2 = A^{\textcircled{W}}A^2$. *From* $\mathcal{N}\left((A^k)^*A^2\right) \subseteq \mathcal{N}(X)$, *we have* $X = Z(A^k)^*A^2$ *for some* $Z \in \mathbb{C}_{n,n}$. *Then*

$$X = Z(A^k)^*A^2 = Z(A^k)^*A^2(A^{\oplus})^2A^2 = XA^{\textcircled{W}}A = XA^k(A^{\textcircled{W}})^{k+1}A, \tag{3}$$

and $XAA^{k-1}(A^{\textcircled{W}})^{k+1}A = A^-A^{\textcircled{W}}A^2A^{k-1}(A^{\textcircled{W}})^{k+1}A = A^-A^k(A^{\textcircled{W}})^{k+1}A$. *Thus,* $\mathcal{R}(X) \subseteq \mathcal{R}\left(A^-A^k\right)$.

$(iii) \Rightarrow (iv)$ *From* $\mathcal{N}\left((A^k)^*A^2\right) \subseteq \mathcal{N}(X)$, *we get* $X = Z(A^k)^*A^2$ *for some* $Z \in \mathbb{C}_{n,n}$. *Multiplying the equation* (3) *by* A *from the right side,* $AX = AZ(A^k)^*A^2 = AXA^{\textcircled{W}}A = AXA^k(A^{\textcircled{W}})^{k+1}A = A^{\textcircled{W}}A^2A^{k-1}(A^{\textcircled{W}})^{k+1}A = A^{\textcircled{W}}AA^{\textcircled{W}}A = A^{\textcircled{W}}A$.

$(iv) \Rightarrow (i)$ *From* $\mathcal{R}(X) \subseteq \mathcal{R}\left(A^-A^k\right)$, *we have* $X = A^-A^kZ$ *for some* $Z \in \mathbb{C}_{n,n}$. *Then,* $X = A^-A^kZ = A^-AA^-A^kZ = A^-AX = A^-A^{\textcircled{W}}A$. □

By using matrix equalities, we have the following characterizations.

Theorem 3.9. *Let* $A \in \mathbb{C}_{n,n}$ *with* $Ind(A) = k$. *Then the following statements are equivalent:*
(i) $X = A^{-,\textcircled{W}}$.
(ii) $A^-AX = X$, $XA^k = A^-A^k$ *and* $X = XA^{\textcircled{W}}A$.
(iii) $A^-AXA^{\textcircled{W}}A = X$ *and* $AXA^k = A^k$.
(iv) $AX = A^{\textcircled{W}}A$ *and* $A^-AX = X$.
(v) $X = XA^{\textcircled{W}}A$ *and* $XA^k = A^-A^k$.
(vi) $A^-A^{\textcircled{W}}A^2 = XA$, $A^{k+1}X = A^kA^{\textcircled{W}}A$ *and* $X = XA^{\textcircled{W}}A$.

Proof. $(i) \Rightarrow (ii)$ *Let* $X = A^{-,\textcircled{W}}$. *Then* $XA^k = A^-A^k$ *and*

$$A^-AX = A^-AA^-A^{\textcircled{W}}A = A^-A^{\textcircled{W}}A = X,$$
$$XA^{\textcircled{W}}A = A^-A^{\textcircled{W}}AA^{\textcircled{W}}A = A^-A^{\textcircled{W}}A = X.$$

$(ii) \Rightarrow (iii)$ *From* $XA^k = A^-A^k$, *we have* $AXA^k = AA^-A^k = A^k$. *Post multiplying by* $A^{\textcircled{W}}A$ *from the right side of* $A^-AX = X$, *we get* $A^-AXA^{\textcircled{W}}A = XA^{\textcircled{W}}A = X$.

$(iii) \Rightarrow (iv)$ *Clearly,* $A^-AX = A^-AA^-AXA^{\textcircled{W}}A = A^-AXA^{\textcircled{W}}A = X$. *And* $AX = AA^-AXA^{\textcircled{W}}A = AXA^k(A^{\textcircled{W}})^{k+1}A = A^k(A^{\textcircled{W}})^{k+1}A = A^{\textcircled{W}}A$.

$(iv) \Rightarrow (v)$ *Since* $AX = A^{\textcircled{W}}A$ *and* $A^-AX = X$, *then* $X = A^-A^{\textcircled{W}}A$. *It is easy to check that* $XA^{\textcircled{W}}A = A^-A^{\textcircled{W}}AA^{\textcircled{W}}A = A^-A^{\textcircled{W}}A = X$ *and* $XA^k = A^-A^k$.

$(v) \Rightarrow (vi)$ *Post multiplying by* A *from the right side of* $X = XA^{\textcircled{W}}A$, *we get*

$$XA = XA^{\textcircled{W}}AA = XA^k(A^{\textcircled{W}})^{k+1}A^2 = A^-A^k(A^{\textcircled{W}})^{k+1}A^2 = A^-A^{\textcircled{W}}A^2.$$

Multiplying $X = XA^{\textcircled{W}}A$ *by* A^{k+1} *from the left side, we have* $A^{k+1}X = A^{k+1}XA^{\textcircled{W}}A = A^{k+1}XA^k(A^{\textcircled{W}})^{k+1}A = A^{k+1}A^-A^k(A^{\textcircled{W}})^{k+1}A = A^kA^{\textcircled{W}}A$.

$(vi) \Rightarrow (i)$ *It follows from* $X = XA^{\textcircled{W}}A = XA(A^{\textcircled{W}})^2A = A^-A^{\textcircled{W}}A^2(A^{\textcircled{W}})^2A = A^-A^{\textcircled{W}}AA^{\textcircled{W}}A = A^-A^{\textcircled{W}}A$. □

In order to determine the whole class of all matrices that can be used instead of A^- or $A^{\textcircled{W}}$, we discuss the maximal classes [8, 17] for the 1WG inverse.

Theorem 3.10. *Let* $A \in \mathbb{C}_{n,n}$ *with* $Ind(A) = k$. *If* $A^{-,\textcircled{W}} = A^-AV$, *then*

(i) $AVA = A^{\textcircled{W}}A^2$ *and* $\mathcal{N}\left(\left(A^k\right)^* A^2\right) = \mathcal{N}(AV)$.

(ii) $AV = A^{\textcircled{W}}A$ *with* $V = (A^{\textcircled{W}})^2A + (I - A^-A)Z$ *for any* $Z \in \mathbb{C}_{n,n}$.

Proof. (i) It is obvious that $AVA = AA^-AVA = AA^{-,\textcircled{W}}A = A^{\textcircled{W}}A^2$. *Next,* $AA^-AV = AV = AA^{-,\textcircled{W}} = A^{\textcircled{W}}A$, *from Theorem 3.5, we have*

$$\mathcal{N}(AV) = \mathcal{N}\left(AA^{-,\textcircled{W}}\right) = \mathcal{N}\left(A^{\textcircled{W}}A\right) = \mathcal{N}\left(\left(A^k\right)^* A^2\right).$$

(ii) It is easy to see that $AV = AA^{-,\textcircled{W}} = A^{\textcircled{W}}A$. *The general solution of* $AV = 0$ *is given by* $(I - A^-A)Z$ *for arbitrary* $Z \in \mathbb{C}_{n,n}$. *And* $(A^{\textcircled{W}})^2A$ *is a particular solution of* $AV = A^{\textcircled{W}}A$, *the general solution of* $AV = A^{\textcircled{W}}A$ *is given by* $(A^{\textcircled{W}})^2A + (I - A^-A)Z$ *for any* $Z \in \mathbb{C}_{n,n}$. □

Theorem 3.11. *Let* $A \in \mathbb{C}_{n,n}$ *with* $Ind(A) = k$. *Then*

(i) $A^{-,\textcircled{W}} = UA^{\textcircled{W}}A$ *if and only if* $U = A^- + W(I - A^{\textcircled{W}}A)$ *for any* $W \in \mathbb{C}_{n,n}$.

(ii) $A^{-,\textcircled{W}} = UAV$ *if and only if* $U = A^- + W(I - A^{\textcircled{W}}A)$, $V = (A^{\textcircled{W}})^2A + (I - A^-A)Z$ *for any* $W, Z \in \mathbb{C}_{n,n}$.

Proof. (i) If $U = A^- + W(I - A^{\textcircled{W}}A)$, *multiplying the equation by* $A^{\textcircled{W}}A$ *from the right side, then*

$$UA^{\textcircled{W}}A = A^-A^{\textcircled{W}}A + W(I - A^{\textcircled{W}}A)A^{\textcircled{W}}A = A^-A^{\textcircled{W}}A = A^{-,\textcircled{W}}.$$

Conversely, let $A^{-,\textcircled{W}} = UA^{\textcircled{W}}A$. *Obviously,* A^- *is a particular solution of* $A^{-,\textcircled{W}} = UA^{\textcircled{W}}A$. *If* W *is an arbitrary solution of* $UA^{\textcircled{W}}A = 0$ *we can express* W *as* $W = W - WA^{\textcircled{W}}A = W(I - A^{\textcircled{W}}A)$. *Consequently,* $U = A^- + W(I - A^{\textcircled{W}}A)$, *where* $W \in \mathbb{C}_{n,n}$, *is the general solution of* $A^{-,\textcircled{W}} = UA^{\textcircled{W}}A$.

(ii) It follows from (i) and Theorem 3.10 (ii). □

4. Relationships with other generalized inverses

Next, we provide a canonical form for the 1WG inverse. Then, we present some necessary and sufficient conditions for which the 1WG inverse coincides with other generalized inverses by using the core-EP decomposition.

Theorem 4.1. *Let $A \in \mathbb{C}_{n,n}$ be a matrix written as in* (1). *For the inner inverse of A of form $A^{-} = U\begin{bmatrix} X_1 & X_2 \\ X_3 & N^{-} \end{bmatrix}U^{*}$, the 1WG inverse of A is given by*

$$A^{-,\textcircled{W}} = U\begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix}U^{*}. \tag{4}$$

where $X_1 = T^{-1} - T^{-1}SX_3$, $TX_2N + SN^{-}N = 0$, $NX_3 = 0$.

Proof. Clearly,

$$\begin{aligned} A^{-,\textcircled{W}} = A^{-}A^{\textcircled{W}}A &= U\begin{bmatrix} X_1 & X_2 \\ X_3 & N^{-} \end{bmatrix}\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}\begin{bmatrix} T & S \\ 0 & N \end{bmatrix}U^{*} \\ &= U\begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix}U^{*}. \end{aligned}$$

□

Lemma 4.2. [2, 5, 9, 27] *Let $A \in \mathbb{C}_{n,n}$ be a matrix written as in* (1). *Suppose $\triangle = [TT^{*} + S(I_{n-t} - N^{\dagger}N)S^{*}]^{-1}$, $\triangle_1 = [TT^{*} + S(P_N - P_{N^{\diamond}})]^{-1}$ and $\tilde{T}_k = \sum_{j=0}^{k-1} T^{j}SN^{k-1-j}$. Then*

(i) the Moore-Penrose inverse of A is

$$A^{\dagger} = U\begin{bmatrix} T^{*}\triangle & -T^{*}\triangle SN^{\dagger} \\ (I_{n-t} - N^{\dagger}N)S^{*}\triangle & N^{\dagger} - (I_{n-t} - N^{\dagger}N)S^{*}\triangle SN^{\dagger} \end{bmatrix}U^{*}.$$

(ii) The WG inverse of A is

$$A^{\textcircled{W}} = U\begin{bmatrix} T^{-1} & T^{-2}S \\ 0 & 0 \end{bmatrix}U^{*}. \tag{5}$$

(iii) The generalized group (GG) inverse of A is

$$A^{\textcircled{W}_2} = U\begin{bmatrix} T^{-1} & T^{-2}S + T^{-3}SN \\ 0 & 0 \end{bmatrix}U^{*}. \tag{6}$$

(iv) The BT inverse of A is

$$A^{\diamond} = U\begin{bmatrix} T^{*}\triangle_1 & -T^{*}\triangle_1 SN^{\diamond} \\ (I_{n-t} - N^{\dagger}N)S^{*}\triangle_1 & N - (P_N - P_{N^{\diamond}})S^{*}\triangle_1 SN \end{bmatrix}U^{*}. \tag{7}$$

Lemma 4.3. [23, 24] *Let $A \in \mathbb{C}_{n,n}$ be a matrix written as in* (1). *Then*

(i) the 1D inverse of A is

$$A^{-,D} = U\begin{bmatrix} Y_1 & Y_1(T^{k})^{-1}\tilde{T}_k \\ Y_3 & Y_3(T^{k})^{-1}\tilde{T}_k \end{bmatrix}U^{*}.$$

(ii) The ICEP inverse of A is

$$A^{-,\textcircled{\dagger}} = U\begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix}U^{*},$$

where $\tilde{T}_k = \sum_{j=0}^{k-1} T^{j}SN^{k-1-j}$, $Y_1 = T^{-1} - T^{-1}SY_3$, $NY_3 = 0$, $Z_1 = T^{-1} - T^{-1}SZ_3$, $NZ_3 = 0$.

Remark 4.4. *Let $A \in \mathbb{C}_{n,n}$ be a matrix written as in* (1) *and A^- be a fixed inner inverse of A. Then*

$$AA^{\textcircled{W}}A^- = A^{\textcircled{\#}}AA^- = U\begin{bmatrix} T^{-1} & X_2 + T^{-1}SN^- \\ 0 & 0 \end{bmatrix}U^* = A^{\textcircled{\#},-}.$$

The expression $AA^{\textcircled{W}}A^-$ can not be considered as a new generalized inverse of A.

Theorem 4.5. *Let $A \in \mathbb{C}_{n,n}$ be a matrix written as in* (1)*. Then*
(i) $A^{-,\textcircled{W}} = A^{\dagger,WG}$ if and only if $X_1 = T^*\triangle$, $X_3 = (I_{n-t} - N^\dagger N)S^*\triangle$.
(ii) $A^{-,\textcircled{W}} = A^{\textcircled{W}_2}$ if and only if $X_1 = T^{-1}$, $X_3 = 0$.
(iii) $A^{-,\textcircled{W}} = A^{\diamond}A^{\textcircled{W}}A$ if and only if $X_1 = T^*\triangle_1$, $X_3 = (I_{n-t} - N^\dagger N)S^*\triangle_1$,
where X_1, X_3, $\triangle$, $\triangle_1$ are defined in Theorem 4.1 and Lemma 4.2.

Proof. *(i) It follows from* (4) *and* (5) *that*

$$\begin{aligned} A^{-,\textcircled{W}} = A^{\dagger,WG} &\Leftrightarrow \begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix} = \begin{bmatrix} T^*\triangle & T^*\triangle(T^{-1}S + T^{-2}SN) \\ (I_{n-t} - N^\dagger N)S^*\triangle & (I_{n-t} - N^\dagger N)S^*\triangle(T^{-1}S + T^{-2}SN) \end{bmatrix} \\ &\Leftrightarrow X_1 = T^*\triangle,\ X_3 = (I_{n-t} - N^\dagger N)S^*\triangle. \end{aligned}$$

(ii) According to (4) *and* (6)*, we obtain that*

$$\begin{aligned} A^{-,\textcircled{W}} = A^{\textcircled{W}_2} &\Leftrightarrow \begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix} = \begin{bmatrix} T^{-1} & T^{-2}S + T^{-3}SN \\ 0 & 0 \end{bmatrix} \\ &\Leftrightarrow X_1 = T^{-1},\ X_3 = 0. \end{aligned}$$

(iii) Using (4) *and* (7)*, we get*

$$\begin{aligned} A^{-,\textcircled{W}} = A^{\diamond}A^{\textcircled{W}}A &\Leftrightarrow \begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix} = \begin{bmatrix} T^*\triangle_1 & T^*\triangle_1(T^{-1}S + T^{-2}SN) \\ (I_{n-t} - N^\dagger N)S^*\triangle_1 & (I_{n-t} - N^\dagger N)S^*\triangle_1(T^{-1}S + T^{-2}SN) \end{bmatrix} \\ &\Leftrightarrow X_1 = T^*\triangle_1,\ X_3 = (I_{n-t} - N^\dagger N)S^*\triangle_1. \end{aligned}$$

□

Theorem 4.6. *Let $A \in \mathbb{C}_{n,n}$ be a matrix written as in* (1)*. Suppose the inner inverse A^- in $A^{-,\textcircled{W}}$, $A^{-,\textcircled{\#}}$ and $A^{-,D}$ is of the same form. Then*
(i) $A^{-,\textcircled{W}} = A^{-,D}$ if and only if $X_1 = Y_1$, $X_3 = Y_3$, $SN^2 = 0$.
(ii) $A^{-,\textcircled{W}} = A^{-,\textcircled{\#}}$ if and only if $X_1 = Z_1$, $X_3 = Z_3$, $TS + SN = 0$,
where X_1, X_3 are defined in Theorem 4.1.

Proof. *(i) According to Theorem 4.1 and Lemma 4.3 (i) we have*

$$\begin{aligned} A^{-,\textcircled{W}} = A^{-,D} &\Leftrightarrow \begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix} = \begin{bmatrix} Y_1 & Y_1(T^k)^{-1}\tilde{T}_k \\ Y_3 & Y_3(T^k)^{-1}\tilde{T}_k \end{bmatrix} \\ &\Leftrightarrow X_1 = Y_1,\ X_3 = Y_3,\ X_3(T^{-1}S + T^{-2}SN) = Y_3(T^k)^{-1}\tilde{T}_k, \\ &(T^k)^{-1}\tilde{T}_k = T^{-1}S + T^{-2}SN \\ &\Leftrightarrow X_1 = Y_1,\ X_3 = Y_3,\ SN^2 = 0. \end{aligned}$$

(ii) From Theorem 4.1 and Theorem 4.3 (ii) we obtain

$$\begin{aligned} A^{-,\textcircled{W}} = A^{-,\textcircled{\#}} &\Leftrightarrow \begin{bmatrix} X_1 & X_1(T^{-1}S + T^{-2}SN) \\ X_3 & X_3(T^{-1}S + T^{-2}SN) \end{bmatrix} = \begin{bmatrix} Z_1 & 0 \\ Z_3 & 0 \end{bmatrix} \\ &\Leftrightarrow X_1 = Z_1,\ X_3 = Z_3,\ X_3(T^{-1}S + T^{-2}SN) = 0, \\ &T^{-1}S + T^{-2}SN = 0 \\ &\Leftrightarrow X_1 = Z_1,\ X_3 = Z_3,\ TS + SN = 0. \end{aligned}$$

□

5. Successive matrix squaring algorithm for the 1WG inverse

The successive matrix squaring (SMS) algorithm from [25, 30] is an efficient algorithm in order to compute different generalized inverses. Using this algorithm, we investigated the 1WG inverse in the following result.

Since

$$\begin{aligned} A^{-}A^{k}(A^{k+2})^{\dagger}A^{2}(AA^{-,Ⓦ}) &= A^{-}A^{k}(A^{k+2})^{\dagger}A^{3}A^{-}A^{k}(A^{k+2})^{\dagger}A^{2} \\ &= A^{-}A^{k}(A^{k+2})^{\dagger}A^{2}, \end{aligned}$$

we get

$$\begin{aligned} A^{-,Ⓦ} &= A^{-,Ⓦ} - \beta(A^{-}A^{k}(A^{k+2})^{\dagger}A^{2}AA^{-,Ⓦ} - A^{-}A^{k}(A^{k+2})^{\dagger}A^{2}) \\ &= (I_n - \beta A^{-}A^{k}(A^{k+2})^{\dagger}A^{3})A^{-,Ⓦ} + \beta A^{-}A^{k}(A^{k+2})^{\dagger}A^{2}. \end{aligned}$$

According to the following matrices

$$P = I_n - \beta A^{-}A^{k}(A^{k+2})^{\dagger}A^{3},\ \ Q = \beta A^{-}A^{k}(A^{k+2})^{\dagger}A^{2},\ \ \beta > 0.$$

It is evident that $A^{-,Ⓦ}$ is the unique solution of $X = PX + Q$. Then an iterative procedure for computing $A^{-,Ⓦ}$ can be defined as follows

$$X_1 = Q,\ \ X_{m+1} = PX_m + Q. \tag{8}$$

This algorithm can be implemented in parallel by considering the block matrix

$$T = \begin{pmatrix} P & Q \\ 0 & I_n \end{pmatrix},\ \ T^m = \begin{pmatrix} P^m & \sum_{i=0}^{m-1} P^i Q \\ 0 & I_n \end{pmatrix}.$$

In the second equation, the top right block is X^m, the mth approximation to $A^{-,Ⓦ}$. And the matrix power T^m can be computed by the successive squaring

$$T_0 = T,\ \ T_{i+1} = T_i^2,\ \ i = 0, 1, \ldots, J,$$

where the integer J is such that $2^J \geq m$.

Next, the sufficient condition for the convergence of the iterative process is given [18, 31].

Theorem 5.1. *For $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$ and $rank(A^k) = r$. Then, the approximation*

$$X_{2^m} = \sum_{i=0}^{2^m-1}(I_n - \beta A^{-}A^{k}(A^{k+2})^{\dagger}A^{3})^{i}\beta A^{-}A^{k}(A^{k+2})^{\dagger}A^{2},$$

defined by the iterative process (8) *converges to the 1WG inverse $A^{-,Ⓦ}$, if the spectral radius $\rho(I_n - X_1A) \leq 1$. And the following error estimation holds*

$$\|A^{-,Ⓦ} - X_{2^m}\| \leq \|A^{-,Ⓦ}\|\|(I_n - X_1A)^{2^m}\|.$$

Therefore,

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{-,Ⓦ} - X_{2^m}\|} \leq \rho(I_n - X_1A) \leq 1.$$

Proof. It is evident that $A^{-,\textcircled{W}}AA^{-,\textcircled{W}} = A^{-,\textcircled{W}}$ and $X_{2^m}AA^{-,\textcircled{W}} = X_{2^m}$. By the mathematical induction, we have

$$I_n - X_{2^m}A = (I_n - X_1A)^{2^m}.$$

As a consequence,

$$\begin{aligned}\|A^{-,\textcircled{W}} - X_{2^m}\| &= \|A^{-,\textcircled{W}} - X_{2^m}AA^{-,\textcircled{W}}\| \leq \|A^{-,\textcircled{W}}\|\|I_n - X_{2^m}A\| \\ &= \|A^{-,\textcircled{W}}\|\|(I_n - X_1A)^{2^m}\|,\end{aligned}$$

and since $\lim_{m\to\infty}\|B^n\|^{\frac{1}{n}} = \rho(B)$, we can get

$$\lim_{m\to\infty} sup \sqrt[2^m]{\|A^{-,\textcircled{W}} - X_{2^m}\|} \leq \rho(I_n - X_1A) \leq 1.$$

If β is a real parameter such that $\max_{1\leq i\leq s}|1-\beta\lambda_i| < 1$, where λ_i $(i = 1, 2, \ldots, s)$ are the nonzero eigenvalues of $A^-A^k(A^{k+2})^\dagger A^3$. Then

$$\rho(I_n - X_1A) = \rho(I_n - \beta A^-A^k(A^{k+2})^\dagger A^3) \leq 1.$$

□

Example 5.2. Let$A = \begin{pmatrix}1&1&0\\0&0&1\\0&0&0\end{pmatrix}$ with $A^- = \begin{pmatrix}0&-1&0\\1&1&0\\0&1&0\end{pmatrix}$, $Ind(A) = 2$. Now, $P = I_3 - \beta A^-A^2(A^4)^\dagger A^3$, $Q = \beta A^-A^2(A^4)^\dagger A^2$, $\beta = 0.6$. The eigenvalues λ_i of QA are included in the set{0, 0, 0.6}.The nonzero eigenvalues λ_i satisfy

$$\max_i|1-\beta\lambda_i| = 1 - 0.36 = 0.64 < 1.$$

Then we obtain the satisfactory approximation for $A^{-,\textcircled{W}}$ after the 6th iteration of the successive matrix squaring algorithm.

$$(T^2)^6 \approx \begin{pmatrix}1.0000&0&0&0&0&0\\-1.0000&0&-1.0000&1.0000&1.0000&1.0000\\0&0&1.0000&0&0&0\\0&0&0&1.0000&0&0\\0&0&0&0&1.0000&0\\0&0&0&0&0&1.0000\end{pmatrix}.$$

The upper right corner of $(T^2)^6$ is an approximation of the 1WG inverse.

$$A^{-,\textcircled{W}} = \begin{pmatrix}0&0&0\\1.0000&1.0000&1.0000\\0&0&0\end{pmatrix}.$$

6. Applications

In this section, we investigate the relationship between the 1WG inverse $A^{-,\textcircled{W}}$ and a nonsingular bordered matrix [21, 22].

Theorem 6.1. *Let $A \in \mathbb{C}_{n,n}$ with $Ind(A) = k$. Suppose $P \in \mathbb{C}_{n,r}$ and $Q^* \in \mathbb{C}_{n,r}$ are of full column rank such that*

$$\mathcal{N}\left((A^k)^*A^2\right) = \mathcal{R}(P),\ \mathcal{R}\left(A^-A^k\right) = \mathcal{N}(Q).$$

Then the bordered matrix

$$X = \begin{pmatrix} A & P \\ Q & 0 \end{pmatrix}$$

is nonsingular and

$$X^{-1} = \begin{pmatrix} A^{-,Ⓦ} & (I_n - A^{-,Ⓦ}A)Q^{\dagger} \\ P^{\dagger}(I_n - AA^{-,Ⓦ}) & -P^{\dagger}(A - AA^{-,Ⓦ}A)Q^{\dagger} \end{pmatrix}.$$

Proof. *Since* $\mathcal{R}\left(A^{-,Ⓦ}\right) = \mathcal{R}\left(A^{-}A^{k}\right) = \mathcal{N}(Q)$, *we have* $QA^{-,Ⓦ} = 0$. *Moreover,*

$$\mathcal{R}\left(I_n - AA^{-,Ⓦ}\right) = \mathcal{N}\left(AA^{-,Ⓦ}\right) = \mathcal{N}\left(A^{-,Ⓦ}\right) = \mathcal{N}\left((A^k)^*A^2\right) = \mathcal{R}(P) = \mathcal{R}\left(PP^{\dagger}\right).$$

Thus, $PP^{\dagger}(I_n - AA^{-,Ⓦ}) = I_n - AA^{-,Ⓦ}$.

For

$$Y = \begin{pmatrix} A^{-,Ⓦ} & (I_n - A^{-,Ⓦ}A)Q^{\dagger} \\ P^{\dagger}(I_n - AA^{-,Ⓦ}) & -P^{\dagger}(A - AA^{-,Ⓦ}A)Q^{\dagger} \end{pmatrix},$$

we obtain

$$\begin{aligned} XY &= \begin{pmatrix} AA^{-,Ⓦ} + PP^{\dagger}(I_n - AA^{-,Ⓦ}) & A(I_n - A^{-,Ⓦ}A)Q^{\dagger} - PP^{\dagger}(A - AA^{-,Ⓦ}A)Q^{\dagger} \\ QA^{-,Ⓦ} & Q(I_n - A^{-,Ⓦ}A)Q^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} AA^{-,Ⓦ} + I_n - AA^{-,Ⓦ} & A(I_n - A^{-,Ⓦ}A)Q^{\dagger} - PP^{\dagger}(A - AA^{-,Ⓦ}A)Q^{\dagger} \\ QA^{-,Ⓦ} & QQ^{\dagger} - QA^{-,Ⓦ}AQ^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} I_n & A(I_n - A^{-,Ⓦ}A)Q^{\dagger} - (A - AA^{-,Ⓦ}A)Q^{\dagger} \\ 0 & QQ^{\dagger} \end{pmatrix} \\ &= \begin{pmatrix} I_n & 0 \\ 0 & I_r \end{pmatrix}. \end{aligned}$$

Similarly, one can verify $YX = I_{n+r}$. *Thus,* $Y = X^{-1}$. □

Theorem 6.2. *Let* $A \in \mathbb{C}_{n,n}$ *with* $Ind(A) = k$. *The equation*

$$Ax = A^{Ⓦ}Ab, \ b \in \mathbb{C}_n, \tag{9}$$

is consistent and its general solution is

$$x = A^{-,Ⓦ}b + (I_n - A^{-}A)y, \ \forall y \in \mathbb{C}_n. \tag{10}$$

Proof. *If x has a form* (10), *then*

$$Ax = A(A^{-,Ⓦ}b + (I_n - A^{-}A)y) = A^{Ⓦ}Ab + Ay - AA^{-}Ay = A^{Ⓦ}Ab.$$

On the other hand, let y be another solution of (9). *It is easy to see* $A^{-,Ⓦ}b = A^{-}A^{Ⓦ}Ab = A^{-}Ay$. *We have*

$$y = A^{-,Ⓦ}b + y - A^{-,Ⓦ}b = A^{-,Ⓦ}b + y - A^{-}Ay = A^{-,Ⓦ}b + (I_n - A^{-}A)y.$$

Thus, the solution has the form (10). □

7. A binary relation based on the 1WG inverse

In this section, we study the binary relation for the 1WG inverse, which is similar to 1D inverse [26], ICEP inverse [8] and 1MP partial order [24].

Definition 7.1. *Let $A, B \in \mathbb{C}_{n,n}$. Then, we say A is below B under the binary relation $\leq^{-,\textcircled{w}}$ if $A^{-,\textcircled{w}}A = A^{-,\textcircled{w}}B$ and $AA^{-,\textcircled{w}} = BA^{-,\textcircled{w}}$. The relation is denoted by $A \leq^{-,\textcircled{w}} B$.*

It is easy to see that the binary relation $\leq^{-,\textcircled{w}}$ is reflexive. Using the following examples, we can observe it is neither symmetric nor anti-symmetric.

Example 7.2. *Let* $A = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. *The inner inverse of A and B are*

$$A^{-} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{4} & 1 \\ 1 & 1 & 1 \\ 0 & \frac{1}{2} & 1 \end{pmatrix}, \ B^{-} = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

Then, by calculating we get

$$A^{\textcircled{w}} = B^{\textcircled{w}} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A^{-,\textcircled{w}} = B^{-,\textcircled{w}} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{4} \\ 1 & 0 & \frac{1}{2} \\ 0 & 0 & 0 \end{pmatrix}.$$

We obtain $A^{-,\textcircled{w}}A = A^{-,\textcircled{w}}B$ and $AA^{-,\textcircled{w}} = BA^{-,\textcircled{w}}$, so $A \leq^{-,\textcircled{w}} B$. Similarly, $B^{-,\textcircled{w}}B = B^{-,\textcircled{w}}A$ and $BB^{-,\textcircled{w}} = AB^{-,\textcircled{w}}$, i.e., $B \leq^{-,\textcircled{w}} A$. However, $A \neq B$. The relation $\leq^{-,\textcircled{w}}$ is not anti-symmetric.

Example 7.3. *Let* $A = \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$, *and*

$$A^{-} = \begin{pmatrix} 1 & 0 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \ B^{-} = \begin{pmatrix} -1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}.$$

We can verify that

$$A^{\textcircled{w}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B^{\textcircled{w}} = \begin{pmatrix} 1 & 2 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ A^{-,\textcircled{w}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \ B^{-,\textcircled{w}} = \begin{pmatrix} -1 & -2 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

It is easy to check $A \leq^{-,\textcircled{w}} B$. Since $BB^{-,\textcircled{w}} \neq AB^{-,\textcircled{w}}$, we can not get $B \leq^{-,\textcircled{w}} A$. Thus, the relation $\leq^{-,\textcircled{w}}$ is not symmetric.

Next, we discuss the conditions that make it becoming a partial order.

Theorem 7.4. *Let $A, B \in \mathbb{C}_{n,n}$. Then*
(i) $A^{-,\textcircled{w}}A = A^{-,\textcircled{w}}B$ if and only if $A^{\textcircled{w}}A^2 = A^{\textcircled{w}}AB$.
(ii) $AA^{-,\textcircled{w}} = BA^{-,\textcircled{w}}$ if and only if $A^{\textcircled{w}} = BA^{-}A^{\textcircled{w}}$.

Proof. (i) Let $A^{-,\textcircled{w}}A = A^{-,\textcircled{w}}B$. Then $A^{\textcircled{w}}A^2 = AA^{-}A^{\textcircled{w}}A^2 = AA^{-,\textcircled{w}}A = AA^{-,\textcircled{w}}B = A^{\textcircled{w}}AB$. Conversely, pre multiplying by A^{-} on both sides of $A^{\textcircled{w}}A^2 = A^{\textcircled{w}}AB$, we can obtain $A^{-,\textcircled{w}}A = A^{-,\textcircled{w}}B$.

(ii) Let $AA^{-,\textcircled{w}} = BA^{-,\textcircled{w}}$. Post multiplying by $A^{\textcircled{w}}$ from the right side of the equation, we have $AA^{-,\textcircled{w}}A^{\textcircled{w}} = A^{\textcircled{w}}$, and $BA^{-,\textcircled{w}}A^{\textcircled{w}} = BA^{-}A^{\textcircled{w}}$. Then, $A^{\textcircled{w}} = BA^{-}A^{\textcircled{w}}$. Conversely, let $A^{\textcircled{w}} = BA^{-}A^{\textcircled{w}}$. It is easy to check $A^{\textcircled{w}}A = BA^{-}A^{\textcircled{w}}A$. Thus, $AA^{-,\textcircled{w}} = BA^{-,\textcircled{w}}$. □

Theorem 7.5. *Let $A, B \in \mathbb{C}_{n,n}$. Then the following statements are equivalent:*
(i) $A \leq^{-,\textcircled{W}} B$.
(ii) $A^{\textcircled{W}}A^2 = BA^{-,\textcircled{W}}A = A^{\textcircled{W}}AB$.
(iii) $A^{\textcircled{W}}A^2 = A^{\textcircled{W}}AB$ and $A^{\textcircled{W}}A = BA^{-,\textcircled{W}}$.

Proof. *(i)⇒(ii) Let $A \leq^{-,\textcircled{W}} B$. Then $A^{\textcircled{W}}A^2 = AA^{-,\textcircled{W}}A = BA^{-,\textcircled{W}}A$. We also have $A^{\textcircled{W}}AB = AA^{-,\textcircled{W}}B = AA^{-,\textcircled{W}}A = A^{\textcircled{W}}A^2$.*
(ii)⇒(iii) Clearly, $A^{\textcircled{W}}A = AA^{-,\textcircled{W}} = AA^{-,\textcircled{W}}AA^{-,\textcircled{W}} = A^{\textcircled{W}}A^2A^{-,\textcircled{W}} = BA^{-,\textcircled{W}}AA^{-,\textcircled{W}} = BA^{-,\textcircled{W}}$.
(iii)⇒(i) Since $A^{\textcircled{W}}A = BA^{-,\textcircled{W}}$, then $A^{\textcircled{W}} = BA^{-,\textcircled{W}}A^{\textcircled{W}} = BA^{-}A^{\textcircled{W}}$. From Theorem 7.4, we get $A \leq^{-,\textcircled{W}} B$. □

Theorem 7.6. *Let $A, B \in \mathbb{C}_{n,n}$ and A be a matrix written as in (2.1). Then the following statements are equivalent:*
(i) $A \leq^{-,\textcircled{W}} B$.
(ii) $B = U\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}U^*$ with $X_1(B_1 + (T^{-1}S + T^{-2}SN)B_3) = X_1T$, $X_1(B_2 + (T^{-1}S + T^{-2}SN)B_4) = X_1(S + T^{-1}SN + T^{-2}SN^2)$, $X_3(B_1 + (T^{-1}S + T^{-2}SN)B_3) = X_3T$, $X_3(B_2 + (T^{-1}S + T^{-2}SN)B_4) = X_3(S + T^{-1}SN + T^{-2}SN^2)$, $B_1X_1 + B_2X_3 = I_n$, $B_3X_1 + B_4X_3 = 0$, where $X_1 = T^{-1} - T^{-1}SX_3$, $NX_3 = 0$.

Proof. *(i) ⇒(ii) Suppose that $B = U\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}U^*$ is partitioned as the sizes of the blocks of A. Then $A^{-,\textcircled{W}}$ can be expressed by using Theorem 4.1. Since $A \leq^{-,\textcircled{W}} B$, then from $A^{-,\textcircled{W}}A = A^{-,\textcircled{W}}B$, we get*

$$
\begin{aligned}
X_1T = X_1(B_1 + (T^{-1}S + T^{-2}SN)B_3),\ & X_3T = X_3(B_1 + (T^{-1}S + T^{-2}SN)B_3), \\
X_1(S + (T^{-1}S + T^{-2}SN)N) =& X_1(B_2 + (T^{-1}S + T^{-2}SN)B_4), \\
X_3(S + (T^{-1}S + T^{-2}SN)N) =& X_3(B_2 + (T^{-1}S + T^{-2}SN)B_4).
\end{aligned}
$$

Moreover, it follows from $AA^{-,\textcircled{W}} = BA^{-,\textcircled{W}}$ that $B_1X_1 + B_2X_3 = I_n$, $B_3X_1 + B_4X_3 = 0$.
(ii)⇒(i) Obviously, the binary relation is established. □

Lemma 7.7. *[23] Let $A, B \in \mathbb{C}_n^{CM}$. If $A \leq^{-,\textcircled{W}} B$, then $A \# \leq B$, where $\# \leq$ is the left sharp partial order.*

Proof. *Suppose that $A \leq^{-,\textcircled{W}} B$ for $A, B \in \mathbb{C}_n^{CM}$. Then $A^{-,\textcircled{W}}A = A^{-,\textcircled{W}}B$ and $AA^{-,\textcircled{W}} = BA^{-,\textcircled{W}}$. Since $A \in \mathbb{C}_n^{CM}$, we have $A^{\textcircled{W}} = A^{\#}$. From $A^{-,\textcircled{W}}A = A^{-,\textcircled{W}}B$, we get $A^{-}A^{\#}AA = A^{-}A^{\#}AB$ and $A^{-}AA^{\#}A = A^{-}AA^{\#}B$. Multiplying $A^{-}AA^{\#}A = A^{-}AA^{\#}B$ by A^2 from the left side, we have $A^2 = AB$. On the other hand, similarly from $AA^{-,\textcircled{W}} = BA^{-,\textcircled{W}}$, we obtain $A = BA^{-}A$, then $\mathcal{R}(A) \subseteq \mathcal{R}(B)$. From Definition 6.3.1 in reference [15], we get $A \# \leq B$. Thus, A is a predeccessor of B under the left sharp partial order.* □

8. Conclusion

In this section, we conclude the main contents of the paper and put forward some prospects. The 1WG inverse of square matrices and its representations are presented. The characterization for the 1WG inverse is investigated from geometrical point of view, algebraic and geometrical point of view. Also, we characterize the 1WG inverse by using matrix equalities. Based on the core-EP decomposition, the relationships between the 1WG inverse and other generalized inverses are investigated. A successive matrix squaring (SMS) algorithm for the 1WG inverse is proposed. Application of the 1WG inverse in solving some system of linear equations are presented too. The binary relation for the 1WG inverse is also discussed.

Possible further research goals include mainly the following topics.

- To explore the proposed inverses for rectangular matrices by using the weight concept.
- One possibility is to study continuity and perturbations results and applications for the 1WG inverse.
- An extension of the 1WG inverse from the matrix case to Hilbert spaces operators or elements in a ring can be considered in future research.

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